Topological Direct Projective Modules

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Abstract

In this search obtain the following results on a topological direct projective modules:

1. A topological direct sum of two topological direct projective modules is topologically direct projective .
2. Topological quasi projectivity of module implies topological direct projectivity of it is topological direct summands .
3. Topological quotient module of topological quasi projective module by topological stable sub module is topological direct projective module .

المستخلص

في بحثنا هذا حصلنا على النتائج الاتية في المقاسات الاسقاطية المباشرة التوبولوجية:

1. الجمع المباشر التوبولوجي لمقاسين توبولوجيين يمتلكان صفة الإسقاط المباشر التوبولوجي يكون مقاس اسقاط مباشر توبولوجي.
2. صفة الإسقاط المباشر التوبولوجي تنتقل من المقاس التوبولوجي الى كل مركبة جمع مباشرتوبولوجي من مركباته.
3. مقاس القسمة التوبولوجي الناتج من قسمة مقاس اسقاط مباشر توبولوجي على مقاس جزئى مستقرتوبولوجي منه يكون مقاس اسقاط مباشر توبولوجي.

Introduction:

Kaplanski is the first scientist who introduced the definition of topological module and topological sub module in 1955,after that, a topological module studied by many scientists like Alberto Tolono and Nelson. In this research a necessary and sufficient conditions for a topological module to be topological direct projective has been established ,the kind of topology don't determinant . The results in this research algebraically satisfied , now we will satisfying it algebraically and topologically.

This research divided into two sections. In section one includes some necessary definitions while section two includes propositions , references [1],[2]and[3]are used to construct definitions (1-15),(1-16),(1-17) and (1-18).

Section " ONE" Some Definitions:

In this section we introduce some necessary definitions.

Definition (1–1) [1]: A non empty set E is said to be a topological group if:

1. E is a group;
2. τ is a topology on E ;
3. A mappings $\pi : E \times E \to E$ and $\mu : E \to E$ are continuous, where μ defined as $\mu(x) \cong x^{-1}$, and $E \times E$ is the product of two topological spaces.

Definition (1–2) [1]: Let E be a topological group .A subset M of E is said to be a topological subgroup of E if:

1. M is a subgroup of E;
2. M is a subspace of E.

Definition (1–3) [1]: A non empty set *R* is said to be a topological ring if:
1. R is a ring;
2. τ is a topology on R;
3. A mappings $\pi : R \times R \to R$ defined as $\pi(x, y) = x - y$ and $\mu : R \to R$ defined as $\mu(x) \cong x^{-1}$ are continuous.

Definition (1–4) [4]: Let *R* be a topological ring. A non empty set *E* is said to be a topological module on *R* if:
1. E is a module on *R*;
2. E is a topological group;
3. A mapping $\pi : R \times E \to E$ defined as $\pi(\lambda, X) \cong \lambda X$ is continuous.

Definition (1–5) [4]: Let *E* be a topological module on a topological ring *R*. A subset *M* of *E* said to be a topological sub module of *E* if:
1. *M* is a sub module of *E*;
2. *M* is a topological subgroup of *E*.

Definition (1–6) [5]: Let E, E^* be two topological modules on a topological ring *R*, a mapping $f : E \to E^*$ is called a topological module morphism if :
1. *f* is a homomorphism;
2. *f* is continuous.

Definition (1–7) [5]: A topological module epimorphism is an onto topological module morphism.

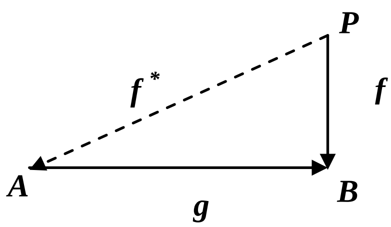
Definition (1–8) [5]: A topological module monomorphism is a one to one topological module morphism.

Definition (1–9) [5]: Let E, E' be two topological modules and $f : E \to E'$ a topological morphism. A topological kernel of *f* is the set $Kerf \cong \{t \in E : f(t) \cong e'\}$, where e' the identity element of E'.

