# On New Algebraic Systems

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## Abstract:

The purpose of this paper is to give some results theorems , propositions and corollaries concerning new algebraic systems flower , garden and farm with accustomed algebraic systems groupoid , group and ring.


**Keywords:** Groupoid , Group , Ring ,Semiflower , Flower , Garden ,Farm , Lahhamian Group.

## Introduction:

Many authors work for development and findout some mathematics systems .Al- Lahham [1] one of them , he was introduced new algebraic system flower , garden and farm ,he gave some results about them as a finite farm ( S, $*$ ,$\lozenge$ ) such that $|S|$ >2, is not a field .Al-Lahham introduced type of groups is called a Lahhamian group, during his work depended on [2] and [3] . We can say he find some relations between his systems with another accustomed algebraic systems.

## Preliminaries :

Throughout this paper S denote to a groupoid is an order pair (S, $*$ )where S is a non empty set and $*$ is a binary operation on S [2] unless explicily stated .A binary operation $*$ on non empty set S is ATL if for all a,b,c$\in$S ,a$*$ (b$*$c)=c$*$ (b$*$a).Asemiflower is an order pair (S, $*$ ) where S is a non empty set and $*$ is an ATL binary operation on S .Aflower is an order pair (S, $*$ ) where S is a non empty set and $*$ is a binary operation on S satisfying the following axioms :

( i )a $*$ (b$*$c) = c $*$ (b$*$ a ) for all a,b,c $\in$S (ATL law).

( ii )There exists an element e in S such that a$*$e =a for all a$\in$S(e is a right identity of S).

( iii ) a$*$a=e for all a$\in$S.Agroup (S, $*$ ) is called Lahhamian group if a$*$a = e for all a $\in$ S , where e the identity of S.

Agarden is a triple (S, $*$,$\lozenge$) ,where S is non empty set has at least two elements , $*$ and $\lozenge$ are two binary operations on S, ($\lozenge$:S$\times$S$^*$$\rightarrow$S)and S$^*$=S-{0},(0is the right identity in (S, $*$)),satisfies the following axioms:

( i ) (S, $*$)is a flower .( ii) $\lozenge$is ATL (i.e.a$\lozenge$(b$\lozenge$c)= c$\lozenge$(b$\lozenge$a)for all a,b,c$\in$S$^*$).

( iii )The distributive law ,(a$*$b) $\lozenge$c=(a$\lozenge$c) $*$ (b$\lozenge$c) for all a,b$\in$S, c$\in$S$^*$, holds in S. A farm ( S, $*$, $\lozenge$)is a garden such that ( S$^*$,$\lozenge$)is aflower.A function p[$\lambda$]from a flower S to S is a right (left)translation of S if p(a)b= a(P(b)) [$\lambda$(ab)= $\lambda$(a)b ]for all a,b $\in$ S .The operation o is composition of functions.

Now ,we will mention some results which we need to a chieve our work which apperared in[1]

### Lemma 1 [Lemma 4]

Let (S, $*$, $\lozenge$)be a garden ,then for all a $\in$S$^*$, 0$\lozenge$a=0.

### Lemma 2[Theorem6]

A flower (S, $*$) is a Lahhamian group if and only if it is commutative .

### Lemma 3[Theorem4]

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Let S be a flower , then every right translation of S commutes with every left translation of S.

**Proposition 5[Proposition 2]**

If (S,.)is an abelian group ,then S with the new binary operation $*$ defined as follows : $a * b=a.b^{-1}$ for all $a,b \in S$ is a flower .

**Lemma 4[Lemma2]**

If (S, $*$) is a flower then for all $a,b \in S, a * b=e * (b * a)$.

**Proposition 6[ Proposition1]**

If (S, $*$) is a flower then , $(b * c) * a=(b * a) * c$ for all $a,b,c \in S$.

**Proposition 7[ Proposition3]**

If S is a Lahhamian group then S is a flower.

**Proposition 8[Theorem 8]**

Let (S,*) be a flower ,then the following conditions are equivalent:

(i) S is a Lahhamian group.
(ii) S has an identity.
(iii) S is commutative.
(iiii) S is associative.

