Finite Difference Method for Two-Dimensional Fractional Partial Differential Equation with parameter

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Abstract

In this paper, we introduce and discuss an algorithm for the numerical solution of two-dimensional fractional partial differential equation with parameter. The algorithm for the numerical solution of this equation is based on implicit and an explicit difference method. Finally, numerical example is provided to illustrate that the numerical method for solving this equation is an effective solution method.

Key words: Fractional derivative, two finite difference methods, fractional partial differential equation.

Introduction

In recent years there has been a great deal of interest in fractional partial differential equations [1, 2, 3, 4, 5]. These equations arise quite naturally in continuous time random walk with spatial and temporal memories.

More and more works by researchers from various fields of science and engineering deal with dynamical systems described by fractional partial differential equations, which have been used to represent many natural processes in physics[6], finance[7,8], and hydrology[9,10].

In this paper, we find the numerical solution of two-dimensional fractional partial differential equation with parameter of the form:

$$\frac{\partial u(x,y,t)}{\partial t} = \lambda\left[a(x,y)\frac{\partial^{\gamma}u(x,y,t)}{\partial x^{\gamma}} + b(x,y)\frac{\partial^{\beta}u(x,y,t)}{\partial y^{\beta}}\right] \quad \text{....... (1)}$$

subject to the initial condition

$$u(x,y,0) = f(x,y), \text{ for } x_0 < x < x_R \text{ and } y_0 < y < y_R \quad \text{..........(2)}$$

and the boundary conditions

$$\begin{aligned} &u(x_0,y,t) = 0, \text{ for } y_0 < y < y_R \text{ and } 0 \le t \le T \\ &u(x,y_0,t) = 0, \text{ for } x_0 < x < x_R \text{ and } 0 \le t \le T \\ &u(x_R,y,t) = g(y,t), \text{ for } y_0 < y < y_R \text{ and } 0 \le t \le T \\ &u(x,y_R,t) = k(x,t), \text{ for } x_0 < x < x_R \text{ and } 0 \le t \le T \end{aligned} \quad \text{.........(3)}$$

where a, b and f are known functions of x and y, g is a known function of y and t, k is a knwon function of x and t. γ and β are given fractional number. λ is a scalar parameters.

We use a variation on the classical explicit and implicit Euler method. We prove that these methods by using a novel shifted version of the usual grunwaled finite difference an approximation for the non-local fractional derivative operator.

1. Two Finite Difference Methods for Solving Two-Dimensional Fractional Partial Differential Equation with Parameter

In this section, we propose two finite difference methods, i.e., an implicit finite difference method and explicit finite difference method for solving two-dimensional fractional partial differential equation with parameter (1)-(3).

The finite difference method starts by dividing the x-interval $[x_0, x_R]$ into n subintervals to get the grid points $x_i = x_0 + i\Delta x$, where $\Delta x = (x_R - x_0)/n$ and i=0,1,…,n. And the y-interval $[y_0, y_R]$ into m subintervals to get the grid points $y_j = y_0 + j\Delta y$, where $\Delta y = (y_R - y_0)/m$ and j=0,1,…,m.

Also, the t-interval [0,T] is divided into M subintervals to get the grid points $t_s = s\Delta t$, s = 0, …,M, where $\Delta t = T/M$.

The problem here is to find the eigenpair (λ,u) which satisfy eq.(1)-(3).

This equation can be written as an eigenvalue problem Au=λBu, where $A = \frac{\partial}{\partial t}, B = a(x,t)\frac{\partial^\gamma}{\partial x^\gamma} + b(x,y)\frac{\partial^\beta}{\partial y^\beta}$.

Firstly, present the following implicit finite difference method for the initial-boundary value problem of the two-dimensional fractional partial differential equation with parameter. By the shifted Grunwald estimate to the γ, β - the fractional derivative, [11]:

$$\frac{\partial^\gamma u(x,y,t)}{\partial x^\gamma} = \frac{1}{(\Delta x)^\gamma}\sum_{k=0}^{M} g_{\gamma,k}u(x-(k-1)\Delta x, y, t) + O(\Delta x)$$

$$\frac{\partial^\beta u(x,y,t)}{\partial y^\beta} = \frac{1}{(\Delta y)^\beta}\sum_{k=0}^{M} g_{\beta,k}u(x, y-(k-1)\Delta y, t) + O(\Delta y) \quad \text{……….(4)}$$

