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SOLVABILITY OF THE OPERATOR MATRIX OF VOLTERRA INTEGRAL EQUATIONS OF THE FIRST KIND

قابلية حل مؤثر المصفوفة لمعادلات فولتيرا التكاملية من النوع الاول

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Abstract:

In this paper, we have presented and discussed an algorithm for analytical method to solve matrix of linear and nonlinear systems of Volterra integral equations of the first kind. Algorithm of the analytical method for the system based on the Laplace transform and generalized inverse. The proposed algorithm has been applied in example to demonstrate the efficiency and simplicity of the algorithm.

المستخلص:

في هذا البحث قمنا بعرض ومناقشة خوارزمية بطريقة تحليلية لحل مصفوفة أنظمة خطية وغير الخطية لمعادلات فولتيرا التكاملية من النوع الاول. الخوارزمية للطريقة التحليلية لهذه المصفوفة تقوم على تحويل لابلاس والمعكوس العام. كما وتم تطبيق الخوارزمية على مثال للتدليل على كفائتها وبساطتها.

1- Introduction:

Biazar, Babolian and Islam [3] used adomian decomposition method to solve linear and non-linear systems of Volterra integral equations of the first kind. Amaal and Sudad in [2] introduced an algorithm to solve a system of linear Volterra integral equations of the first kind by using generalized inverse.

In this paper, we introduce an algorithm to solve a matrix of systems of integral Volterra equations.

2- The Analytic Algorithm:

2.1 Laplace Transform, [6]:

A system of linear Volterra integral equations of the first kind can be reduced to a matrix form by Laplace transform as follows:

$$\int_0^x \sum_{j=1}^n k_{ij}(x, t)u_j(t)\,dt = f_i(x),\ i = 1, 2, \ldots, m$$

where f_i, $i = 1, 2, \ldots, m$; are known functions, $k_{ij}(x, t)$, $i =1, 2, \ldots, m$, $j =1, 2, \ldots, n$; are the kernel of the i-th integral equation and u_j, $j = 1, 2, \ldots, n$ are unknown functions.

Take the Laplace transform to the both sides, yields:

$$\sum_{j=1}^n L\left\{\int_0^x k_{ij}(x, t)u_j(t)\,dt\right\} = L\{f_i(x)\},\ i = 1, 2, \ldots, m \quad \text{.................................(1)}$$

Note that, the term $L\{\int_0^x k_{ij}(x, t)u_j(t)\,dt\}$ in the left hand side of the equation (1) could not be evaluated unless $k_{ij}(x, t)$, i =1, 2, …, m; j = 1, 2, …, n; are the difference kernel, that is, $k_{ij}(x, t) = k_{ij}(x - t)$ or constant kernel.

If $k_{ij}(x, t)$ are difference kernels in equation (1). Then can be used the convolution property of Laplace transform, we get:

$$\sum_{j=1}^{n} K_{ij}(s)U_j(s) = F_j(s),\ i = 1, 2, \ldots, m \quad \ldots\ldots(2)$$

Where $K_{ij}(s) = L\{k_{ij}(x, t)\}$, $U_j(s) = L\{u_j(x)\}$, $F_i(s) = L\{f_i(x)\}$, i =1, 2, …, m; j = 1, 2, …, n.

If $k_{ij}(x, t) = c$, where c is any constant, then:

$$\sum_{j=1}^{n} \frac{cU_j(s)}{s} = F_i(s),\ i = 1, 2, \ldots, m \quad \ldots\ldots(3)$$

Consequently equations (2) and (3) are both systems of linear equation is $U_j(s)$, j = 1, 2, …, m. Solving it by generalized inverse to find $U_j(s)$, j =1, 2, …, m.

Finally, using inverse Laplace transform on $u_j(x)$, j = 1, 2, …, m to obtain the solution of the original system of linear Volterra integral equation.

2.2 Generalized Inverse:

Throughout the paper, H and K are Hilbert spaces over the same field. We denote the set of all bounded linear operators from H into K by $\mathcal{L}(H, K)$ and $\mathcal{L}(H)$, when H = K. For $A \in \mathcal{L}(H, K)$, let R(A) and N(A) be the range of A and the null space of A, respectively. $A \in \mathcal{L}(H, K)$ is g-invertible, if there exists an operator $\bar{A} \in \mathcal{L}(H, K)$, such that $A\bar{A}A = A$. In this case $\bar{A}$ is called a g-inverse, or an inner generalized inverse of A. Recall that $A \in \mathcal{L}(H, K)$ is g-invertible if and only if R(A) and N(A), respectively, are closed and complemented subspaces of K and H. In this case, the Moore-Penrose generalized inverse of A, denoted by A^+ [5] is the unique operator $A^+ \in \mathcal{L}(H, K)$, which satisfies:

$AA^+A = A$, $A^+AA^+ = A^+$

$(AA^+)^* = AA^+$ and $(A^+A)^* = A^+A$

It is well-known that if $A \in \mathcal{L}(H, K)$, then using the following decompositions:

$H = N \oplus N^\perp$ and $K = A(N) \oplus A(N)^\perp$

[where ($\perp$) is aperpendicular rotation, (*) is adjoinnt rotation and($\oplus$) is direct sum rotation]

Then:

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \in \mathcal{L}(H, K)$$

is equal to:

$$\begin{bmatrix} G_1^*(F_1^* A_1 G_1^*)^{-1} F_1^* & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{L}(H, K)$$

Such that $A_1 = F_1 G_1$, where F_1 G_1 are two bounded operators.

