# Mathematical operator of Dirac delta function and the calculation of nucleon density distributions of some light nuclei

# مؤثر دالة ديراك الرياضي و حساب توزيعات كثافة النيكليون لبعض النوى الخفيفة

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## Abstract:

An analytical expression for the nucleon density distributions is derived based on the use of occupation numbers of the states and the single particle wave functions of the harmonic oscillator potential with size parameters chosen to reproduce the observed root mean square radii for all considered nuclei. The derived expression has been employed for calculating the nucleon density distributions. Considering the higher shells leads to reproduce remarkable agreement between the calculated and experimental results of the nucleon density distributions throughout the whole range of $r$.

[Keywords: Delta function, Nucleon density distributions, root mean square radii, occupation numbers of higher states]

## الخلاصة:

تم اشتقاق صيغة رياضية تحليلية لتوزيع كثافة النيكليون للنوى الواقعة ضمن القشرة النووية $1p$. تعتمد الصيغة الرباضية المشتقة على كل من اعداد الأشغال للحالات النووية والدوال الموجية للجهد المتذبذب التوافقي ذي أعلومات حجمية اختيرت لكي تعيد انتاج مربع متوسط انصاف اقطار النوى قيد الدراسة. لقد تم استخدام الصيغة الرباضية المشتقة في حساب توزيع كثافة النيكليون. لقد وجد ان ادخال القشرات النووية العالية في الحسابات تؤدي الى توافق ممتاز بين النتائج المحسوبة والنتائج العملية لتوزيع كثافة النيكليون ولكل قيم المسافة $r$ .

## Introduction:

The nucleon density distribution ($NDD$) is considered as an important quantity which was studied experimentally over a wide range of nuclei. This interest in the $NDD$ is related to the basic characteristics such as the shape and size of nuclei, their binding energies, and other quantities that are connected with the $NDD$. $NDD$ with $Z=N$ is related to the charge density distribution ($CDD$) by $NDD = 2\,CDD$. By measuring the elastic cross sections, one obtains information about the distribution of the charges within the nucleus. Various theoretical methods [Antonov, A.N. et. al] are used for calculations of $NDD$, among them the theory of finite Fermi system, the Hartree-Fock method with skyrme effective interaction. These methods describe correctly the density distributions in closed shell nuclei $^{4}He$, $^{16}O$ and $^{40}Ca$. An analytical expression were derived for the one and two body terms in the cluster expansion of $CDD$ and elastic electron scattering form factors of $1s$, $1p$ and $2s-1d$ shell nuclei [Massen, S.E. et. al., Moustakidis, H.C. et. al.]. This expression was used for the systematic study of the effect of the short-range correlations on the $CDD$. Hence, the studies [Massen, S.E. et. al. Moustakidis, H.C. et. al.] depend on the harmonic oscillator size parameter ($b$) and the correlation parameter ($\beta$), since $b$ and $\beta$ were determined by fitting the theoretical charge electron scattering form factors to the experimental one. Many particle shell model wave functions were used to calculate the elastic electron scattering from factors of the even-even nuclei with mass number $A = 20-36$ that have positive parity states [Brown, B.A. et. al.]. Gul'karov et al. calculated the $CDD$ of some 1$p$-shell nuclei. The aim of this study is to derive an analytical expression for the $NDD$, based on the use of the single particle wave functions of the harmonic oscillator potential and the occupation numbers of the states. The derived expression, which is applicable through out the whole region of 1$p$- shell nuclei, is used to calculate the ($NDD$) of $^{6}Li, ^{7}Li, ^{10}B, ^{11}B, ^{14}N,$ and $^{15}N$ nuclei. The calculated $NDD$ of considered nuclei demonstrates a remarkable agreement with those of experimental data throughout the whole range of the distance $r$.