Definition (1–10) [5]: Let M be a topological submodule of a topological module E on a topological ring R, the set E/M is said to be a topological quotient module if :
1. E/M is a module on *R*;
2. E/M is a topological group on *R* with quotient topology;
3.A mappings $\pi_M : E/M \times R \to E/M$ is continuous.

Definition (1–11) [5]: Let $\{E_\lambda\}_{\lambda \in \Omega}$ be a family of topological modules on a topological ring *R*, the set $E = \prod_{\lambda \in \Omega} E_\lambda$ with projection morphism is a topological direct product module on *R*, where $f : R \times E \to E$ defined as $f(\alpha(X_\lambda))_{\lambda \in \Omega} = \alpha(X_\lambda)_{\lambda \in \Omega}$ is continuous for all $X_\lambda \in E_\lambda$.

Definition (1–12) [5]: A topological module *P* is said to be a topological projective module if for each topological module morphism $f : P \to B$ and for each topological module epimorphism $g : A \to B$ there is a topological module morphism $f^* : P \to A$ for which the following diagram commutes



Definition (1–13) [5]: A topological module epimorphism $f^*: P \to M$ called a topological projective cover of M if P is a topological module and if f^* is a small topological module epimorphism.

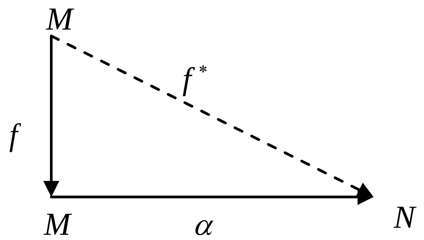
Definition (1–14): Let R be a topological ring, and $0 \to L \to A_1 \to B \xrightarrow{f} A_2 \to 0$ a short exact sequence of topological *R*-module morphism, then the sequence called a topological splits sequence if $fof^{-1} \cong I_{A_2}$.

Definition(1–15) : A topological sub module N of a topological module M is said to be a topologically stable if and only if $\alpha(N) \subseteq N$ for every $\alpha \in End(M)$.

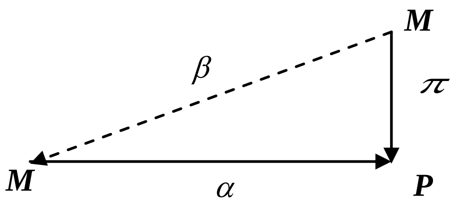
Definition (1–16): A projection morphism is a topological morphism from a topological module onto each one of it is summands.

Definition (1–17): $End(M)$ is the set of all one to one topological morphism from M onto M .

Definition (1–18): A topological module M is said to be a topological quasi projective if for each topological module morphism $\alpha: M \to N$ and for each topological module monomorphism $f: M \to M$ there is a topological module morphism $f^*: M \to N$ for which the following diagram commutes



Definition (1–19): A topological module M is said to be a topological direct projective if given any topological direct summand P of M with a topological projection $\pi: M \to P$ and topological epimorphism $\alpha: M \to P$, then there exists $\beta \in End(M)$ such that $\pi \cong \alpha o \beta$, (or the following diagram is commutative).



Section" TWO" Some Propositions:

In this section we introduce some necessary propositions and the relation between topological direct projective modules and topological quasi projective modules.

Propositions (2-1) : A topological direct sum of two topological direct projective modules is topologically direct projective .

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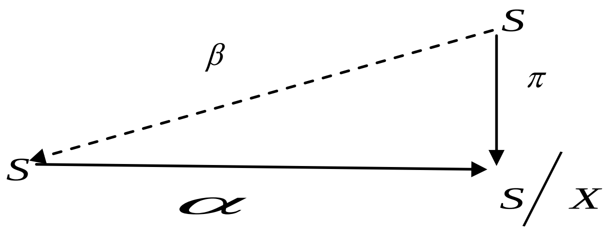
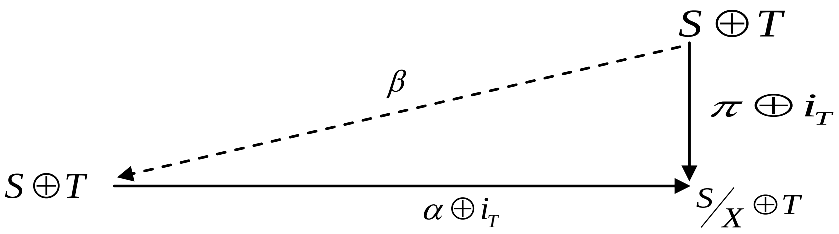
Proof : Let S and T be two topological direct projective modules, and let M be a topological direct sum of S and T, to prove topological direct projectivity of M , as S is topological direct projective, the diagram Commutes,