It is our aim in this paper to give some new results concerning new algebraic systems.

## The Main Results:

**Theorem 1**

Let (S, $*$) be a non–empty set such that a is idempotent element of S , $*$ satisfy ATL law and left cancellation law hold in S,then (S, $*$) is flower .

**Proof** :We have $a * a =a$ , $a \in S$ , $*$ satisfy ATL law .Then we must be prove

(i)$a * e=a$ ,$a \in S$. By replacing e by a, we obtain

$a * e=a * a, a \in S$.Then

$=a$ , $a \in S$.Thus e is a right identity element of S.

(ii) $a * a = e$ , $a \in S$. According to our relation ,we have

$a*a=a$ ,$a \in S$. Since e is a right identity, then $a * a=a * e$, $a \in S$.Then a can be cancelled on the left our equation , we obtain a=e. Then $a * a=e$

Thus (S, $*$) is flower .

**Remark 2**

In Theorem 1 ,(S, $*$) is a semiflower .

**Proposition 3**

If (S, $*$) is a flower then the following are true for all $a,b,c \in S$.

(1) S is closed under $*$ (2) $(e * a) * (a * e)= e$ (e is a right identity element of S) . (3)$( b * a) * a= b$

**Proof:** (1)Since $*$ is binary operation ,then for all $a,b \in S$, we obtain , $a * b \in S$ , replacing b by a ,we have $a * a \in S$ ,by axaiom (iii ) ,we get $e \in S$ , e is a right identity element of S.

(2)We have $(e * a) * (a * e) =e$ .Let $x=(e * a)$ and $y = (a * e)$ ,then

$x * y =e$.We set $x =e * x$ and $y=y * e$ ,then

$(e * x) * (y * e)$

$=(e * (e * a)) * ((a * e) * e)$ . By ATL law ,we get

$=(a * (e * e)) * ((a * e) * e)$

$=a * ((a * e) * e)$. Since e is a right identity element of S

$=a * (a * e)$ .By ATL law,we get

$= e * (a * a)$. By axiom ( iii ),we obtain.

$=e * e =e$.

(3)We have $(b * a) * a =b$,then

Replacing a by $e * a$ ,we obtain

$(b * a) * a=(b * (e * a)) * (e * a)$. By ATL law ,we get

$=(a * (e * b)) * (e * a)$ .By Proposition 6,we obtain

$=(a * (e * a)) * (e * b)$ .By ATL law ,we get

$=b * (e * (a * (e * a))$ .By ATL law ,we obtain

$=b * ((e * a) * (a * e))$. By Part 2,then

$=b * e=b$.

**Theorem 4**

Let (S, $*$) be a flower , if $a * x=y * b$ then x=y for all $a,b,x,y \in S$.

**Proof:** We have $a * x=y * b$ for all $a,b,x,y \in S$ .Then

$a*(a*x)=a*(y*b)$ for all $a,b,x,y\in S$. By ATL law,we obtain
$x*(a*a)=b*(y*a)$ for all $a,b,x,y\in S$.By axiom (iii),we get
$x*e=b*(y*a)$ for all $a,b,x,y\in S$. By axiom (ii) ,we obtain
$x=b*(y*a)$ for all $a,b,x,y\in S$ .Then
$x*(e*e)=(b*(y*a))*(e*e)$ for all $a,b,x,y\in S$. By ATL law,we get
$e*(e*x)=e*(e*(b*(y*a)))$ for all $a,b,x,y\in S$ .By ATL law,we obtain $x*(e*e)=e*((y*a)*(b*e))$ for all $a,b,x,y\in S$.By ATL law,we get
$x*e=(b*e)*((y*a)*e)$ for all $a,b,x,y\in S$. Also by axiom( ii ) ,we obtain
$x=b*(y*a)$ for all $a,b,x,y\in S$. Since e is a right identity element of S ,then
$x=b*(y*(a*e))$for all $a,b,x,y\in S$ .By ATL law,we obtain
$x=b*(e*(a*y))$ for all $a,b,x,y\in S$. By ATL law,we obtain
$x=(a*y)*(e*b)$for all $a,b,x,y\in S$ .Let $b=e*b$ ,then
$x=(a*y)*(e*(e*b))$ for all $a,b,x,y\in S$.By ATL law ,we obtain
$x=(a*y)*(b*e)$for all $a,b,x,y\in S$.Then
$x=(a*y)*b$ for all $a,b,x,y\in S$.By Proposition 6 ,we obtain
$x=(a*b)*y$ for all $a,b,x,y\in S$. Replacing a by b ,we obtain
$x=(b*b)*y$ for all $b,x,y\in S$ .By axiom ( iii ),we obtain $x=e*y$ for all $x,y\in S$. Replacing y by $e*y$ ,we get
$x=e*(e*y)$ for all $x,y\in S$ .By ATL law ,we get $x=y$ for all $x,y\in S$.