to reduce eq.(1) as the following form

$$u_{i,j}^{s+1} = \lambda\left[a_{i,j}\frac{\Delta t}{\Delta x^\gamma}\sum_{k=0}^{i+1} g_{\gamma,k}u_{i-k+1,j}^{s+1} + b_{i,j}\frac{\Delta t}{\Delta y^\beta}\sum_{k=0}^{j+1} g_{\beta,k}u_{i,j-k+1}^{s+1}\right] + u_{i,j}^{s}$$

$$i = 1,\dots,n-1,\ j = 1,\dots,m-1,\ s = 0,\dots,M \quad \text{………(5)}$$

Where $u_{i,j}^s = u(x_i, y_j, t_s)$, $a_{i,j} = a(x_i, y_j)$, $b_{i,j} = b(x_i, y_j)$, $g_{\gamma,k} = (-1)^k\frac{\gamma(\gamma-1)\cdots(\gamma-k+1)}{k!}$, k=0,1,2,… and $g_{\beta,k} = (-1)^k\frac{\beta(\beta-1)\cdots(\beta-k+1)}{k!}$, k=0,1,2,…

Secondly, present the following explicit finite difference method for solving the two-dimensional fractional partial differential equation with parameter eq.(1) with the boundary conditions eq.(3), and the initial condition (2), also by usimg the shifted Grunwald estimate to the γ, β -th fractional derivative given by eq.(4) to reduce as the following form:

$$\frac{u_{i,j}^{s+1} - u_{i,j}^{s}}{\Delta t} = \lambda\left[\frac{a_{i,j}}{\Delta x^\gamma}\sum_{k=0}^{i+1} g_{\gamma,k}u_{i-k+1,j}^{s} + \frac{b_{i,j}}{\Delta y^\beta}\sum_{k=0}^{j+1} g_{\beta,k}u_{i,j-k+1}^{s}\right],$$

$$i = 1,\dots,n-1,\ j = 1,\dots,m-1,\ s = 0,\dots,M \quad \text{……….(6)}$$

Where $u_{i,j}^s = u(x_i, y_j, t_s)$, $a_{i,j} = a(x_i, y_j)$, $b_{i,j} = b(x_i, y_j)$, $g_{\gamma k} = (-1)^k \frac{\gamma(\gamma-1)\cdots(\gamma-k+1)}{k!}$, k=0,1,2,…

and $g_{\beta k} = (-1)^k \frac{\beta(\beta-1)\cdots(\beta-k+1)}{k!}$ k=0,1,2,…

After evaluating eq.(5) and eq.(6) at i=1,…,n-1, j=1,…,m-1 and s=0,…,M one can get a system of algebraic equations which can be solved.

Using any suitable method to get the eigenpair can solve $(\lambda, \{u\}_{\substack{i=1,\dots,n-1 \\ j=1,\dots,m-1 \\ s=0,\dots,M}})$.

Also, from the initial and boundary conditions one can get:

$u_{i,j}^0 = f_{i,j}$, i=0,…, n

$u_{0,j}^s = 0$, j=0,…, m and s=1,…,M

$u_{i,0}^s = 0$, i=0,…, n and s=1,…,M

$u_{R,j}^s = g_j^s$, j=0,…, m and s=1,…,M

$u_{i,R}^s = k_i^s$, i=0,…, n and s=1,…,M

Where $f_{i,j} = f(x_i, y_j, t_s)$, $g_j^s = g(y_j, t_s)$ and $k_i^s = (x_i, t_s)$

2. Numerical example

In this section, present numerical example which confirm our theoretical results.

Example: Consider the two-dimensional fractional partial differential equation with parameter:

$$\frac{\partial u(x,y,t)}{\partial t} = \lambda\left[\frac{\Gamma(2.5)x^{1.5}}{6}\frac{\partial^{1.5}u(x,y,t)}{\partial x^{1.5}} + \frac{\Gamma(1.2)y^{1.8}}{2}\frac{\partial^{1.8}u(x,y,t)}{\partial y^{1.8}}\right]$$

subject to the initial condition

u (x,y,0) = x^3y^2, 0 < x < 1, 0 < y < 1

and the boundary conditions

u (0,y,t) = 0, 0 < y < 1, 0 ≤ t≤ 0.025

u (x,0,t) = 0, 0 < x < 1, 0 ≤ t≤ 0.025

u (1,y,t) = $e^t y^2$, 0 < y < 1, 0 ≤ t≤ 0.025

u (x,1,t) = $e^t x^3$, 0 < x < 1, 0 ≤ t≤ 0.025

This fractional partial differential equation together with the above initial and boundary condition is constructed such that the exact solution is u(x,y,t) = $e^t x^3 y^2$.

