Open Newton Contes Formula for Solving Linear Voltera Integro-Differential Equation of the First Order

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Abstract

In this work, some of numerical methods for solving first order linear Volterra Integro-Differential Equations are presented.

The numerical solution of these equations is obtained by using Open Newton Cotes formula.

The Open Newton Cotes formula is applied to find the optimum solution for this equation.

The computer program is written in (MATLAB) language (version 6)

Key word: Voltera Integro- Differential Equation, Open Newton Cotes Formula

Introduction

In this search we first present the most familiar formula of numerical integration: the Open Newton Cotes formula (O-N). and then we illustrate primarily the use of these rules for evaluating integrals and show how the linear VIDEs of first order is reduced to system of (n) equations in the (n) unknowns of the solution sample values $u(x_i)$, i=0,1,2,…n.

The procedure of the previous technique is called the Open Newton Cotes formula.

In addition a computer program is written, examples with satisfactory results are given.

1. Classification of Integral Equations:

Any functional equation in which the unknown function appears under the sign of integrations is called an integral equation [1].

The general non-linear integral equation can be presented in the form.

$$h(x)u(x) = f(x) + \lambda \int_a^{b(x)} k(x,t,u(t))\,dt \qquad \ldots(1.1)$$

and if

$k(x,t,u(t)) = k(x,t)u(t)$

Then (1.1) is called linear integral equation having the form

$$h(x)u(x) = f(x) + \lambda \int_a^{b(x)} k(x,t)u(t)\,dt \qquad \ldots(1.2)$$

Here the function h(x), f(x) and the kernel function k(x,t) are prescribed, while u(x) is the unknown function to be determined and λ is a scalar parameter[2,3]

If in equation (1.1) and equation (1.2), f(x)=0 the integral equations are said to be homogeneous integral equations otherwise, they are non homogeneous [4].

In the classical theory of integral equations one distinguishes between Volterra equations and Fredholm equations In a Fredholm integral equation the region of integration is fixed i.e. b(x)=b, where as in Volterra integral equation the region is variable [5].

Thus, the equation

$$h(x)u(x) = f(x) + \lambda \int_a^{b} k(x,t)u(t)\,dt\,, \quad a \le x \le b \qquad \ldots(1.3)$$

is an example of a linear Fredholm integral equation and the equation

$$h(x)u(x) = f(x) + \lambda \int_a^x k(x,t)u(t)dt \, , a \le x \qquad \dots(1.4)$$

is an example of a linear Volterra integral equation.

1.1 Voltera Integral Equations:

A linear Volterra integral equation of the first and second kind is defined as in equation (1.4) by letting h(x)=0 and h(x)=1, [6,1]

$$f(x) = \lambda \int_a^x k(x,t)u(t)dt \, , \; a \le x \qquad \dots (1.5)$$

$$u(x) = f(x) + \lambda \int_a^x k(x,t)u(t)dt \, , a \le x \qquad \dots(1.6)$$

A Volterra equation can be looked at as an important special case of a Fredholm equation which arises when k(x,t)=0 for t >x. The distinction between Fredholm and Volterra equation is analogous to the distinction between boundary and initial value problems in ordinary differential equations [6,1].

Definition 1.1:

An integral equation is termed linear if it involves the integral operator

$$L = \lambda \int_a^{b(x)} k(x,t)dt$$

which satisfies the linearity condition

$$L[c_1u_1(t) + c_2u_2(t)] = c_1L[u_1(t)] + c_2L[u_2(t)]$$

where $L[u(t)] = \lambda \int_a^{b(x)} k(x,t)dt,$ and c_1, c_2 are constants [1,7].

Definition 1.2:

The equation (1.1) is said to be linear integral equation of the first kind , if the unknown function is present under the integral sign only, i.e. $h(x) \equiv 0$; linear integral equation of the second kind also has the unknown function outside the integral, i.e. $h(x) \ne 0$ for $a \le x \le b$; and if h(x) vanishes somewhere but not identically, the equation is of third kind [8].

1.2 Singular and Weakly Singular Equations:

An integral equation may be called singular if either

(a) its kernel k(x,y) is not bounded

(b) the range of integration is infinite e.g., $0<x<\infty$ or $-\infty < x < \infty$.

And it is said to be Weakly-singular if the kernel becomes infinite at y=x. [9,10]

2.3 Structure of kernel:

(a) Linear integral equation with a kernel k(x,t)=k(t,x) is said to be Symmetric. This property plays a key role in the theory of Fredholm integral equations.