## Mathematical Part:

The $NDD$ of a system of A point nucleons can be expressed by using the one body density operator $\hat{\rho}(\vec{r})$ which is given by means of Dirac delta function as [Antonov, A.N. et. al].

$$\hat{\rho}(\vec{r}) = \sum_{i=1}^{A} \delta(\vec{r} - \vec{r}_i) \tag{1}$$

The expectation value of the one body density operator can be written in terms of the many particle wave function ($\Psi$) as:

$$< \Psi(\vec{r}_1, \vec{r}_2 ...., \vec{r}_A) \,|\, \hat{\rho}(\vec{r}) \,|\, \Psi(\vec{r}_1, \vec{r}_2 ...., \vec{r}_A) >= \sum_{i=1}^{A} \int_0^{\infty} \Psi^*(\vec{r}_1, \vec{r}_2 ...., \vec{r}_A)\delta(\vec{r} - \vec{r}_i)\Psi(\vec{r}_1, \vec{r}_2 ...., \vec{r}_A).d\vec{r}_1, d\vec{r}_2,.., d\vec{r}_A \tag{2}$$

The nuclear many particle wave functions $\Psi$ may be expressed as a Slater determinant depending on the single particle wave functions $\phi_i(\vec{r}_j)$ [ de Shalit A. et. al.]

$$\Psi(\vec{r}_1, \vec{r}_2, ...., \vec{r}_A) = \frac{1}{\sqrt{A!}} \det \phi_i(\vec{r}_j) \tag{3}$$

Integrating eq. (2) over the coordinates, and using the orthonormality property of $\phi_i(\vec{r}_j)$ yields

$$\rho_o(r) = \left\langle \Psi(\vec{r}_1, \vec{r}_2 ..., \vec{r}_A) \middle| \hat{\rho}(\vec{r}) \middle| \Psi(\vec{r}_1, \vec{r}_2 ..., \vec{r}_A) \right\rangle = \sum_{i=1}^{A} \int_0^{\infty} \phi_i^*(\vec{r}_1)\delta(\vec{r} - \vec{r}_1)\phi_i(\vec{r}_1)d\vec{r}_1 . \tag{4}$$

The nucleon density distribution, eq. (4), of a system containing $A$ nucleons may be obtained in terms of the single particle wave function as [Antonov, A.N. et. al]

$$\rho_o(r) = \sum_{i=1}^{A} |\phi_i(\vec{r})|^2 \tag{5}$$

In order to obtain an explicit expression for the one body density matrix elements, we shall use a harmonic oscillator basis and define our single particle states as

$$\phi_i(\vec{r}) = \psi_{nlm}(\vec{r})\chi_{s_i}\chi_{t_i}, \tag{6}$$

where $i \equiv (n, l, m, s, t)$ represents the principle quantum number ($n$), orbital angular momentum quantum number ($l$), projection to the $l$ quantum number ($m$), spin quantum number ($s$), and isospin quantum number ($t$) of $i^{th}$ single particle state. $\psi_{nlm}(\vec{r})$, $\chi_{s_i}$ and $\chi_{t_i}$ are the space, spin and isospin wave functions, respectively. The space wave function is a harmonic oscillator wave function, given by

$$\psi_{nlm}(\vec{r}) = R_{nl}(r)\, Y_{lm}(\theta_i, \varphi_i), \tag{7}$$

where $R_{nl}(r)$ is the harmonic oscillator radial wave function and $\Upsilon_{lm}(\theta_i,\phi_i)$ is the spherical harmonic wave function. Thus eq. (5) can be written as

$$\rho_o(r)=\sum_{nlm}\left|R_{nl}(r)\,\Upsilon_{lm}(\theta_i,\varphi_i)\right|^2\sum_{s_it_i}\left|\chi_{s_i}\chi_{t_i}\right|^2 \tag{8}$$

$$\rho_o(r)=4\sum_{nlm}\left|R_{nl}(r)\,\Upsilon_{lm}(\theta_i,\varphi_i)\right|^2 = 4\sum_{nl}\left|R_{nl}(r)\right|^2\sum_{m}\left|\Upsilon_{lm}(\theta_i,\phi_i)\right|^2 \tag{9}$$

where the factor 4 in eq. (9 ) takes into account the spin – isospin degeneracy, and the second term that concerns with the summation over the projection $m$ is given by [Edmonds A. R. et. al.]

