A modified associated flow rule in classical plasticity
قاعدة انسياب مطورة في اللدونـة التقليدية

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الخلاصة :

لقد تم في هذا البحث توسعة نظرية الانسياب في اللدونة التقليدية لتشمل التطبيقات ذات الأبعاد المايكروية في الصناعات الدقيقة التي ينشط فيها تأثير المشتقة الاتجاهية للانفعال اللدن. و ذلك من خلال اقتراح قاعدة للتقسية المعتمدة على الانفعال بالاستفادة من نظرية الانخلاعات في علم المواد، حيث إن هذه القاعدة المقترحة تمكن الباحث من الحصول على علاقات الإجهاد-الانفعال للمواد ذات الابعاد المايكروية المتعرضة للاجهادات اللدنة. لقد تم تدعيم صحة التوسعة المقترحة من خلال دراسة السلوك اللدن لعدة نماذج نظرية ذات ابعاد مايكروية اعتماداً على قاعدة التقسية المقترحة في هذا البحث و مقارنة النتائج المستحصلة من هذه الدراسة مع نتائج متوفرة في ادبيات اللدونة ذات الابعاد المايكروية، حيث تبين المقرنة توافقاً كبيراً، مما يدعم صحة التوسعة لنظرية الانسياب التقليدية المقترحة في هذا البحث و قابلية قاعدة الانسياب المطورة لوصف السلوك اللدن للمواد المايكروية الابعاد.

Abstract:

In this paper the flow theory in classical plasticity is extended to micro-scale plasticity, which is one of the most important active fields in micro-scale industries, where the directional derivative of plastic strain affects significantly on material's hardening. This is accomplished throughout proposing a new strain-hardening rule based on some facts from dislocations theory, where this proposed hardening rule enables the researcher from obtaining constitutive stress-strain relations for materials of micro-scale size undergoing plastic loading. Simulations are performed to show the plastic behavior of several theoretical models with micro-scale sizes based on the modified flow rule developed in this paper, and the results obtained from these simulations show good agreement with published results available in the literature. This agreement proves the validity of the extension of the classical flow theory proposed in this paper and the ability of the modified flow rule to describe the plastic behavior of materials at the micron scale.

Nomenclature

$\forall$	For all
$\exists$	There exist
ε^P	Plastic Strain
σ_0	Initial Stress
η^P	Gradient of plastic strain
ζ	Material internal variable that characterizes its strength
ψ_p	Generalized plastic strain
S	Set of plastic strain increment components
H	Set of measures of strain hardening components.
e_i	Unit vector of the *i-th* axis direction.

1 Introduction

The materials are -generally- classified into associated and non-associated materials, and the basis of this classification is the normality property. Accordingly, the materials which its plastic strain increment $d\boldsymbol{\varepsilon}^{P}$ is normal to the yield surface are known as associated materials, while the other materials which do not exhibit this property are called non-associated materials. The conventional basis of normality property is Drucker's postulate [1] which has been employed extensively in the classical theory of plasticity in order to formulate a constitutive relationship between the plastic strain increment and the stress gradient of the yield function. This relationship is known as the associated flow rule. On the other hand, many researchers employed the principles of thermodynamics in the formulation of the constitutive relationships in plasticity theory ([2-4]). In spite of the employment of the work concept in Drucker's postulate, it is not based on thermodynamical laws and it is restricted to strain-hardening – and, in the limit, perfectly plastic- materials to prove the validity of the associated flow rule to be applied in this class and does not introduce the physical interpretation of this flow rule.

The development of a flow theory for the fields of strain gradient plasticity, which are expanding steadily to involve a wide range of problems ([5-11]), has been attempted by several groups based on different approaches. Qiu and his group [12] established a flow theory of mechanism-based strain gradient plasticity based on the same framework of the deformation theory of mechanism-based strain gradient plasticity connected with the Taylor model in dislocation mechanics. While Gudmundson [13] presented a theoretical frame work applicable to incremental plasticity model in strain gradient plasticity by using an expression for plastic dissipation. On the other hand, a nonlocal flow rule in the form of a tensorial second-order partial differential equation for the plastic strain has been formulated by Gurtin and Anand based on a developed small-deformation theory of strain gradient plasticity [14]. Abu Al-Ruba et al [15] presented a formulation of small strain higher-order gradient plasticity theory based on thermodynamical principles incorporated with the theory of dislocation mechanics.