## Theorem 5

If (S,.) is an abelian group ,then S with the new binary operation $*$ defined as follows: $a*b=a^{-1}.b$ for all $a,b\in S$ is a flower ,where e is identity element for * on S.

**Proof**: (i) $c*(b*a)=c^{-1}.(b*a)$ for all $a,b,c\in S$.

Since S is belain then

$c^{-1}.(b^{-1}.a)=a.(b^{-1}.c^{-1})$ for all $a,b,c\in S$. Since e is identity element of S ,we obtain

$=(a.e).(b.c)^{-1}$

$=((b.c).(a.e)^{-1})^{-1}$for all $a,b,c\in S$.

$=((b.c).(a^{-1}.e^{-1}))^{-1}$ for all $a,b,c\in S$.

$=(a^{-1}.e^{-1})^{-1}.(b.c)^{-1}$ for all $a,b,c\in S$.Since e is identity element of S,we get $=((a^{-1}.e).e^{-1})^{-1}.(b^{-1}.e.c^{-1}.e)$ for all $a,b,c\in S$. Then

$=((a^{-1}.e)^{-1}.(e^{-1})^{-1}).(b^{-1}.(c^{-1}.e))$ for all $a,b,c\in S$.

$=((a^{-1}.e)^{-1}.e).(b*(c*e))$ for all $a,b,c\in S$.

$=((a*e)^{-1}.e).(b*(c*e))$.Since e is identity elemnt of S,we get

$=(a*e)^{-1}.(b*(c*e))$ for all $a,b,c\in S$.

$=(a*e)*(b*(c*e))$ .Since e is identity for $*$ on S.

$=a*(b*c)$ for all $a,b,c\in S$.

Then $*$ satisfying ATL law.

(ii) Satisfying from our hypothesis (e is a right identity element for $*$ on S).

(iii) $a*a=a^{-1}.a=e$ for all $a\in S$.Then $(S,*)$ is a flower.

## Proposition 6

Let $(S,*)$be a flower such that $a*b=a.b^{-1}$ for all $a,b\in S$ ,then S is Lahhamian group ,where (S,.)is a group.

**Proof:**We have $a*b=a.b^{-1}$ for all $a,b\in S$.Then we must prove $a*b=b*a$ for all $a,b\in S$ .Then we suppose $a*b\neq b*a$ for some $a,b\in S$ **.**Then
$a.b^{-1}\neq b.a^{-1}$ for some $a,b\in S$. Then
$a.b^{-1}.a\neq b.a^{-1}.a$ for some $a,b\in S$. Then
$a.b^{-1}.a\neq b.e$ for some $a,b\in S$.Since e is a right identity element of S , we obtain
$a.b^{-1}.a\neq b$ for some $a,b\in S$ .Then replacing b by a,we get $a.a^{-1}.a\neq a$ for some $a\in S$ .Then
$a.e\neq a$ for some $a\in S$ ,then $a\neq a$ for some $a\in S$.
This lead to contradiction .
Then $(S,*)$is a commutative ,by Lemma 2,$(S,*)$ is Lahhamian group.