$$\sum_m|\,\Upsilon_{lm}(\theta_i,\varphi_i)\,|^2=\frac{2l+1}{4\pi} \tag{10}$$

Introducing eq. (10) into eq. (9), the density of $A$ point nucleons system becomes

$$\rho_o(r)=4\sum_{nl}\left|R_{nl}(r)\right|^2\frac{2l+1}{4\pi} \tag{11}$$

in the unit of nucleon per unit of volume (nucleon$.fm^{-3}$). The total occupation number of nucleons in the orbit $l$, which is given by $4(2l+1)$, can be given in terms of the sub-orbit $J$ as

$$\sum_{nl}4(2l+1)\equiv\sum_{nlj}2(2J+1)\,. \tag{12}$$

Using eq. (12) in eq. (11), we obtain

$$\rho_o(r)=\frac{1}{4\pi}\sum_{nlj}2(2j+1)\left|R_{nl}(r)\right|^2 \tag{13}$$

or

$$\rho_o(r)=\frac{1}{4\pi}\sum_{nlj\in I}2(2j+1)\left|R_{nl}(r)\right|^2+\frac{1}{4\pi}\sum_{nlj\notin I}N_P\,|\,R_{nl}(r)\,|^2 \tag{14}$$

where $I$ is the closed orbit and $N_p$ is the number of nucleons in the unfilled orbit $nlj$. Eq.(14) gives the $NDD$ by means of the harmonic oscillator radial wave functions $R_{nl}(r)$ and the occupation numbers of the states. The radial part of the harmonic oscillator wave function is given by [Brussard, P.J. et. al.]

$$\begin{aligned}R_{nl}^2(r)=&\left[\left(\pi^{1/2}b^3\right).\{(2l+1)!!\}^2.(n-1)!\right]^{-1}2^{l-n+3}(2l+2n-1)!!\\&\times\left(\frac{r}{b}\right)^{2l}\sum_{k=0}^{n-1}[(-1)^k\{(n-k-1)!k!(2k+2l+1)!!\}^{-1}\\&\times2^k(n-1)!(2l+1)!!\left(\frac{r}{b}\right)^{2k}]^2.\exp(-r^2/b^2),\end{aligned} \tag{15}$$

where $b$ is the harmonic oscillator size parameter and (!) is the factorial. In the present work, the nucleon density distributions $NDD$ of the 1$p$- shell nuclei will be calculated on the basis that the occupation numbers of states differ from the predictions of the simple shell model. We assume that there is a core of filled 1$s$ shell (i.e. filled with 4 particles) and the particle numbers in 1$p$ and 2$s$-shells are equal to $(\mathrm{A}-4-\alpha)$ and $\alpha$ instead of (A – 4) and zero (as in the simple shell model) respectively. (A) represents the mass number of the nucleus and the parameter $\alpha$ represents the deviation of the shell nucleons from the prediction of the simple shell model. Using this assumption and with the help of eqs. (14) & (15), the ground state $NDD$ of 1$p$- shell nuclei is expressed as

$$\rho_0(r)=\frac{2e^{-r^2/b^2}}{\pi^{3/2}b^3}\left[2+\frac{3}{4}\alpha+\frac{A-4-4\alpha}{3}\left(\frac{r}{b}\right)^2+\frac{\alpha}{3}\left(\frac{r}{b}\right)^4\right] \tag{16}$$

it is important to note that, when $\alpha=0$, the $NDD$ of eq. (16) is reduced to that of the prediction of the simple shell model, i.e.

$$\rho_0(r)=\frac{2e^{-r^2/b^2}}{\pi^{3/2}b^3}\left[2+\frac{A-4}{3}\left(\frac{r}{b}\right)^2\right]. \tag{17}$$

The mean square radii (MSR) of the considered nuclei are obtained by [Vries, H. De. et. al.]

$$\langle r^2\rangle=\frac{4\pi}{A}\int_0^\infty \rho_o(r)\,r^4 dr\,, \tag{18}$$

where the normalization of $NDD$ is given by [1].