(b) If k(x, t)=k(a+ b- x,a+ b- t) in linear integral equation, the kernel is called conter-symmetric.

(c) If k(x,t) in (1.1)and (1.2) depends only on the difference (x-t) i.e. if the kernel is of the form k(x,t)=k(x-t).Such a kernel is called difference kernel and the integral equation is called integral equation of convolution type which has the form [1,11]:

$$h(x)u(x) = f(x) + \lambda \int_a^{b(x)} k(x-t)u(t)dt \qquad \dots (1.7)$$

(d) In equation (1.1), k(x,t) is called Separable or Degenerate kernel of rank n if it is of the form:

$$k(x,t) = \sum_{r=1}^{n} a_r(x)b_r(t)$$

where n is finite, it is assumed that the functions $\{a_r\}$ and $\{b_r\}$ are sufficiently smooth functions of these arguments [11].

Hence, in many cases a non degenerate kernel k(x,t) may be approximated by a degenerate kernel as a partial sum of Taylor series expansion of k(x,t) [11]. For example, consider the kernel $k(x,t) = -1 + e^{xt}$ which may be approximated to be finite number of terms of its Taylor's series about $x_0 = 0, y_0 = 0$ i.e.

$$-1 + e^{xt} = -1 + [1 + xt + \frac{x^2t^2}{2!} + \frac{x^3t^3}{3!} + \cdots]$$

Hence, if only 3-terms of the series are considered, we have a degenerate kernel:

$$k(x,t) = xt + \frac{x^2t^2}{2!} + \frac{x^3t^3}{3!} + \cdots\cdots \cong \sum_{r=1}^{3} a_r(x)b_r(t)$$

where

$$a_1(x) = x,\ a_2(x) = x^2,\ a_3(x) = x^3$$

$$b_1(t) = t,\ b_2(t) = \frac{t^2}{2!},\ b_3(t) = \frac{t^3}{3!}$$

Definition 1.3:

An ordinary differential equation is an equation that involves at most the n^{th} derivative of an unknown function [13]. i.e. if the unknown function u is a function of x then we write the differential equation of order n as:

$$\frac{d^n u(x)}{dx^n} = g(x, u, u', u'', \ldots\ldots, u^{(n-1)})$$

where g is a given function of variable $x, u, u', u'', \ldots\ldots, u^{(n-1)}$

1.4 Integro-Differential Equation: [14,15]

An integro-differential equation is an equation involving one (or more) unknown functions u(x), together with both differential and integral operations on x.

A linear integro-differential equation of order n is an equation of the form

$$u^{(n)}(x) + \sum_{i=0}^{n-1} p_i(x) u^{(i)}(x) = f(x) + \lambda \int_a^{b(x)} k(x,t) u(t)\, dt$$

Here, $k(x,t), f(x), p_i(x) (i = 0,1,\ldots\ldots, n-1)$ are known functions, u(x) is the unknown function, and λ is a scalar parameter[11].

2. Numerical Solution of VIDE Using Open Newton Cotes formula(O-N).

In this search, we use Open Newton Cotes formula (O-N) to find the numerical solution of 1st order linear VIDE, in the form:

$$u'(x) + p(x)u(x) = f(x) + \int_a^x k(x,y)u(y)\, dy, x \in I = [a,b] \qquad \ldots (1.8)$$

with the initial condition $u(a) = u_0$, where the functions f and p are assumed to be continuous on I and k denotes given continuous functions.

Were the interval [a, b] is divided in to n equal subintervals, where h=(b-a)/n, $y_0 = a, y_n = b$ and $y_j = a + j * h, j = 0,1,\ldots,n.$ we set $x_i = y_j, i = 0,1,\ldots,n, u'(x_i) = u_i, p(x_i) = p_i, f(x_i) = f_i$, $u(x_i) = u_i$ and $k(x_i, y_j) = k_{ij}$.