Then using the topological direct projectivity of S,T the above diagram can be embedded in the commutative diagram

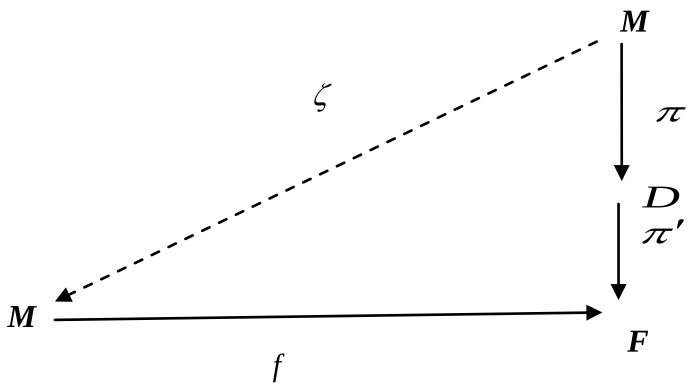


It is clear that, $M = S \oplus T$ is a topological direct projective module,
This completes the proof .

Proposition (2-2): A topological direct summand of a topological direct projective module is topologically direct projective .

Proof

Let M be a topological direct projective module, and let D be a topological direct summand of M , to prove topological direct projectivity of D, we consider any topological direct summand F of D, let $\pi : M \to D$ and $\pi' : D \to F$ be the topological projection morphism, further let $i : D \to M$ be a topological injection morphism and $f : M \to F$ be a topological epimorphism, as M is topological direct projective, the diagram



Commutes, i.e. (there exists $\zeta \in End(M)$ such that $\pi' o \pi \cong f o \zeta$)
Let $h \cong \zeta o i$,where $h : D \to M$, then we must have $\pi' \cong f o h$, now from above diagram

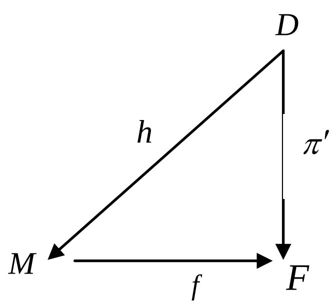
$\pi' o \pi \cong f o \zeta$

$\pi' o \pi o i \cong f o \zeta o\ i$

$\pi' o i_D \cong f o h$

$\pi' \cong f o h$

Then the following diagram commutative



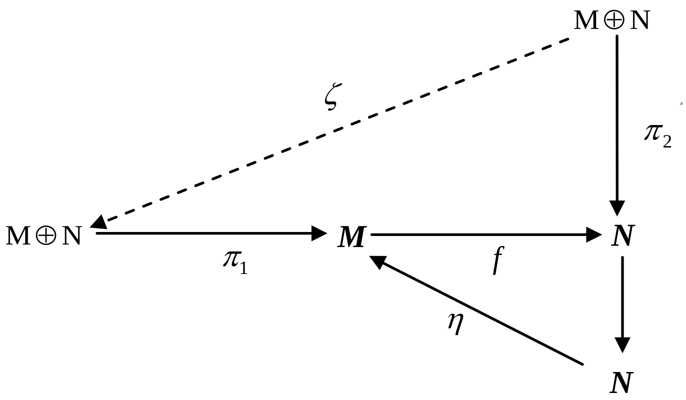
Then D is topological direct projective.

Lemma(2-3): A sufficient condition for a topological exact sequence $0 \to L \to M \xrightarrow{f} N \to 0$ to be topological splits is that $M \oplus N$ be topological direct projective.