$$A=4\pi\int_0^\infty \rho_o(r)\ r^2\ dr\,. \tag{19}$$

Introducing the form of eq. (16) into eq. (18), we obtain the MSR of 1$p$- shell nuclei as

$$\langle r^2\rangle=b^2\left[\frac{\alpha}{A}+\frac{5}{2}+\frac{4}{A}\right] \tag{20}$$

In eq. (16) the parameter $\alpha$ is determined from the central $NDD$ $\rho_o(\mathrm{r}=0)$, i.e.

$$\rho_o(0)=\frac{2}{\pi^{3/2}\ b^3}\left[2+\frac{3}{4}\alpha\right] \tag{21}$$

Since $\rho_o(\mathrm{r}=0)$ and the oscillator size parameter $b$ are known, i.e. $\rho_o(0)$ can be taken from experiments and $b$ can be obtained by introducing the experimental MSR of considered nuclei into eq. (20)

## Results and Discussion:

The analytical expression of eq. (16) has been used for the study of *NDD*, $\rho_0(r)$, for $^6Li$, $^7Li$, $^{10}B$, $^{11}B$, $^{14}N$, and $^{15}N$ nuclei. The harmonic oscillator size parameter $b$ is chosen in such a way as to reproduce the experimental root mean square radii $\langle r^2\rangle^{1/2}_{\exp}$ of the considered nuclei, and the parameter $\alpha$ is determined by introducing the experimental $\rho_{\exp}(r=0)$ into eq. (21). It is important to remark that when $\alpha=0$, eq. (16) is reduced to that of the simple shell model prediction. Table (1) displays all parameters needed for calculating the distributions $\rho_0(r)$ and it also displays the calculated values of occupation numbers of $1p$ and $2s$ shells for all considered nuclei.

**Table-1:** Parameters of the distributions $\rho_0(r)$ and calculated occupation numbers of $1p$ and $2s$ shells for all considered nuclei.

| Nucleus | $b(fm)$ | $\rho_{\exp}(0)$ (in $fm^{-3}$) [10] | $<r^2>^{1/2}_{\exp}$ (in $fm$) [10] | Occupation No. of $1p$ $(A-4-\alpha)$ | Occupation No. of $2s$ $(\alpha)$ |
|---|---|---|---|---|---|
| $^{6}Li$ | 1.795 | 0.1548 | $2.54 \pm 0.050$ | 1.341 | 0.659 |
| $^{7}Li$ | 1.715 | 0.1521 | $2.39 \pm 0.030$ | 2.819 | 0.181 |
| $^{10}B$ | 1.685 | 0.1592 | $2.45 \pm 0.012$ | 5.838 | 0.162 |
| $^{11}B$ | 1.610 | 0.1846 | $2.42 \pm 0.012$ | 6.805 | 0.195 |
| $^{14}N$ | 1.700 | 0.1655 | $2.54 \pm 0.020$ | 9.646 | 0.354 |
| $^{15}N$ | 1.780 | 0.1643 | $2.70 \pm 0.030$ | 10.224 | 0.775 |

The dependence of the $NDD$ (in the unit of $fm^{-3}$) on $r$ (in the unit of $fm$) is shown in Figs. 1-3 for ($^{6}Li$, $^{7}Li$), ($^{10}B$, $^{11}B$) and ($^{14}N$, $^{15}N$) nuclei, respectively. The dashed distributions are the calculated $NDD$ obtained by the prediction of the simple shell model of eq. (16) with $\alpha = 0$ or eq. (17). The solid distributions are the calculated $NDD$ when the higher shell (i.e. the $2s-$shell) is included in the calculations and obtained by eq. (16) using the values of $\alpha$ given in Table (1). The solid circles (●) are the experimental data of Ref. [Vries, H. De. et. al.]. It is obvious that the form of the $NDD$ represented by eq. (16) or (17) behaves as an exponentially decreasing function as seen by the solid and dashed distributions for all considered nuclei of Figs. 1-3. These figures show that the probability of finding a nucleon near the central region ($0 \le r \le 2\ fm$) of $\rho_0(r)$ is larger than the tail region ($r > 2\ fm$). Besides, including the higher $2s-$shell through introducing the calculated values of $\alpha$ [presented in Table (1)] into eq. (16) leads to increasing significantly the central region of $\rho_0(r)$ and decreasing slightly the tail region of $\rho_0(r)$ as seen by the solid distributions. This means that the effect of inclusion of higher shells tends to increase the probability of transferring the nucleons from the surface region of the nucleus towards its central region and then makes the nucleus to be more rigid than the case when there is no this effect. Figs. 1-3 also illustrate that the dashed distributions in all considered nuclei are not in good accordance with those of experimental data of Ref. [Vries, H. De. et. al.], especially at the central region of $\rho_0(r)$. But once the higher $2s-$ shell is considered in the calculations, the results for the $NDD$ become in astonishing accordance with those of experimental data throughout the whole values of $r$ as seen by the solid distributions.