The main contribution of this paper can be outlined in the following points:

1) Proposing a strain-hardening/softening rule based on some facts from dislocations theory.
2) Explaining a theoretical basis for the flow theory in classical plasticity based on the proposed hardening rule.
3) Developing an associated flow rule applicable to micro-scale plasticity in the light of the proposed hardening rule.
4) Verifying the modified associated flow rule throughout simulations of micro-scale case studies, and comparing the results obtained from these simulations with published results available in the literature.

Notation: A function $g : S \rightarrow H$ is said to be *bijective* in Abstract Analysis literature if it maps all elements of *S* to all elements of *H* such that each different elements in *S* are mapped to different elements in *H*.

2 Conventional Approach to the Associated Flow Rule

Drucker (1951) has stated in his postulate that over a cycle of loading and unloading the work performed by the additional external agency is non-negative. This postulate leads to the important inequality[1]:

$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}_0).d\boldsymbol{\varepsilon}^{P} \geq 0 \qquad (1)$$

where $\boldsymbol{\sigma}$ is the stress state at the yield surface, $\boldsymbol{\sigma}_0$ is the initial stress state lying inside or on the yield surface and $d\boldsymbol{\varepsilon}^P$ is the incremental plastic strain vector.

As a consequence, Drucker has shown that the plastic strain increment $d\boldsymbol{\varepsilon}^P$ must be normal to the yield surface [1].

On the other hand, the stress gradient of the yield function $\frac{\partial F}{\partial \boldsymbol{\sigma}}$ is proportional to the direction cosines of the normal to the yield surface *F*. Consequently, the plastic strain increment $d\boldsymbol{\varepsilon}^P$ must be directed along the gradient vector $\frac{\partial F}{\partial \boldsymbol{\sigma}}$ yielding the associated flow rule which can be written as:

$$d\varepsilon^P = d\lambda \frac{\partial F}{\partial \sigma} \qquad (2)$$

where $d\lambda$ is an infinitesimal scalar multiplier.

3 Main Results

The main result of this paper is a modified associated flow rule obtained from extending the flow theory in classical plasticity to micro-scale plasticity. We accomplish this task in three steps. In the first step, we propose a strain hardening rule based on some facts from dislocations theory combined with some principles of Abstract Analysis. Then this proposed hardening rule is employed to show a theoretical basis for the associated flow rule in classical plasticity. Based on the same theoretical basis, we conclude a modified associated flow rule applicable for the plasticity at the micron scale. In the resulting modified associated flow rule, we impose a size parameter that accounts for geometrically necessary dislocations (GNDs), associated to the non-uniform straining in polycrystalline aggregate, that affect significantly on material's hardening at the micron scale.

3.1 Proposed Strain Hardening/Softening Rule

In this section, a strain hardening rule is proposed such that a greatest hardening is induced in the direction of the plastic strain with greatest Euclidean norm. Let s_i denotes the projection of $d\boldsymbol{\varepsilon}^P$ in the direction of a unit vector $\boldsymbol{u}_i$, while $\boldsymbol{S}$ represents the set of all possible elements of s_i, where *i* belongs to the set of all positive integers $\boldsymbol{J}$. Therefore:

$$\boldsymbol{S} = \{ s_i : s_i = d\boldsymbol{\varepsilon}^P \, . \, \boldsymbol{u}_i \, , \; i \in \boldsymbol{J} \; \} \qquad (3)$$

Let h_i represents a measure of the hardening induced in $\boldsymbol{u}_i$ direction due to a plastic strain s_i, then $\boldsymbol{H}$ is the set of all h_i s.

Therefore:

$$g : \boldsymbol{S} \rightarrow \boldsymbol{H} \;\; | \; \forall \; s_j > s_l \;\; \exists \;\; h_j > h_l \qquad (4)$$

where *j*, *l* $\in$ $\boldsymbol{J}$, and *g* is a *bijective* function which maps $\boldsymbol{S}$ to $\boldsymbol{H}$.

This proposed hardening rule agrees with the interpretation of strain hardening hypothesis which is based on the back stress induced due to the pile-up of dislocations at the barriers. Then, the greatest hardening is due to a greatest back stress which is corresponding to a plastic strain with greatest Euclidean norm.

Moreover, based on this proposed hardening rule, every component of the plastic strain increment $d\boldsymbol{\varepsilon}^P$ must produce a corresponding hardening and so that $d\boldsymbol{\varepsilon}^P$ has no component tangent to the yield surface. As a consequence $d\boldsymbol{\varepsilon}^P$ must be normal to the yield surface.