Proof:

Assume that $M \oplus N$ is topological direct projective, let $\pi_1 : M \oplus N \to M$ and $\pi_2 : M \oplus N \to N$ be the corresponding topological projection mappings. As $M \oplus N$ is topological direct projective, there exists $\zeta \in End(M \oplus N)$ such that $\pi_2 \cong fo\pi_1 o\zeta$, define

$\eta : N \to M$

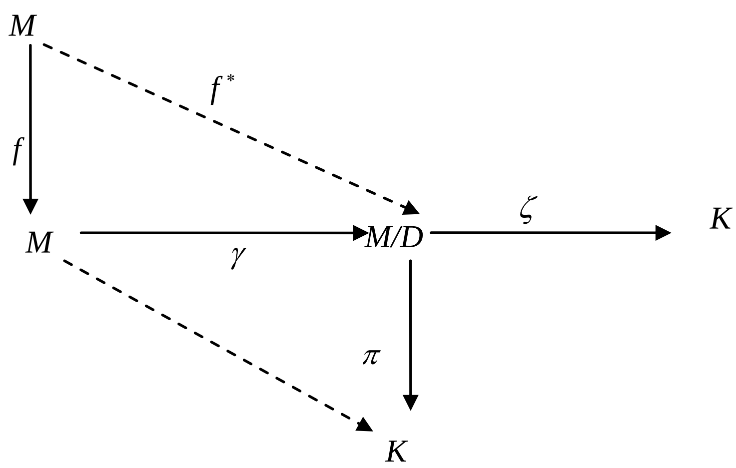


$n \in N \Rightarrow \eta(n) \cong \pi_2(0,n) \cong fo\pi_1 o\zeta(0,n) \cong fo\eta(n) \cong I_N(n)$, Hence the topological sequence splits.

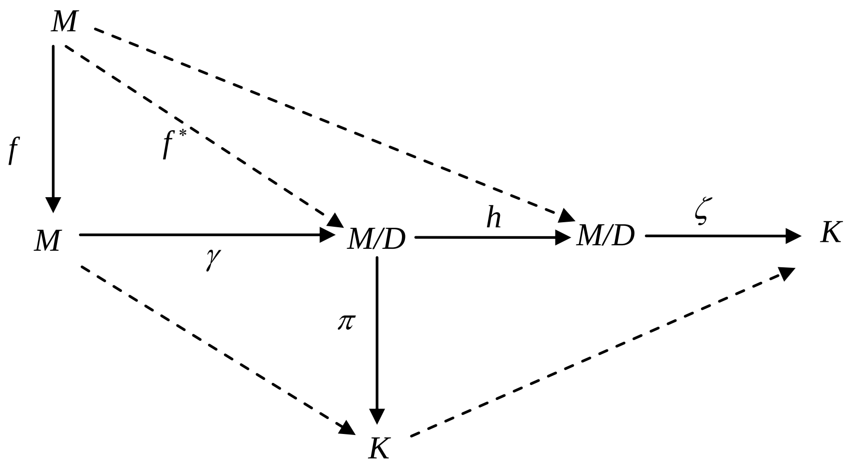
Proposition (2–4): Topological quotient module of topological quasi projective module by topological stable sub module is topological direct projective module .

Proof

Let M be a topological quasi projective module , D is a topological stable sub module of M . For a topological direct summand K of M/D,a topological projection morphism *is* $\pi : M/D \to K$, and a topological canonical morphism is $\gamma : M \to M/D$, Let $\zeta : M/D \to K$ be a topological epimorphism .A topological quasi projectivity of M gives $\pi o\gamma \cong \zeta o\gamma o f$ for some $f \in End(M)$.As the following diagram



As D is a topological stable Sub module then $Ker\gamma \subseteq Ker(\gamma of)$. Hence there exists $h \in End(M/D)$ with $\gamma of \cong ho\gamma$ giving $\pi o\gamma \cong \zeta o ho\gamma$,whence $\pi \cong \zeta oh$. As the following diagram



This proves that M/D is topological direct projective.