The solid distributions of $^{11}B$ [Fig. 2] and $^{15}N$ [Fig. 3] show a slight deviation from the experimental data especially at the regions of $0.8 \le r \le 1.8\ fm$ for $^{11}B$ and $1.6 \le r \le 2.2\ fm$ for $^{15}N$. Considering the higher $2s-$shell in eq. (16) improves strongly the calculated $NDD$ of $^{11}B$ and $^{15}N$, but this higher shell is not enough for resolving completely the problem of slight deviation. This deviation may be attributed to other reasons or to the necessity of considering other higher shells, such as the $1d-$shell or $1f-$shell. In this case the occupation numbers of $1d$ or $1f$ shell must be different from zero. This deviation doesn't affect the very well accordance with the experimental data throughout the whole values of $r$.

## Conclusion:

Through the use of Dirac delta function, the Introduction of an additional parameters $\alpha$, that reflects the difference of the occupation numbers of the states from the prediction of the simple shell model, leads to a very good agreement between the calculated and experimental results of the nucleon density distributions throughout all range of $r$.

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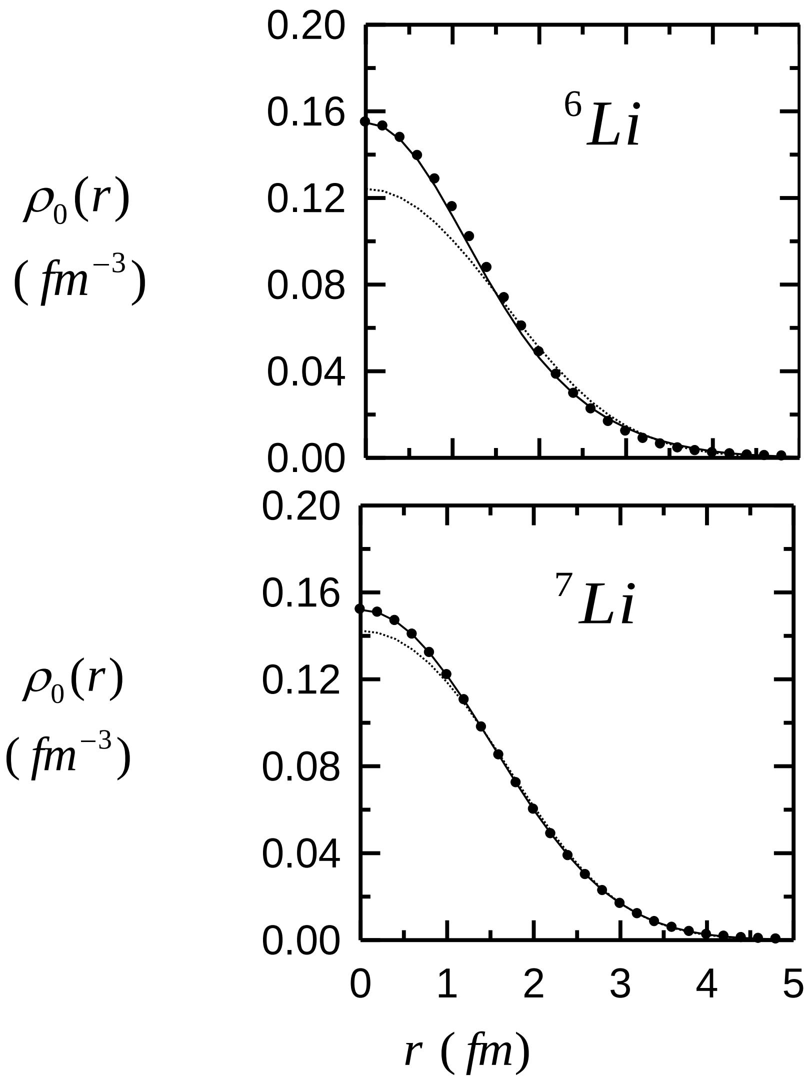