In a similar manner, a strain softening rule is proposed such that a greatest softening is induced in the direction of the plastic strain with greatest Euclidean norm. Let $\bar{h}_i$ represents a measure of the strain softening induced in $\boldsymbol{u}_i$ direction due to a plastic strain s_i, while $\overline{\boldsymbol{H}}$ is the set of all $\bar{h}_i$ s. Then:

$$\bar{g}:\boldsymbol{S} \rightarrow \overline{\boldsymbol{H}} \;\; | \;\forall\; s_j > s_l \;\; \exists \;\; \bar{h}_j > \bar{h}_l \qquad (5)$$

where $\bar{g}$ is a *bijective* function which maps $\boldsymbol{S}$ to $\overline{\boldsymbol{H}}$.

This proposed softening rule agrees with the softening mechanism in some polymeric materials in which the yield strength is decreased with the increased plastic strain due to a morphological change in molecular chains and entanglement of amorphous parts [16]. Then, the greatest softening is caused by a plastic strain with greatest Euclidian norm which is corresponding to a greatest morphological change in molecular chains and entanglement of amorphous parts.

Similarly, every component of $d\boldsymbol{\varepsilon}^P$ must produce a corresponding softening and hence $d\boldsymbol{\varepsilon}^P$ must be normal to the yield surface.

3.2 A Theoretical Basis for the Classical Associated Flow Rule

In the present analysis, two independent approaches are introduced for deducing the associated flow rule based on the proposed hardening rule presented in this paper. In the first one, the shape of the yield surface “after” the application of an incremental stress is considered. While in the second one, the behavior of the yield surface “during” the application of an incremental stress is considered.

3.2.1 First Approach

The yield criteria are empirical relationships proposed for predicting the plastic yielding of a material subjected to a combination of stresses. One of the fundamental experimental observations is that the plastic yielding depends on the stress deviators and is not influenced by the hydrostatic stress. Therefore, the yield criterion could be represented by the following general form:

$$f(\boldsymbol{\sigma}) = \bar{f}(\boldsymbol{\zeta}) \qquad (6)$$

where $f(\boldsymbol{\sigma})$ is a function of the stress state which causes plastic yielding and $\bar{f}(\boldsymbol{\zeta})$ represents the distance of the yield surface from the hydrostatic line, which is a function of a set of internal variables $\boldsymbol{\zeta}$. According to the distortion-energy criterion [17], the function $f(\boldsymbol{\sigma})$ can be expressed as:

$$f(\boldsymbol{\sigma}) = \sqrt{\frac{2}{3}}\left\{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1\right\}^{1/2} \qquad (7)$$

while:

$$\bar{f}(\boldsymbol{\zeta}) = \sqrt{2/3}\; Y(\boldsymbol{\zeta}) \qquad (8)$$

where σ_1, σ_2 and σ_3 are the principal stresses, while $Y(\boldsymbol{\zeta})$ is the yield stress of the material in uniaxial tension.

If the material is strain-hardening one such that it obeys the hardening rule proposed in this paper, then the maximum hardening induced due to plastic strain increment $d\boldsymbol{\varepsilon}^P$ will be directed along $d\boldsymbol{\varepsilon}^P$ itself because the maximum projection of a vector is the vector itself. On the other hand, the greatest hardening is corresponding to the farthermost point a way from the hydrostatic line. Consequently, a local maximum is obtained in the direction of $d\boldsymbol{\varepsilon}^P$ as shown in Fig. (1). For this local maximum, the function $f(\boldsymbol{\sigma})$ in (7) has its maximum value, and as a consequence:

$$\frac{\partial \bar{f}(\boldsymbol{\zeta})}{\partial s} = \frac{\partial f(\boldsymbol{\sigma})}{\partial s} = 0 \qquad (9)$$

where *ds* is an incremental distance on the yield surface. Therefore:

$$\frac{\partial f}{\partial \sigma_1}\frac{d\sigma_1}{ds} + \frac{\partial f}{\partial \sigma_2}\frac{d\sigma_2}{ds} + \frac{\partial f}{\partial \sigma_3}\frac{d\sigma_3}{ds} = 0 \qquad (10)$$