## Theorem 7

Every flower is commutative. .

**Proof:** Suppose $(S,*)$ is a flower ,then we have $a*(b*c)=c*(b*a)$ for all $a,b,c\in S$. Then by ATL law ,we obtain

$c*(b*a)=a*(b*c)$ for all $a,b,c \in S$ .Then $e*(c*(b*a))=e*(a*(b*c))$ for all $a,b,c\in S$,e is a right identity element of S.By ATL law,we get $(b*a)*(c*e)=(b*c)*(a*e)$ for all $a,b,c\in S$. Since e is a right identity, then $(b*a)*c=(b*c)*a$ for all $a,b,c \in S$. Then

Replacing a by $(b*c)$and c by $(b*a)$, we obtain

$(b*(b*c))*(b*a)=(b*(b*a))*(b*c)$ for all $a,b,c\in S$.

By ATL law ,we get

$(c*(b*b))*(b*a)=(a*(b*b))*(b*c)$ for all $a,b,c\in S$ .Then

$(c*e)*(b*a)=(a*e)*(b*c)$for all $a,b,c\in S$.

Replacing a by b ,we obtain $(c*e)*(b*b)=(b*e)*(b*c)$ for all $b,c\in S$.According to axiom (iii),we get $(c*e)*e=e$.Since e is a right identity of S. Then c=e for all $c\in S$,then $c*b=e*b$ for all $c,b\in S$. By Theorem 4 ,We get b=e for all $b\in S$. Then

$c*b=b*e$ for all $c,b\in S$. Since c=e ,then $c*b=b*c$ for all $c,b\in S$.Then $(S,*)$is commutative.

## Corollary 8:

Let (S,o) be a flower with left and right translation of S ,then (S,o)is Lahhamian group.

**Proof:**By Lemma 3 ,left and right translation are commutative i.e. (S,o)is commutative flower ,by Lemma 2 , we get (S,o)is Lahhamian group.

## Theorem 9

The Boolean ring $(S,*,\Diamond)$is garden ,where $(S,*)$ is Lahhamian group.

**Proof:** (i)By Proposition 7 , $(S,*)$ is flower .

(ii) For all $a,b,c\in S^*$,we have $a\Diamond(b\Diamond c)$,replacing b by c ,we obtain

$a\Diamond(b\Diamond c)=a\Diamond(c\Diamond c)$. Since $c\Diamond c=c,c\in S^*$, S is Boolean ring, then

$=a\Diamond c$ ,By replacing c by a ,we obtain

$= a\Diamond a =a$.

Also , we have $c\Diamond(b\Diamond a)$,by same method ,we obtain $c\Diamond(b\Diamond a)=a$.

Thus ,we obtain $a\Diamond(b\Diamond c)=c\Diamond(b\Diamond a)$ for all $a,b,c\in S^*$.

i.e. $\Diamond$ is ATL law.

## Theorem 10

Let $(S,*,\diamond)$ be a garden ,then

(1) $(a\diamond b)*(c\diamond d)=o$ for all $a,b,c,d\in S^*$.

(2)$c*(b\diamond a)=0$ for all $c\in S,a,b\in S^*$ .

(3)$o\diamond o=o$.

(4)$(a*b)\diamond(c*d)=o$ for all $a,b,c,d\in S$.

**Proof:**(1) We have $(a\diamond b)*(c\Diamond d)$ ,replacing a by c and b by d , we obtain $(a\diamond b)*(c\diamond d)=(c\diamond d)*(c\diamond d)=o$ for all $c,d\in S^*$ .(o is a right identity element of (S,*)).

(2) By replacing c by $(b\diamond a)$ ,we get

$c*(b\diamond a)=(b\diamond a)*(b\Diamond a)=o$ for all $c\in S,a,b\in S^*$ . (o is a right identity of $(S,*)$).

(3)Since o is a right identity of $(S,*)$, then we have

$o\diamond o=(a*a)\diamond(a*a)$ for all $a \in S$. Since o is a right identity of (S,*),then for all $a \in S$.

