Um-Salama Science Journal **Vol.5(2)2008**

A Fixed Point Theorem for L-Contraction in Generalized D-Metric Spaces

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Date of acceptance 5/3/2008

Abstract

We define L-contraction mapping in the setting of D-metric spaces analogous to L-contraction mappings [1] in complete metric spaces. Also, give a definition for general D- matric spaces.And then prove the existence of fixed point for more general class of mappings in generalized D-metric spaces.

Keywords: Fixed point, L-contraction mappings, D-metric spaces.

1. Preliminaries

In [2] Dhage introduced the concept of D- metric spaces as followes

Definition 1.1 [2] Let X be a nonempty set . A function D : $X \times X \times X \to \Re^+$($\Re^+$ is the set of all non negative real numbers) is called a **D-metric spaces** on X if

i. $D(x,y,z) = 0$ if and only if $x = y = z$ (coincidence)

ii. $D(x,y,z) = D(p\{x,y,z\})$, where p is a permutation of x, y, z (symmetry)

iii. $D(x,y,z) \leq D(x,y,a) + D(x,a,z) + D(a,y,z)$ for all $x, y, z, a \in X$ (telrahedral inequality).

A nonempty set X , together with D-metric , is called D- metric space and denoted by (X,D). Some specific examples of D-metrics appeared in [3] and [4] .

Definition 1.2 [2] A sequence of points of a D-metric space X is said to be **D-convergent to a point** $x \in X$ if for each $\in > 0$ there exists an $n_0 \in N$ such that for all $m, n \geq n_0$, $D(x_m,x_n,x) < \in$.

Definition 1.3 [2] A sequence of points of a D-metric space X is said to be **D-cauchy sequence** if for $\in > 0$ there exists an $n_0 \in N$ such that for all m , n , $p \geq n_0$, $D(x_m,x_n,x_p) < \in$.

Definition 1.4 [5] A D-metric space X is said to be **complete** every D-cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$.

Definition 1.5[1] Let E be a Banach space. A subset K is called **a cone** if it is

closed, convex and $t\,K \subset K$ for $t \in \Re^+$ and $K \cap (-K) = 0$.

Given a cone K in E we define a partial ordinary in E by writing.

$x < y$ if and only if $y - x \in K$ (1)

Definition 1.6 [1] A subset K in a Banach space E is called a **normal** if there exists $\delta > 0$ such that $0 < x < y$ implies $\|x\| \leq \delta \|y\|$.

2. Main Results

Firstly we define the following

Definition 2.1 A set X is said to be a **general D-metric space** if there exists a function D : $X \times X \times X \to K$,where K is a normal cone in a Banach space, such that

i. $D(x,y,z) = 0 \in K \Leftrightarrow x = y = z$

ii. $D(x,y,z) = D(p\{x,y,z\})$ for all x,y,z in X, (p is a permutation of x,y,z)

iii. $D(x,y,z) < D(x,y,z) + D(x,a,z) + D(a,y,z)$ for all x,y,z,a inX"<denote the partial ordering induced by K

Remark

If we put $K= \Re^+$ in the definition (2.1) then the general D-metric function will be D-metric function and then the examples (1.8), (1.20) and (1.21) in [4] show that in general D-metric space : (a) D-metric does not always define a topology , (b) even D-metric define a topology , it need not be Hausdorff (therefore the limit need not be unique),

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and (c) even D-metric define a topology ,the D-metric function need not be continuous even in a single variable .Now for uniqueness limit in general D-metric spaces, we reform the concept of continuity of general D-metric function and then give a result which guarantee the uniqueness limit if exist

Definition 2.2

A general D-metric function is called in three variables if the sequence$\{D(x_n.y_n,z_n)\}$ in K converges toD(x,y,z) whenever x,y,z$\in$X and $\{x_n\}$,$\{y_n\}$and$\{z_n\}$are sequences in Xconverge to x,y and z,respectively with respect to general D-metric .

Proposition 2.3 Let(X,D)be a general D-metric space and D be continuous in three variables,then every convergent sequence in(X,D)has a unique limit.

Proof It is easy to prove this result since $\{D(x_n.y_n,z_n)\}$in K and any convergent sequence in Banach space has a unique limit .

Definition 2.4 [1] Let K be a normal cone in a Banach space E,the function ℓ:K$\rightarrow$ $\Re^+$is a sublinear positively homogenous functional if for any u, $v \in$ K then $\ell(u + v) \leq \ell(u) + \ell(v)$ and $\ell(t\,u) = t\,\ell(u)$,for $t \geq 0$ such that $\ell^{-1}(0) = 0$.