The left hand side of (10) can be thought of as the dot product of two vectors as follows:

$$\left(\frac{\partial f}{\partial \sigma_1}\boldsymbol{e_1} + \frac{\partial f}{\partial \sigma_2}\boldsymbol{e_2} + \frac{\partial f}{\partial \sigma_3}\boldsymbol{e_3}\right) . \left(\frac{d\sigma_1}{ds}\boldsymbol{e_1} + \frac{d\sigma_2}{ds}\boldsymbol{e_2} + \frac{d\sigma_3}{ds}\boldsymbol{e_3}\right) = 0 \qquad (11)$$

where $\boldsymbol{e_1}$, $\boldsymbol{e_2}$ and $\boldsymbol{e_3}$ represent the unit vectors along the direction of the principal stresses σ_1, σ_2 and σ_3 respectively. Hence that the unit vector in the direction tangent to the yield surface can be written as: $\boldsymbol{e_t} = \frac{d\sigma_1}{ds}\boldsymbol{e_1} + \frac{d\sigma_2}{ds}\boldsymbol{e_2} + \frac{d\sigma_3}{ds}\boldsymbol{e_3}$

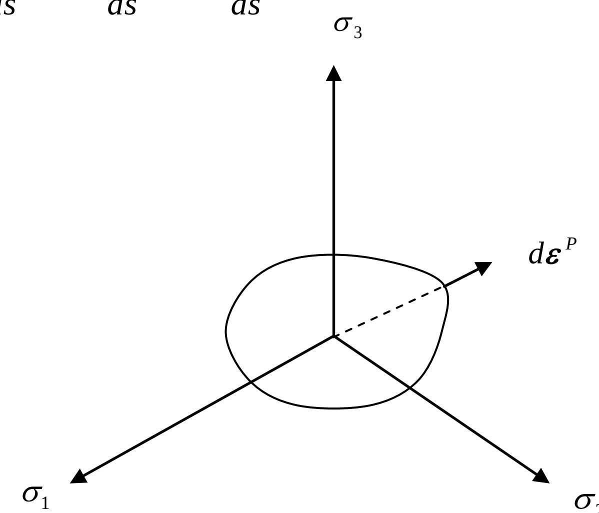

Fig. (1). Representation of local maximum yield strength.

Then (11) can be rewritten as:

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} \ . \ \boldsymbol{e_t} = 0 \qquad (12)$$

Moreover, the plastic strain increment $d\boldsymbol{\varepsilon}^P$ must be normal to the yield surface at the point of local maximum, otherwise there will be a component of $d\boldsymbol{\varepsilon}^P$ that doesn't produce a hardening, leading to a violation of the proposed hardening rule in which every component of $d\boldsymbol{\varepsilon}^P$ must produce an extension of the yield surface in the deviatoric plane. Therefore:

$$d\boldsymbol{\varepsilon}^P \ . \ \boldsymbol{e_t} = 0 \qquad (13)$$

and hence that $\frac{\partial f}{\partial \boldsymbol{\sigma}}$ must be directed along $d\boldsymbol{\varepsilon}^P$ yielding the associated flow rule (2).

If the material is strain-softening one such that it obeys the softening rule proposed in this paper, then the maximum softening induced due to plastic strain increment $d\boldsymbol{\varepsilon}^P$ will be directed along

$d\boldsymbol{\varepsilon}^P$ itself. Moreover, the greatest softening is corresponding to the nearest point from the hydrostatic line. As a consequence, a local minimum is obtained in the direction of $d\boldsymbol{\varepsilon}^P$ as shown in Fig. (2).

Then, at the point of local minimum yield strength, the function $f(\boldsymbol{\sigma})$ in (7) has its minimum value, and in a similar manner for the case of strain-hardening, the following can be obtained:

$$\frac{\partial f}{\partial \sigma} \,.\, \boldsymbol{e_t} = 0 \qquad \text{and} \qquad d\boldsymbol{\varepsilon}^P \,.\, \boldsymbol{e_t} = 0$$

yielding the associated flow rule.

3.2.2 Second Approach

During the plastic yielding, and according to (6), the stress function $f(\boldsymbol{\sigma})$ takes the value of $\overline{f}(\boldsymbol{\zeta})$ which represents a measure of the yield strength, so that the hardening induced during the

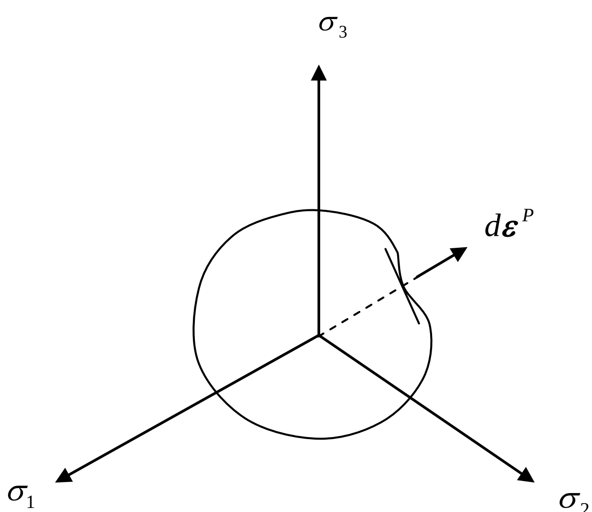

Fig. (2). Representation of local minimum yield strength.