In this case, the Moore-Penrose generalized inverse of A has the following matrix decomposition:

$$A^+ = \begin{bmatrix} G^*(F^* A_1 G^*)^{-1} F^* & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{L}(A(N) \oplus N, K)$$

Now, in finite dimensional spaces, let A be m×n matrix of the rank(r), where r(A) < min{m, n} and $A_1 = F_1 G_1$, such that F and G are two matrices also have rank(r). Then the generalized inverse A^+ of A can be obtained from the following relation [1]:

$$A^+ = G^T(F^T A G^T)^{-1} F^T \qquad \text{.....(4)}$$

Now, construct the matrices F and G.

Firstly, the general Gaussian elimination procedure [4] applied to the matrix A, we obtain a new matrix have the rows below the matrix all elements are zeros and the other rows up the matrix represent the matrix G. The number of rows of G is the rank of matrix A.

To find the matrix F. Firstly, we write the identity matrix (I) of order m×m if n < m or n×n if m < n, then we apply the same operations which may be applied on the matrix A (to get the matrix G), but we begin from the last to the first operation with change sign the addition or the subtraction.

Finally, we find the matrix F from the first columns, such that the number of columns in F equal to the number of rows in G.

The solvability of the equation:

$$AU = B, \text{ where } A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \in \mathcal{L}(H, K)$$

and

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \in \mathcal{L}(H) \qquad \text{.....(5)}$$

are given operators, and U is a Laplace transform matrix.

The above system has a solution:

$$U = A^+ B$$

$$= \begin{bmatrix} G^*(F^* A_1 G^*)^{-1} F^* & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

$$= \begin{bmatrix} G^*(F^* A_1 G^*)^{-1} F^* B_1 & G^*(F^* A_1 G^*)^{-1} F^* B_2 \\ 0 & 0 \end{bmatrix}$$

This analytic algorithm will be illustrated by the examples in the ne xt section.

Example:

Consider the following matrix of system of linear Volterra integral equations with exact solutions $f(x) = x^2$ and $g(x) = x$.

$$\begin{bmatrix} \begin{bmatrix} \int_0^x [f(y)-g(y)]dy \\ \int_0^x [f(y)+(x-y)g(y)]dy \\ \int_0^x (x-y)f(y)dy \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \frac{X^3}{3}-\frac{X^2}{2} \\ \frac{X^3}{2} \\ \frac{X^4}{12} \end{bmatrix} \ldots 0 \\ 0\ldots\ldots\ldots\ldots\ldots\ldots 0 \end{bmatrix}$$

The algorithm start by taking the Laplace transform with using (2 and 3) and simplify, we get:

$$\begin{bmatrix} \begin{bmatrix} F(s)-G(s) \\ sF(s)+G(s) \\ F(s) \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \frac{1}{s^2} \\ \frac{3}{s^2} \\ \frac{2}{s^3} \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}$$

The above linear system can be described by the following matrix form:

$AU = B$

$$A_1 = \begin{pmatrix} 1 & -1 \\ s & 1 \\ 1 & 0 \end{pmatrix}, U_1 = \begin{pmatrix} F(s) \\ G(s) \end{pmatrix} \text{ and } B_1 = \begin{pmatrix} \frac{2-s}{s^3} \\ \frac{3}{s^2} \\ \frac{2}{s^3} \end{pmatrix}$$

Then equation (5) can be solved for the vector U_1, of coefficients by generalized inverse can be summarized by the following:

Step (1):

Consider the matrices F_1G_1, such that $A = F_1G_1$, as follows:

$$\begin{bmatrix} \begin{pmatrix} 1 & -1 \\ s & 1 \\ 1 & 0 \end{pmatrix} & 0 \\ 0 & 0 \end{bmatrix} \xrightarrow[R_3-R_1]{R_2-sR_1} \begin{bmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1+s \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3-\frac{1}{1+s}R_2} \begin{bmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1+s \\ 0 & 0 \end{pmatrix} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} G_1 \\ 0 \end{pmatrix} & 0 \\ 0 & 0 \end{bmatrix}$$

Then:

$$G_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1+s \end{pmatrix}$$

Now, to find F_1, let the identity matrix of order 3×3

$$\begin{bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & 0 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_3-\frac{1}{1+s}R_2} \begin{bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{1+s} & 1 \end{pmatrix} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\xrightarrow[R_3+R_1]{R_2+sR_1} \begin{bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ 1 & \frac{1}{1+s} & 1 \end{pmatrix} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} F_1 & 0 \\ & 0 \\ & 1 \end{pmatrix} & 0 \\ 0 & 0 \end{bmatrix}$$

Where:

$$F_1 = \begin{pmatrix} 1 & 0 \\ s & 1 \\ 1 & \frac{1}{1+s} \end{pmatrix}$$

Step (2):

Using equation (3) generalized inverse A^+ of matrix A and show that:

$$A^+ = \begin{pmatrix} \frac{1}{s^2+2s+3} & \begin{pmatrix} 1+s & 1+s & 2 \\ -(s^2+s+1) & 2+s & 1-s \end{pmatrix} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Step (3):

$$U = A^{+}B$$

$$= \begin{pmatrix} \begin{pmatrix} \frac{2}{s^3} \\ \frac{1}{s^2} \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{(6)}$$

$$= \begin{pmatrix} \begin{pmatrix} F(s) \\ G(s) \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix}$$

Step (4):

Using the inverse Laplace transformation for both sides of equation (6), we have:

$$\begin{pmatrix} \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} x^2 \\ x \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix}$$

3- References:

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