Proposition 2.5 Let K be a normal cone in a Banach space E and (X, D) be a general D-metric space.If D*:X$\times$X$\times$X$\rightarrow\Re^+$is the function defined by

$D^*(x,y,z) = \ell(D(x,y,z))$ (2)

Where ℓ as in the definition(2.2) then (X,D) is D-metric space.

Proof: By conditions i ,ii and iii of definition (1.1) and definition (2.2) ,one can prove $\ell(u) = \|u\|$ Naidu[4],show that the metric function D is not continuous even in one variable,therefore,through this paper the D-metric is assumed to be continuous in three variables.Note that,if D is continuous in three variables then the limit is unique in exists [4] .

Theorem 2.6 Let X be a generalized D-metric space which is complete in the metric defined by(2)with D iscontinuous in three variables and ,if T:X $\rightarrow$ X satisfies

$D(Tx,Ty,Tz) \leq L(D(x,y,z) + D(Tx,Ty,Tz))$

Where L is bounded positive linear operator in E with spectral radius less than1/2, then there is a unique fixed point, $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n$.

proof :

$let\ x_0 \in X,\ put\ x = T^2x_0,\ y = Tx_0,\ z = x_0$

$D(T^3x_0, T^2x_0, Tx_0) \subseteq LD(T^3x_0, T^2x_0, Tx_0) +$

$LD(T^2x_0, Tx_0, x_0) \Rightarrow (I-L)D(T^3x_0, T^2x_0, Tx_0) \subseteq LD(T^2x_0, Tx_0, x_0)$

$\sin ce\ \ r(L) \subset \frac{1}{2} \subset 1, then (1-L) is\ invertable [6, pp795]$

$D(T^3x_0, T^2x_0, Tx_0) \subseteq (I-L)^{-1}$

$L(D(T^2x_0, Tx_0, x_0))$

To prove that

$D(T^{n+1}x_0, T^n x_0, T^{n-1}x_0) \subseteq (I-L)^{-(n+1)}L^{n+1}$

$[D(T^2x_0, Tx_0, x_0)]\quad n \supseteq 2\ldots3$

Assume that (3) is true for $n = 2$ then

$D(T^{n+1}x_0, T^n x_0, T^{n-1}x_0) \subset LD(T^{n+1}x_0, T^n x_0, T^{n-1}x_0) +$

$LD(T^n x_0, T^{n-1}x_0, T^{n-2}x_0) \Rightarrow$

$(I - L)\ D(T^{n+1}x_0, T^n x_0, T^{n-1}x_0) < L$

$D(T^n x_0, T^{n-1}x_0, T^{n-2}x_0)$

$\Rightarrow D(T^{n+1}x_0, T^n x_0, T^{n-1}x_0) < (I - L)^{-1}\ L$

$D(T^n x_0, T^{n-1}x_0, T^{n-2}x_0)$

$< (I - L)^{-1}\ L\ (I - L)^{-n}\ L^n\ D(T^2x_0, Tx_0, x_0)$ (4)

Since $L(I - L)^{-n} = (I - L)^{-n}\ L$ (4) will be

$D(T^{n+1}x_0, T^n x_0, T^{n-1}x_0) < (I - L)^{-(n+1)}\ L^{n+1}$

$D(T^2x_0, Tx_0, x_0)$ and (3) is proved.

Furthermore, by condition (3) of D-metric, we have:

$D(T^{n+m+1}x_0, T^{n+m}x_0, T^n x_0) <$

$D(T^{n+m+1}x_0, T^{n+m}x_0, T^{m+n-1}x_0) +$

$D(T^{n+m+1}x_0, T^{m+n-1}x_0, T^n x_0) +$

$D(T^{m+n-1}x_0, T^{n+m}x_0, T^n x_0)$

$< (I - L)^{-(m+n+1)}\ L^{(m+n+1)}\ D(T^2x_0, Tx_0, x_0) +$

$D(T^{m+n+1}x_0, T^{m+n-1}x_0, T^n x_0) +$

$D(T^{m+n-1}x_0, T^{m+n}x_0, T^n x_0)$

$< (I - L)^{-(m+n+1)}\ L^{m+n+1}\ D(T^2x_0, Tx_0, x_0) +$

$D(T^{m+n+1}x_0, T^{m+n-1}x_0, T^{m+n}x_0) +$

$D(T^{m+n+1}x_0, T^{m+n}x_0, T^n x_0) +$

$D(T^{m+n}x_0, T^{m+n-1}x_0, T^n x_0) +$

$D(T^{m+n}x_0, T^{m+n-1}x_0, T^n x_0)$.