application of an incremental stress $d\boldsymbol{\sigma}$ can be represented by the directional derivative $D_{\boldsymbol{u}_i} f$, i.e. the hardening induced in a direction of a unit vector $\boldsymbol{u}_i$ can be expressed as:

$$D_{\boldsymbol{u}_i} f = \nabla f \,.\, \boldsymbol{u}_i \;\Big|\; i \in \boldsymbol{J} \qquad (14)$$

Then, the hardening rule proposed in this paper, can be rewritten as:

$$\forall \;\; d\varepsilon^P \,.\, \boldsymbol{u}_j > d\varepsilon^P \,.\, \boldsymbol{u}_l \quad \exists \quad \frac{\partial f}{\partial \sigma} \,.\, \boldsymbol{u}_j > \frac{\partial f}{\partial \sigma} \,.\, \boldsymbol{u}_l \qquad (15)$$

Since the function which maps the set of plastic strain components to the set of the corresponding hardening is a bijective function, therefore:

$$\forall \;\; d\varepsilon^P \,.\, \boldsymbol{u}_m = \max(s_i) \quad \exists \quad \frac{\partial f}{\partial \sigma} \,.\, \boldsymbol{u}_m = \max(h_i) \qquad (16)$$

But:

$$\max(s_i) = d\varepsilon^P \cdot \frac{d\varepsilon^P}{|d\varepsilon^P|} \quad , \quad \max(h_i) = \frac{\partial f}{\partial \sigma} \cdot \frac{\partial f / \partial \sigma}{|\partial f / \partial \sigma|}$$

Then:

$$\boldsymbol{u}_m = \frac{d\varepsilon^P}{|d\varepsilon^P|} = \frac{\partial f / \partial \sigma}{|\partial f / \partial \sigma|} \tag{17}$$

i.e. the gradient vector $\frac{\partial f}{\partial \sigma}$ must be directed along $d\varepsilon^P$, leading to the associated flow rule.

Similarly, for strain softening materials:

$$\forall \quad d\varepsilon^P \cdot \boldsymbol{u}_m = \max(s_i) \quad \exists \quad \frac{\partial f}{\partial \sigma} \cdot \boldsymbol{u}_m = \max(\bar{h}_i) \tag{18}$$

So that:

$$\boldsymbol{u}_m = \frac{d\varepsilon^P}{|d\varepsilon^P|} = \frac{\partial f / \partial \sigma}{|\partial f / \partial \sigma|}$$

3.3 Extension of the Flow Theory in Classical Plasticity

Experimental evidence [5] shows that in applications at micron and submicron scales the hardening evolution is increased with the decreased size due to the presence of plastic strain gradients. To combine the effect of the plastic strain to the effect of its gradient, various proposals have been introduced, leading to the formulation of strain gradient plasticity theories (SGP) ([5-11]).

Since the work concept, which is independent of the strain gradient, is the basis of Drucker's postulate, the effect of strain gradient cannot be embodied in the mathematical formulation of the associated flow rule, leading to a main difficulty in extending the flow theory in classical plasticity to the fields of strain gradient plasticity.

In contrast, based on the present approach, the effect of the plastic strain gradients can be conveniently incorporated to the effect of the plastic strain to produce a generalized associated flow rule applicable to micron and submicron plasticity where the strain gradients play a significant role in hardening evolution.

3.3.1 General Formulation

Let ψ^P denotes the generalized plastic strain vector, which can be expressed in the general incremental form:

$$d\psi^P = \sum_{k=1}^{n} l^{(k)}\, d\rho^{(k)} \tag{19}$$

where $d\rho^{(k)}$ is an incremental vector which represents the response of the material - which may be increment of the plastic strain, its gradients or others- to the applied incremental stress $d\sigma$, such that this response contributes to the generalized-strain hardening of the material, *n* is the total number of response vectors which contribute to the hardening, and $l^{(k)}$ is the length parameter which characterizes the size-dependence of $\rho^{(k)}$ in its contribution to the hardening. The hardening caused by $\rho^{(k)}$ is assumed to obey the hardening rule proposed in this paper. If $\rho_i^{(k)}$ is assumed to denote the projection of $\rho^{(k)}$ in the direction of a unite vector $\boldsymbol{u_i}$, then $\boldsymbol{S}^{(k)}$ is the set of all possible elements of $\rho_i^{(k)}$ where $i \in \boldsymbol{J}$, while $h_i^{(k)}$ is a measure of the hardening in the direction of $\boldsymbol{u}_i$ due to $\rho_i^{(k)}$, and $\boldsymbol{H}^{(k)}$ is the set of all possible elements of $h_i^{(k)}$. Accordingly:

$$g^{(k)} : \boldsymbol{S}^{(k)} \rightarrow \boldsymbol{H}^{(k)} \mid \forall \;\; \rho_j^{(k)} > \rho_l^{(k)} \;\; \exists \;\; h_j^{(k)} > h_l^{(k)} \qquad (20)$$

where $g^{(k)}$ is a *bijective* function which maps $\boldsymbol{S}^{(k)}$ to $\boldsymbol{H}^{(k)}$.

Then, the maximum hardening induced due to incremental stress $d\sigma$ must be directed along $d\psi^P$. The generalized normality can be represented as in Fig. (3), while the generalized associated flow rule can be expressed as:

$$d\psi^P = \sum_{k=1}^{n} l^{(k)}\, d\rho^{(k)} = d\lambda \frac{\partial f}{\partial \sigma} \qquad (21)$$

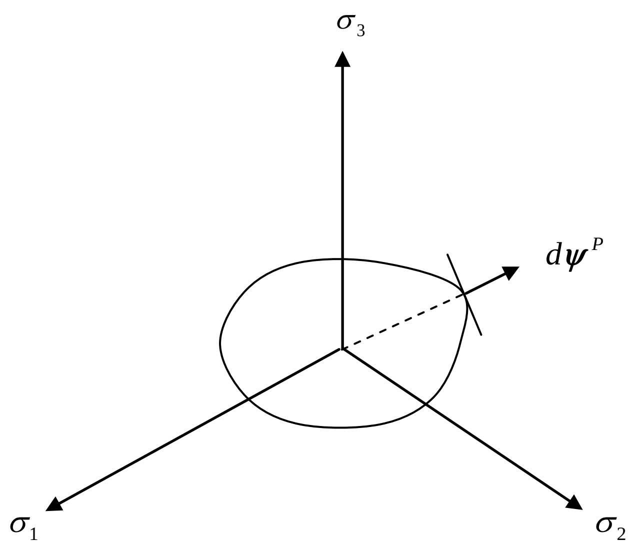

Fig. (3). Representation of the generalized normality.

The parameter $d\lambda$ expressed by:

$$d\lambda = \left[\frac{d\psi^{PT} \,.\, d\psi^P}{(\partial f / \partial\sigma)^T \,.\, (\partial f / \partial\sigma)} \right]^{1/2} \qquad (22)$$

can be determined conveniently depending on the specific application as established in the next section where the application of strain gradient plasticity is considered.

It is clear that, for the case of size-independence, the following condition is satisfied:

$$l^{(k)} = 1$$

3.3.2 Application to Strain Gradient Plasticity

The condition expressed by (20) agrees with the mechanism of hardening due to strain gradient, where the hardening evolution is believed to be attributed to the geometrically necessary dislocations (GNDs) which are associated with the non-uniform straining in polycrystalline aggregate [18]. Then, the greatest plastic strain gradient is corresponding to the greatest density of geometrically necessary dislocations leading to a greatest hardening.

Therefore, (19) can be applied with two material responses; $d\boldsymbol{\varepsilon}^P$ and $d\boldsymbol{\eta}^P$, where η^P is the gradient of plastic strain, i.e.:

$$\eta^P = \nabla \varepsilon^P \tag{23}$$

Then, the generalized normality can be represented as in Fig. (4), while the associated flow rule can be written as:

$$d\psi^P = d\varepsilon^P + l^{(2)} d\eta^P = d\lambda \frac{\partial f}{\partial \sigma} \tag{24}$$

where: $n = 2,\ l^{(1)} = 1,\ d\rho^{(1)} = d\varepsilon^P$ and $d\rho^{(2)} = d\eta^P$,

while $l^{(2)}$ depends on the size of the specific application.