Table1, 2, 3 and 4 give the numerical solution using the two finite difference methods. From table 1, 2, 3 and 4, it can be seen that there isa good agreement between the numerical solution and exact solution.

3. Conclusions

In this paper, a numerical method for solving the two-dimensional fractional partial differential equation with parameter has been described and demonstrated. Furthermore numerical example is presented to illustrate that good agreement between the numerical solution and exact solution has been noted.

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Table (1) The numerical solution of example using the implicit finite difference method for $\Delta x = 0.2, \Delta y = 0.2$ **and** $\Delta t = 0.0125$

Numerical Solution	Exact Solution	Error
0.46300	0.50000	3.70000 E -2
3.61100E-4	3.24025 E -4	-3.70749 E -5
0.01100	1.03688 E -2	-6.31197 E -4
0.08200	7.87381 E -2	-3.26190 E -3
0.30700	0.33180 E -2	-0.30368
4.01100E-4	3.28101 E-4	-7.29992 E -5
0.01100	1.04992 E -2	-5.00773 E -4
0.07900	7.97285 E -2	7.28504 E -4
0.28800	0.33596	4.79753 E -2

Table (2) The numerical solution of example using the implicit finite difference method for $\Delta x = 0.25, \Delta y = 0.25$ **and** $\Delta t = 0.0125$

Numerical Solution	Exact Solution	Error
0.39600	0.50000	0.04000
1.31700E-3	9.88846 E -4	-3.28154 E-4
0.03300	3.16431 E -2	-1.35692 E-3
0.22900	0.24029	1.12896 E-2
1.41900E-3	1.00128 E -3	-4.17716 E-4
0.03300	3.20411 E -2	-9.58902 E-4
0.22100	0.24331	2.23121 E-2

Table (3) The numerical solution of example using the explicit finite difference method for $\Delta x = 0.2, \Delta y = 0.2$ **and** $\Delta t = 0.0125$

Numerical Solution	Exact Solution	Error
0.50000	0.50000	0.00000
3.20000E-4	3.24025 E -4	4.02510 E -6
0.01000	1.03688 E -2	3.68803 E -4
0.07800	7.87381 E -2	7.38100 E -4
0.32800	0.33180	3.80171 E -3
2.96300E-4	3.28101 E -4	3.18008 E -5
0.01000	1.04992 E -2	4.99227 E -4
0.07700	7.97285 E -2	2.72850 E -3
0.30400	0.33598	3.19753 E -2

Table(4) The numerical solution of example using the explicit finite difference method for $\Delta x = 0.25, \Delta y = 0.25$ **and** $\Delta t = 0.0125$

Numerical Solution	Exact Solution	Error
0.50000	0.50000	0.00000
9.76600E-4	9.88846 E -4	1.22461 E-5
0.03100	3.16431 E -2	6.43077 E-4
0.23700	0.24029	3.28961 E-3
9.73200E-4	1.00128 E -3	2.80843 E-5
0.03100	3.20411 E -2	1.04110 E-3
0.22600	0.24331	1.73121 E-2

طريقة الفروق المنتهيةّ للمعادلة التفاضليةِ الجزئيـةِ الكسرية الثنائية الأبعادِ مع متغير

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الخلاصة

في هذا البحث قدمنا وناقشنا خوارزمية للحل العددي لمعادلة التفاضليةِ الجزئيـةِ الكسرية الثنائية الأبعادِ مع متغيرِ. وان خوارزمية الحل العددي لتلك المعادلة قائمة على اساس طريقة الفروق المنتهية الضمنية و الصريحة.

اخيرا قدمنا مثالا عدديا والذي وضح ان الطريقة العددية لحل هذه المعادلة هي طريقة ذو حل مؤثر فعال .

الكلمات المفتاحية : الاشتقاق الكسري،طريقتا الفروق المنتهية، المعادلة التفاضليةِ الجزئيةِ الكسرية.