$\Rightarrow 2\ D(T^{m+n+1}x_0, T^{n+m}x_0, T^n x_0) < (I - L)^{-(m+n+1)}\ L^{m+n+1}\ D(T^2x_0, Tx_0, x_0) +$

$(I - L)^{-(m+n+1)}\ L^{m+n+1}\ D(T^2x_0, Tx_0, x_0) + 2$

$D(T^{m+n}x_0, T^{m+n-1}x_0, x_0)$

$\Rightarrow$ D($T^{m+n-1}x_0$, $T^{m+n}x_0$, $T^n x_0$) < $(I-L)^{-(m+n+1)} L^{m+n+1}$ D($T^2 x_0$, Tx_0 , x_0)

Continue, we get:

D($T^{m+n-1}x_0$, $T^{m+n}x_0$, $T^n x_0$) < [$(I-L)^{-(m+n+1)} L^{m+n+1} + (I-L)^{-(m+n)} L^{m+n} + \cdots +$ $(I-L)^{-(n+2)} L^{n+2}$] D($T^2 x_0$, Tx_0 , x_0) +

D($T^{n+1}x_0$, $T^n x_0$, $T^{n-1}x_0$). < [$(I-L)^{-(m+n+1)} L^{m+n+1} + \cdots + (I-L)^{-(n+2)} L^{n+2}$]

D($T^2 x_0$, Tx_0 , x_0) + $(I-L)^{-(n+1)} L^{n+1}$

D($T^2 x_0$, Tx_0 , x_0)

< $(I-L)^{-(n+1)} L^{n+1}(\sum (I-L)^{-m} L^m$

D($T^2 x_0$, Tx_0 , x_0))= $((I-L)^{-1})^{n+1} x^1$

where x^1 is the unique solution of the $x=(I-L)^{-1}$ L x + D($T^2 x_0$, Tx_0 , x_0).

By spectral mapping theorem [6,pp 798]

$(I-L)^{-1}$ L is such that $r((I-L)^{-1} L) < 1$

Hence,

D($T^{m+n+1}x_0$, $T^{m+n}x_0$, $T^n x_0$) < $((I-L)^{-1}) x^1$

Being K is normal we have:

$$\left\| D(T^{m+n+1}x_0, T^{m+n}x_0, T^n x_0 \right\| \le \delta \left\| ((I-L)^{-1} L) x^1 \right\| (5)$$

Since the right-hand side of (5) going to zero when $n \to \infty$, we obtain that $\{T^{n+1}x_0\}$ is Cauchy with respect to the metric D. Being (X,D) complete, we denote by y the limit of $\{T^{n+1}x_0\}$. The following inequalities hold:

($T^{n+1}x_0$, $T^n x_0$, Ty) < L D($T^{n+1}x_0$, $T^n x_0$, Ty) + L D($T^n x_0$, $T^{n-1}x_0$, y)

$\Rightarrow$D($T^{n+1}x_0$, $T^n x_0$, Ty)<$(I-L)^{-1}$

D($T^n x_0$, $T^{n-1}x_0$, y)Finally using the normality of K:

$$\left\| D(T^{n+1}x_0, T^n x_0, Ty \right\| \le \delta \left\| (I-L)^{-1} L \right\| \left\| D(T^n x_0, T^{n-1}x_0, y \right\| (6)$$

Letting $n \to \infty$ in (6) we obtain

D(y, y, Ty) ≤ 0, mean Ty = y, and this complete the proof.

A consequence of theorem (2.6) one can prove the following result

Corollary (2.7) Let (X,D), ℓ and L as in theorem (2.10). If the mapping T : X $\to$ X satisfies the condition

D(Tx,Ty,Tz) < L(D(x,y,z) for all $x, y, z \in$ X

Then T has a unique fixed point in X.

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مبرهنة حول النقطة الصامدة لتطبيق L- الأنكماشية في فضاءات D- المترية المعممة

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الخلاصة

بأسلوب مشابه لما ورد في [1] من تعريف لتطبيق L- الأنكماشي في الفضاءات المترية الأعتيادية سنعرف تطبيق L- الأنكماشية في فضاءات D-المترية وكذلك نعطي تعريفاً لفضاءات D-المترية المعممة ثم نبرهن وجود نقطة صامدة لنمط من التطبيقات الأكثر عمومية في فضاءات D-المترية المعممة.