$=((a*a)*o)\diamond(a*a)$. By distributive law,we obtain

$=((a*a)\diamond(a*a))*(o\diamond(a*a))$ for all $a\in S$.

Since o is a right identity of (S,*),then

$=((a*a)\diamond(a*a))*((a*a)\diamond(a*a))=o$ for all $a\in S$.

(4)For all $a,b,c,d \in S$ we have $(a*b)\diamond(c*d)$ . By take a=b,we obtain

$(b*b)\diamond(c*d)=o\diamond(c*d)$.Since o is a right identity of $(S,*)$, then

$=((c*d)*(c*d))\diamond(c*d)$ .By distributive law ,we get

$=((c*d)\diamond(c*d))*((c*d)\diamond(c*d))$

$=o$ (o is a right identity of $(S,*)$).

## Theorem 11

In a garden $(S,*,\diamond)$ for all $a,b\in S$, $c\in S^*$,then $c\diamond(a*b)=0$.

**Proof:** We have $c\diamond(a*b)$,then $(c*o)\diamond(a*b)$for all $a,b\in S,c\in S^*$.By distributive law ,we get

$(c*o)\diamond(a*b)=(c\Diamond(a*b)*(o\Diamond(a*b))$for all $a,b\in S,c\in S^*$.

Let a=b ,we obtain

$=(c\diamond(a*a))*(o\diamond(a*a))$ for all $a\in S,c\in S^*$.

Since $(S,*)$ is a flower then by axiom ( iii ),we get

$=(c\diamond o)*(o\diamond o)$. By distributive law,we obtain

$=(c*o)\diamond o$ (o is right identity of $(S,*)$). By Lemma1,we get $=(c*(o\diamond y))\diamond(o\diamond y)$for all $c,y\in S^*$ .By distributive law ,we obtain $=(c\diamond(o\diamond y))*((o\diamond y)\diamond(o\diamond y))$ for all $c,y\in S^*$.Then

$=(c\diamond(o\diamond y))*o$ for all $c,y\in S^*$. Since o is a right identity of (S,*),then

$=c\diamond(o\diamond y)$ for all $c,y\in S^*$. By Lemma 1 , we obtain

$=c\diamond o$ for all $c\in S^*$. By Theorem 10(3),we obtain

$=c\diamond(o\diamond o)$ for all $c\in S^*$. By ATL law,we get

$=o\diamond(o\diamond c)$ for all $c\in S^*$ . By Lemma1,we obtain

$=o\Diamond o$.By Theorem 10(3),we get

$=o$

As a corollary of Theorem 11, we can write

## Corollary 12

Let $(S,*,\diamond)$be a garden ,then for all $c\in S^*$,$c\diamond o=o$.

**Proof:**We have $c\diamond o$,for all $c\in S^*$,by Theorem 10(3),we obtain

$c\diamond o=c\diamond(o\diamond o)$ for all $c\in S^*$.By ATL law ,we get

$=o\diamond(o\diamond c)$ for all $c\in S^*$.By Lemma1,we obtain

$=o\diamond o$ .Again by Theorem 10(3),we get

$=o$

## Proposition 13

In a farm $(S,*,\diamond)$ for all $a,b,c\in S$, then $c\diamond(a*b)=o$

**Proof:**We can write the proof of our proposition by same method in Theorem 11**.**

From precedence Theorem 10, Theorem 11and Corollary 12, we can write the following remark.

## Remark 14

Let $(S,*,\Diamond)$be a garden ,then

$(a\Diamond b)*(c\Diamond d)=(a*b)\Diamond(c*d)=c*(b\Diamond a)=c\Diamond(a*b)=c\diamond o=o\Diamond o=o$ .

## Theorem 15

If $(S,*,\Diamond)$ be a ring ,then S with the new operations $\Diamond$ and $*$ defined as follows :

$a*b=a.b^{-1}$for all $a,b\in S$.

$a\Diamond b=a$ for all $a,b\in S^*$.Is a garden ,where (S,.) is a belian group.