If the distortion-energy criterion expressed by:

$$\frac{1}{6}\left\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right\} = \frac{1}{3}Y^2 \tag{25}$$

is considered, then (24) can be expressed as :

$$d\boldsymbol{\psi}^P = \frac{2}{3} d\lambda \begin{Bmatrix} \sigma_1 - \frac{1}{2}(\sigma_2 + \sigma_3) \\ \sigma_2 - \frac{1}{2}(\sigma_3 + \sigma_1) \\ \sigma_3 - \frac{1}{2}(\sigma_1 + \sigma_2) \end{Bmatrix} \tag{26}$$

Then, the parameter $d\lambda$ can be explicitly calculated by the equation:

$$d\lambda = \frac{3 d\overline{\psi}^P}{2\overline{\sigma}} \tag{27}$$

where $\overline{\sigma}$ is the effective stress and $d\overline{\psi}^P$ is defined in a similar manner for defining the classical effective strain, i.e.:

$$d\overline{\psi}^P = \frac{\sqrt{2}}{3}\left[(d\psi_1^P - d\psi_2^P)^2 + (d\psi_2^P - d\psi_3^P)^2 + (d\psi_3^P - d\psi_1^P)^2\right]^{1/2} \tag{28}$$

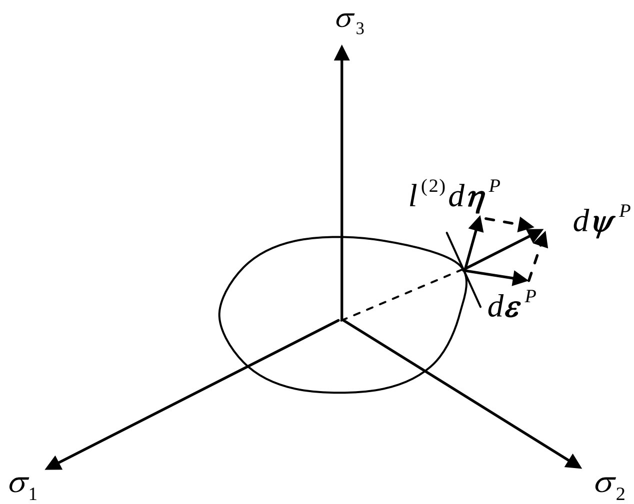

Fig. (4). Representation of normality in SGP.

Hence the parameter $d\lambda$ can be evaluated from a ($\bar{\sigma}$-$\bar{\psi}^{P}$) curve for an increment of the generalized plastic strain $d\bar{\psi}^{P}$, in a similar manner for evaluating the proportionality constant in Levy-Mises equation in classical plasticity [19], as shown in Fig. (5), where:

$$d\lambda = \frac{3}{2} cot\theta \qquad (29)$$

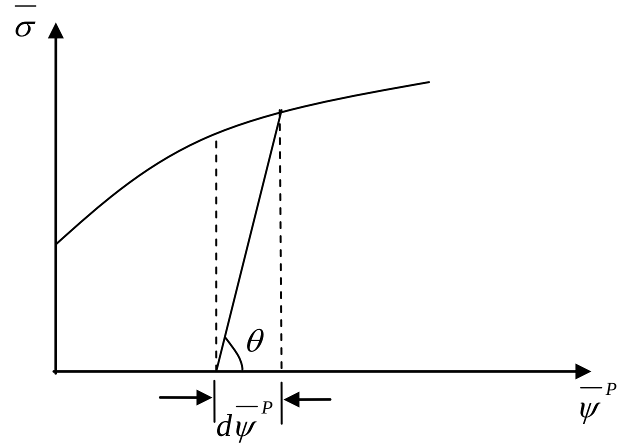

Fig. (5). Method of evaluating $d\lambda$ in Eq.(27).

5 Simulation

The modified associated flow rule developed in this paper based on a newly proposed hardening rule is demonstrated in this section throughout a case study. In this case study, we consider uniaxial tensile loading of a given material for several micro-scale diameters. Then we investigate the plastic behavior of these cases based on the modified flow rule developed in this paper, and compare this behavior with a published result available in the literature to show the validity of the proposed flow rule.

Consider a material whose simple tensile test gives the following stress-strain relation

$$\sigma = \left(10^{17}\varepsilon_{p} + Y_{t}^{2}\right)^{1/2} \qquad (30)$$

where Y_t is the material's yield stress in tension. We assume $Y_t = 80\,MPa$.

Furthermore, assume that yield occurs after an elastic strain $\varepsilon_0 = 0.01$. Such a material can be described by the stress-strain curve shown in Fig. (6).

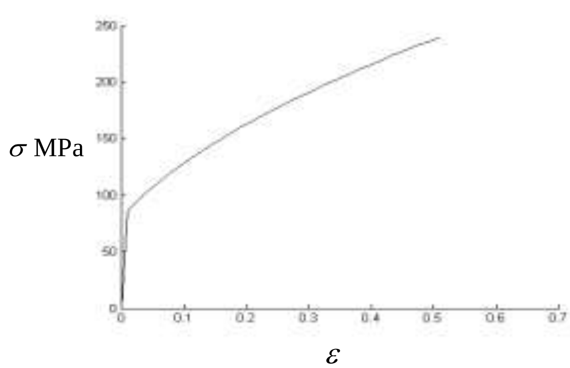

Fig.(6). Material's stress-strain curve.