2.1 Open Newton Cotes formula(O-N).

Open Newton Cotes formula is used with n-subinterval to approximate the integral in equation (1.6), hence

If n = 0 Midpoint Method

Let h = (b-a)/(2m+2) and x_j = a + (j + 1)h for j = 1,….,2m+1 for n = 2m subinterval is

$$\int_a^b f(x)dx = 2h\sum_{j=0}^{M} f(x_{2j}) + \frac{b-a}{6}h^2 f''(M) \quad for\,som\,M \in (a,b) \qquad \ldots(1.9)$$

$$u_i' + p_i u_i = f_i + 2h[k_{i0}u_0 + 2k_{i1}u_1 + \ldots.. + 2k_{ii-1}u_{i-1} + k_{ii}u_i] \qquad \ldots(1.10)$$

since

$$u_i'(x) = \frac{u_i(x) - u_{i-1}(x)}{h} \qquad \ldots(1.11)$$

Substituting equation (1.11) into equation (1.9) we have

$$(1 + hp_i - \frac{h^2}{2}k_{ii})u_i - u_{i-1} = hf_i + 2h^2[k_{i0}u_0 + 2k_{i1}u_1 + \ldots.. + 2k_{ii-1}u_{i-1}] \qquad \ldots(1.12)$$

which are (n+1) equation in u_i that represents the approximate solution to equation (1.8) at $x = a + i * h, (i = 0,1,\ldots., n)$.

$$\begin{bmatrix} 1+hp_1-2h^2k_{11} & 0 & 0 & 0 & \cdots & 0 \\ -1-2h^2k_{21} & 1+hp_2-2h^2k_{22} & 0 & 0 & \cdots & 0 \\ -2h^2k_{31} & -1-2h^2k_{32} & 1+hp_3-2h^2k_{33} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \vdots \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} hf_1 + 2h^2k_{10}u_0 + u_0 \\ hf_2 + 2h^2k_{20}u_0 \\ hf_3 + 2h^2k_{30}u_0 \\ \vdots \end{bmatrix}$$

The Algorithm (AQ)

The numerical solution of (1^{st} order VIDEs), by using Open Newton Cotes formula (O-N), is obtained as follows:

Step 1:

Put h=(b-a)/n, $n \in N$

Step 2:

Set $u_0 = u(a)$ (which is the initial condition) is given.

Step 3:

Compute u_i' by using $u_i' = \frac{u_i - u_{i-1}}{h}$

Step 4:

Use step –1, -2 and –3 in equation (1.10) to find $u_i, (i = 1,2,\ldots\ldots, n)$ we get

$$(1 + hp_i - 2h^2k_{ii})u_i - u_{i-1} = hf_i + 2h^2[k_{i0}u_0 + 2k_{i1}u_1 + \ldots. + 2k_{ii-1}u_{i-1}]$$

3. Numerical Examples:

Example 4.1:

Consider the following VIDE:

$$u'(x) = -\frac{1}{2} - \frac{5}{6}x + \int_0^x (xt+1)u(t)dt, \; 0 \le x \le 1$$

The exact solution is u(x) = 1 + x , [16].

Take n=10 h=0.1 and x_i =a+ih, i=0, 1, …., n.

Table (1) illustrates the comparison between the exact and numerical solution depending on the least square error and running time.

Example 4.2:

Consider the following Voltera integro-differential equation:

$$x \geq 0 \quad u'(x) = 2x - \frac{5}{6}x^4 + \int_0^x (x + 2t)u(t)\,dt$$

Table (2) presents results from a computer program that solves this problem over the interval x=0 to x=1 with u(x) = x^2 for which the analytical solution is h=0.1, [16].

4. Discussion and Conclusion.

The approximate solution of linear Voltera integro-differential equation (1.8) is given using the Open Newton Cotes formula. A computer program was written and several examples were solved using these method. We have the option that the result of the O-N formula is better than the results of Homotopy Perturbation method.

Relying on our work the following notes are drawn:

1.The number of subintervals n is restricted to be even for Open Newton Cotes formula.

2.Through the solution of linear Voltera integro-differential equations of the first order, we see that

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Table (1) Results of (O-N) Formula for example 1

X	EXACT	O- N	$\|exact-(O-N)\|$
0.0	1.0	1.00000000000	0.00000000000
0.1	1.1	−0.059535959719	1.040464040280
0.2	1.2	−0.130123733624	1.069876266376
0.3	1.3	−0.213768960894	1.086231039105
0.4	1.4	−0.313206121581	1.086793878418
0.5	1.5	−0.432225024164	1.067774975835
0.6	1.6	−0.576122808453	1.023877191546
0.7	1.7	-0.752340013843	0.947659986156
0.8	1.8	−0.971371157918	0.828628842081
0.9	1.9	−1.248090330516	0.651909669483
1.0	2.0	−1.603711691947	0.396288308052
L.S.E.		53.40100881831022	53.401008818
R.Time.		0.33000000000000	0.3300000000

Table (1.a) comparison between (O-N) formula and Homotopy Perturbation Method of example (1).