**Proof:**(i) By Proposition 5 ,$(S,*)$ is a flower .

(ii)We have $c\diamond(b\diamond a)$ according to defined of $\diamond$,then $c\Diamond(b\Diamond a)=c\Diamond b$ for all $a,b,c\in S^*$.Since$(a\Diamond b=a)$ ,then

$=(c\Diamond a)\Diamond(b\Diamond a)$ for all $a,b,c\in S^*$.

Replacing a by c and c by a ,we obtain

$=(a\Diamond c)\Diamond(b\Diamond c)$ for all $a,b,c\in S^*$ .Since $(a\Diamond c=a)$ according to defined of $\Diamond$, we obtain

$=a\Diamond(b\Diamond c)$ for all $a,b,c\in S^*$

Thus $\Diamond$ is ATL law.

(iii) We have $(a*b)\Diamond c=a*b$ for all $a,b\in S$ ,$c\in S^*$. According to defined of $\Diamond$,we obtain

$=(a\Diamond c)*(b\Diamond c)$ for all $a,b\in S,c\in S^*$.

Thus the distributive law hold in S.

Then $(S,*,\Diamond)$is a garden .

## Theorem 16

If $(S,*,\Diamond)$be a ring ,then S with new operation $\Diamond$ and $*$ defined as follows:

$a*b=a.b^{-1}$ for all $a,b\in S$ and $a\Diamond b=a$ for all $a,b\in S^*$. Is a farm, where (S,.) is a belian group.

**Proof:** According to Theorem 15 ,we have $(S, *)$ is a flower and the distributive law hold in S,therefore ,we must prove $(S^*, \Diamond)$is a flower ,then

(i) Form Theorem 15, $\Diamond$ is ATL law of $S^*$.

(ii) According to $a \Diamond b=a$ for all $a,b \in S^*$,there exists an element e in $S^*$ such that $a \Diamond e=a$ for all $a \in S^*$ , i.e. e is a right identity element of $S^*$.

(iii) According to $a \Diamond b=a$ for all $a,b \in S^*$,then $a \Diamond a=a$ for all $a \in S^*$,therefore,

$a \Diamond a=a \Diamond (a \Diamond e)$.By ATL law ,we get $= e \Diamond (a \Diamond a)$ for all $a \in S^*$.According to defined $\Diamond$ ,we obtain $=e \Diamond a$ for all $a \in S^*$.According to defined of $\Diamond$ ,we obtain $=e$. Then $(S^*, \Diamond)$ is a flower

Thus ,we obtain $(S, * , \Diamond)$ is a farm.

As a corollary of Theorem 8 and Proposition 8 ,we write the following theorem.

### Theorem 17

If $(S, *)$ is a flower .Then

(i) S is a Lahhamian group.

(ii)S is has an identity.

(iii) S is commutative.

(iv)S is a ssociative.

The following conjecture appeared in [1] with out proof ,we give the proof .

### Al-Lahham Conjecture 18

If $(S, *)$is a groupoid with a right identity element e ,such that $x * x= e$ for all $x \in S$,then for all elements $a,b,c \in S$:

$a * (b * c)=c * (b * a)$ if and only if $(a * b) * c=(a * c) * b$.

**Proof:** For all $a,b,c \in S$, we have $a * (b * c)=c * (b * a)$ .Then we will deal with $(S, *)$ as a flower. By Proposition 6 ,we get

$(a * c) * c=(a * c) * b$.

Also , we have for all $a,b,c \in S$,such that $(a * b) * c=(a * c) * b$.

Since $x * x =e$ for all $x \in S$ .

Then $(S, *)$is Lahhamian group ,by Proposition 7 ,$(S, *)$ is flower,then $a * (b * c)= c * (b * a)$is satisfy.

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**الخلاصة :**


الغرض الرئيسي من هذا البحث اعطاء بعض النتائج مبرهنات وتمهيديات وملاحظات بخصوص هذه البنى الجبرية الحديثة ،الزهرة والحديقة والمزرعة مع بنى جبرية مألوفة ،نصف الزمرة والزمرة والحلقة.