**Fig. 1:** The dependence of the nucleon density distributions $\rho_0(r)$ (in $fm^{-3}$) on $r$ (in $fm$) for $^6Li$ and $^7Li$ nuclei. The dashed distributions are the calculated $NDD$ obtained by the prediction of the simple shell model using eq. (16) with $\alpha = 0$ or eq. (17). The solid distributions are the calculated $NDD$ when the higher shell (i.e. the $2s-$shell) is included in the calculations and obtained by eq. (16) using the values of $\alpha$ given in Table (1). The solid circles (●) are the experimental data of Ref. [Vries, H. De. et. al.].

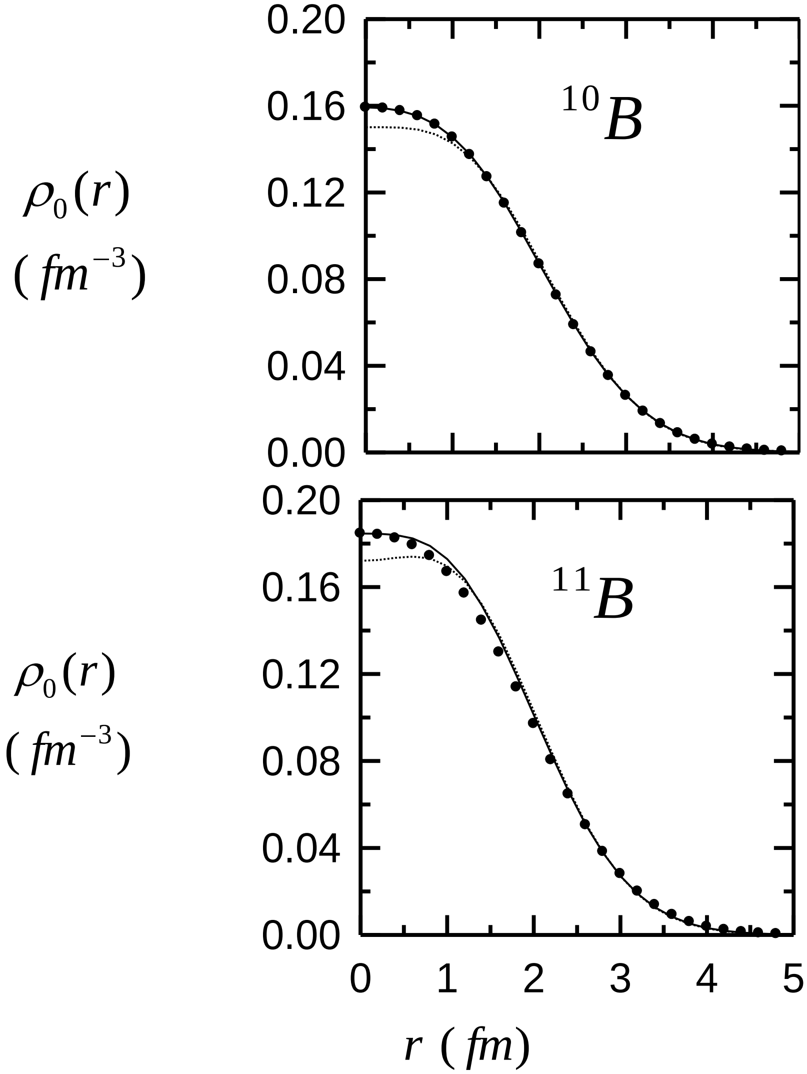


**Fig. 2:** The dependence of the nucleon density distributions $\rho_0(r)$ (in $fm^{-3}$) on $r$ (in $fm$) for $^{10}B$ and $^{11}B$ nuclei. The dashed distributions are the calculated $NDD$ obtained by the prediction of the simple shell model using eq. (16) with $\alpha = 0$ or