Proposition (2–5): For a topological ring R the following conditions are equivalent:

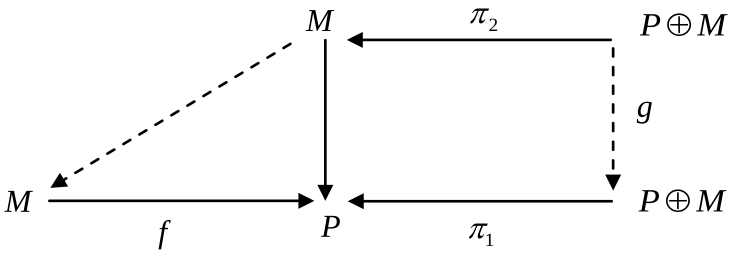
(*i*) The topological direct sum of two topological direct projective modules is topological direct projective,

(ii) Every topological direct projective module is topological projective.

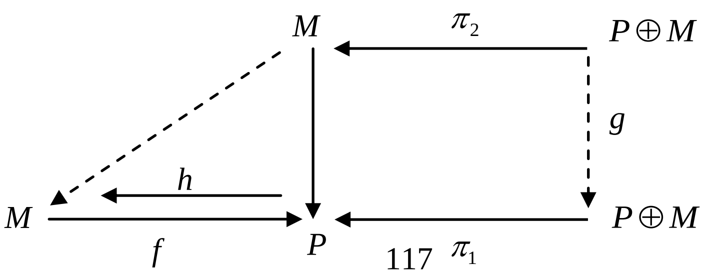
Proof

(*i*) ⇒ (*ii*)

f Let P be a topological projective module and M be a topological direct projective module, let $f: P \to M$ be a topological epimorphism, let $\pi_1 : P \oplus M \to P$, $\pi_2 : P \oplus M \to M$ be the topological projection morphisms on P and M respectively. As $P \oplus M$ is topological direct projective, there exists $g \in End(P \oplus M)$ such that $\pi_2 \cong fo\pi_1 og$, As the following diagram



let $h: M \to P$ be given by $h(m) \cong \pi_1 og(0,m)$,then $m \in M$ implies $m \cong \pi_2(0,m) \cong fo\pi_1 og(0,m) \cong foh(m) \cong I_M(m)$, As the following diagram



thus *M* being a topological direct summand of a topological projective module, that implies *M* itself is topological direct projective.

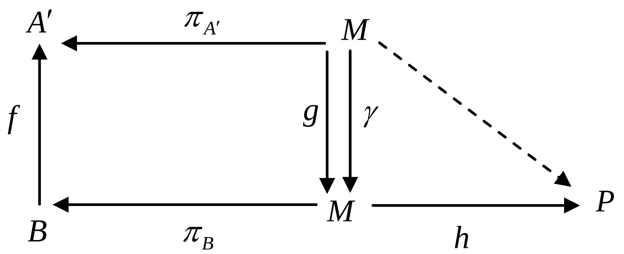
(ii) **⇒(*i*)**

Directly from definition of topological projective module and definition of topological projective module.

Proposition (2 –6): Let M be a topological direct projective module with topological direct summands A and B ,if $A \oplus B$ is topological direct summand then so is $A \cap B$.

Proof

Let $M \cong A \oplus A' \cong B \oplus B'$, with out loss of generality we can assume that $M \cong A \oplus B$, let $\pi_{A'} : M \to A'$ and $\pi_B : M \to B$ be the topological canonical projection morphisms ,let $\pi_{A'} \setminus B \cong f : B \to A'$ then it can easily be seen that *f is* also topological epimorphism .As *M* is a topological direct projective, there exists $g \in End(M)$ such that $\pi_{A'} \cong fo\pi_B og$. As the following diagram



Let $g' \cong g / A$ i $.e$ $g' : A \to M$, then $g \cong \pi_B og' \cong i_A$ f being retraction, as $f : B \to A$ is a topological epimorphism implies that $B \setminus Kerf \cong A' \cong M \setminus A$, but $Ker f \cong Ker\pi_{A'} \cap B \cong A \cap B$. Thus we have $B \setminus A \cap B \cong A'$, then f being retraction implies that $B \cong A' \oplus (A \cap B)$. Hence $A \cap B$ also topological direct summand of M .

References:

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