According to Von-Mises yield criterion, we have

$$F = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y_t^2 \qquad (31)$$

Based on this criterion, we can obtain the relation (30) from the proposed flow rule expressed in (24) as follows. For simplicity, we employ the notation $l = l^{(2)}$. For uniaxial tensile loading, we have $\sigma = \sigma_1$ and $\sigma_2 = \sigma_3 = 0$. Then $F = 2\sigma^2 = 2Y_t^2$ and $\frac{\partial f}{\partial \sigma} = 4\sigma$. Letting $d\lambda = 5\times 10^{-18} d\sigma$, then based on (24), combined with the fact that $l = 0$ for macro scale case, we have

$$2\sigma \times 10^{-17} d\sigma = d\varepsilon_p \qquad (32)$$

Integrating (32), we get

$$\int_{Y_t}^{\sigma} 2\sigma \times 10^{-17} d\sigma = \int_0^{\varepsilon_p} d\varepsilon_p$$

Which yields the relation expressed in (30). Hence for micro scale cases we have $l > 0$ and therefore

$$2\sigma \times 10^{-17} d\sigma = l\, d\eta_p + d\varepsilon_p \qquad (33)$$

The value of the size parameter *l* can be calculated experimentally. We assume values of *l* for different diameters in Table.1. Hence the value of *l* increases with the decreased values of the size represented by specimen diameter *D.*

D	l
200 μm	0.3
150 μm	0.6
90 μm	1
30 μm	1.5

Table.1. Values of *l* for different sizes.

We further assume that $\eta_p = (\varepsilon_p)^{1/2}$, and so that we have $d\eta_p = \frac{}{\sqrt{\varepsilon_p}} d\varepsilon_p$, and (33) becomes

$$2\sigma \times 10^{-17} d\sigma = \left(l\ \frac{0.5}{\sqrt{\varepsilon_p}} + 1 \right) d\varepsilon_p \qquad (34)$$

Integration of (34) gives

$$\sigma = \left\{ \left(l\ \sqrt{\varepsilon_p} + \varepsilon_p \right) \times 10^{17} + Y_t^2 \right\}^{1/2} \qquad (35)$$

Simulation of relation (35) is shown in Fig. (7) for the different specimen diameters listed in Table.1.

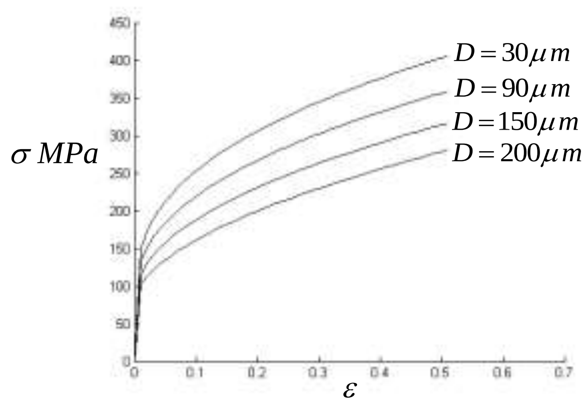

Fig. (7). Effect of size on stress-strain curve at the micron scale.

The effect of size on material's hardening is shown clearly in this figure, where material's strength increases with the decreased diameter. This observation agrees with the general plastic behavior for micro-scale applications as shown, for example, in [8]. This agreement shows the validity of the proposed flow rule and its ability to describe plastic behavior of materials at the micron scale.

6 Conclusions

The difficulty of extending the classical associated flow rule to strain gradient plasticity can be ascribed to a fundamental restriction in Drucker's postulate. Moreover, this postulate is not applicable to strain-softening materials. In this paper, a new strain-hardening/softening rule is proposed based on some facts from dislocations theory, and this proposed hardening rule is employed to develop a modified associated flow rule applicable to micro-scale plasticity, which is one of the most important fields in micro-scale industries. This extension of the classical flow theory to micro-scale plasticity is of independent interest, where it provides a unifying framework for a plasticity theory applicable to macro and micro-scale applications. Moreover, the simple stress-strain relations obtained from this extension can be easily numerically solved with less efforts compared with the complicated relations available in the literature of mechanism-based strain gradient plasticity.

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