eq. (17). The solid distributions are the calculated $NDD$ when the higher shell (i.e. the $2s-$shell) is included in the calculations and obtained by eq. (16) using the values of $\alpha$ given in Table (1). The solid circles (●) are the experimental data of Ref. [8 Vries, H. De. et. al.].

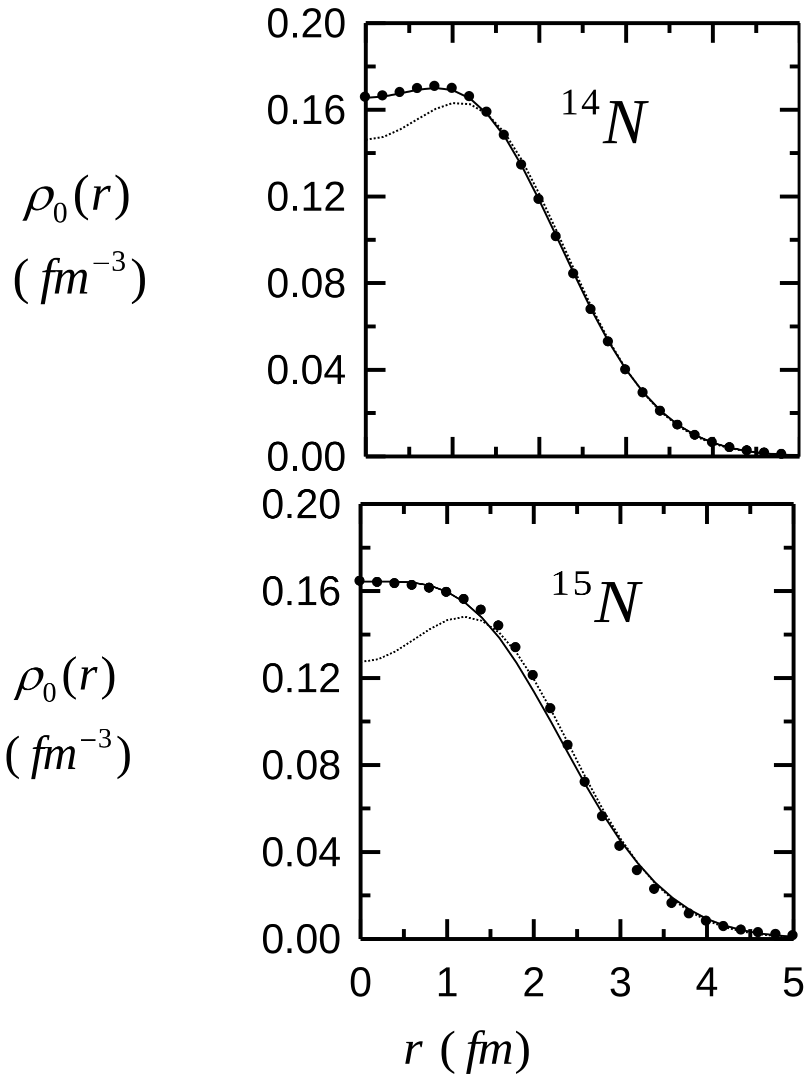


**Fig. 3:** The dependence of the nucleon density distributions $\rho_0(r)$ (in $fm^{-3}$) on $r$ (in $fm$) for $^{14}N$ and $^{15}N$ nuclei. The dashed distributions are the calculated *NDD* obtained by the prediction of the simple shell model using eq. (16) with $\alpha = 0$ or eq. (17). The solid distributions are the calculated *NDD* when the higher shell (i.e. the $2s-$shell) is included in the calculations and obtained by eq. (16) using the values of $\alpha$ given in Table (1). The solid circles (●) are the experimental data of Ref. [Vries, H. De. et. al.].