METHOD / *Nodes*	O- N	Homotopy Perturbation	SIMPSONS1/3	SIMPSONS13/8
0.0	1.00000000000	1.00000000000	1.00000000000	1.00000000000
0.1	-0.059535959719	-0.059535959719	-0.058629411863	-0.058629411863
0.2	-0.130123733624	-0.130123733624	-0.126532083085	-0.126532083085
0.3	-0.213768960894	-0.213768960894	-0.204566898710	-0.204556471062
0.4	-0.313206121581	-0.313206121581	-0.293815208714	-0.293816208897
0.5	-0.432225024164	-0.432225024164	-0.395769114158	-0.395757665364
0.6	-0.576122808453	-0.576122808453	-0.512372661710	-0.512361184126
0.7	-0.752340013843	-0.752340013843	-0.646289429429	-0.646277987576
0.8	-0.971371157918	-0.971371157900	-0.801011403831	-0.800999920591
0.9	-1.248090330516	-1.248090330516	-0.981254168189	-0.981226479926
1.0	-1.603711691947	-1.603711691947	-1.193202937519	-1.193192880315
L.S.E.	53.40100881831	53.400975910829	47.059404783003	47.0589476850
R.T.	0.330000000000	0.550000000000	0.440000000000	0.440000000000

Table (2) Results of (O-N) Formula for example 2

X	EXACT	Open Newton Cotes formula.	$\|exact-(O-N)\|$
0.0	0.00	0.000000000000	0.00000000000
0.1	0.01	0.020112340710	1.079887659290
0.2	0.04	0.060870350306	1.139129649694
0.3	0.09	0.123471139122	1.176528860878
0.4	0.16	0.210064039206	1.189935960794
0.5	0.25	0.324135025802	1.175864974198
0.6	0.36	0.471049431039	1.128950568961
0.7	0.49	0.658846458868	1.041153541132
0.8	0.64	0.899428134734	0.900571865266
0.9	0.81	1.210363739718	0.689636260282
1.0	1.00	1.617657119881	0.382342880119
L.S.E.		0.659596059903	
R.T.		0.330000000000	

Table (2.a) comparison between (O-N) formula and Homotopy Perturbation Method of example (2).

METHOD / *Nodes*	O- N	Homotopy Perturbation	SIMPSONS1/3	SIMPSONS13/8
0.0	0.00000000000	0.00000000000	0.00000000000	0.00000000000
0.1	0.020112340710	0.020112340710	0.020021699215	0.020021699215
0.2	0.060870350306	0.060870350306	0.060115379036	0.060115379036
0.3	0.123471139122	0.12347113912	0.120446218308	0.120432870642
0.4	0.210064039206	0.210064039206	0.201204441266	0.201205986374
0.5	0.324135025802	0.324135025802	0.302718695118	0.302703459057
0.6	0.471049431039	0.471049431039	0.425348164531	0.425332499117
0.7	0.658846458868	0.658846458868	0.569633374553	0.569617082912
0.8	0.899428134734	0.899428134734	0.736177654792	0.736160153872
0.9	1.210363739718	1.210363739718	0.925843695363	0.925801402953
1.0	1.617657119881	1.617657119881	1.139636574700	1.139616212933
L.S.E.	0.659596059903	0.659596059903	0.058689234914	0.058663452819
R.T.	0.330000000000	0.980000000000	0.820000000000	0.380000000000

طريقة نيوتن كوتس لحل معادلات فولتيرا التكاملية التفاضلية من الرتبة الاولى

عاطفة جليل صالح

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الخلاصة

يوجد في هذا العمل بعض الطرائق العددية لحل معادلة فولتيرا التكاملية-التفاضلية الخطية ذي الرتبة الاولى.
الحلول العددية لهذه المعادلات تم التوصل اليها بأستعمال طريقة نيوتن كوتس.
طريقة نيوتن كوتس طبقت لايجاد الحل الامثل لهذه المعادلة.
البرامج الحسابية قد كتبت بلغة ماتلاب (version 6).

الكلمات المفتاحية: معادلات فولتيرا التفاضلية –التكاملية– صيغ نيوتن– كوتس المفتوحة