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Numerical Solution of Mixed Volterra – Fredholm Integral Equation Using the Collocation Method

Nahdh S. M. Al-Saif*

Ameen Sh. Ameen

Department of applied Mathematics, College of Science, University of Anbar, Al- Anbar, Iraq

* Corresponding author: nn_ss_m68@yahoo.com, amensh66@yahoo.com

*ORCID ID: <https://orcid.org/0000-0003-3517-2251>, <https://orcid.org/0000-0001-8638-3556>

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Abstract:

Volterra – Fredholm integral equations (VFIEs) have a massive interest from researchers recently. The current study suggests a collocation method for the mixed Volterra - Fredholm integral equations (MVFIEs). "A point interpolation collocation method is considered by combining the radial and polynomial basis functions using collocation points". The main purpose of the radial and polynomial basis functions is to overcome the singularity that could associate with the collocation methods. The obtained interpolation function passes through all Scattered Point in a domain and therefore, the Delta function property is the shape of the functions. The exact solution of selective solutions was compared with the results obtained from the numerical experiments in order to investigate the accuracy and the efficiency of scheme.

Key words: Mixed Volterra - Fredholm integral equations, Radial basis function.

Introduction:

Consider the general mixed Volterra - Fredholm integral equation has the form

$$u(s, t) = f(s, t) + \int_0^t \int_{\Omega} K(s, t, x, y, u(x, y)) dx dy \quad (1)$$

$(s, t) \in I = \Omega \times [0, T]$

where $u(s, t)$ is unknown function should to be found, $f(s, t)$ and $K(s, t, x, y, u(x, y))$ are given analytic functions on $I = \Omega \times [0, T]$ and $C(I^2 \times R)$, respectively and Ω is close subset on R , with norm $\|\cdot\|$.

Equations of this type arise in the main branches of modern mathematics that appear in various applied areas including mechanics, physics and engineering ...etc. In recent years different numerical methods have been used to solve (Eq. 1). Hassan (1) investigated a new iterative method to solve the (MVFIEs). In (2) Nili used the Meshless method for solving (MVFIEs) of Urysohn type on non – rectangular regions numerically. Nemati (3) introduced numerical method for the (MVFIEs) using Hybrid Legendre Functions. In (4) Shahooth presented a numerical solution for mixed (MVFIEs) of the second kind using Bernstein polynomial method. Babolian (5) applied block pulse function and the associated operational matrix to solve the (MVFIEs) in 2 - dimensional spaces. In our previous work (6), solution of mixed Volterra -

Fredholm integral equation by designing neural network is given .

The aim of this paper is to apply collocation method for solving mixed Volterra – Fredholm integral equations (MVFIEs) which have the formula (Eq.1).

Radial basis functions (RBF)

Definition of radial function

Consider $R^+ = \{t \in R, t \geq 0\}$ the half – line of non-negative numbers and $\phi: R^+ \rightarrow R$ be a continuous function with $\phi(0) \geq 0$. A (RBF) on R^d is a function of the form

$$\phi(\|t - t_i\|) \quad (2)$$

Where $t, t_i \in R^d$ and $\|\cdot\|$ is the Euclidean distance between t and t_i . If chooses N points $\{t_i\}_{i=1}^N$ in R^d , then

$$s(X) = \sum_{i=1}^N \lambda_i \phi(\|t - t_i\|) \quad ; \lambda_i \in R, X \in R^n \quad (3)$$

Called a radial basis function (RBF) (7).

Point Interpolation Method (PIM) on (RBF)

Let $u(s, t)$ be an approximation function in a domain (with an arbitrarily distributed nodes set) denoted by $(s_i, t_i), i = 1, 2, 3, \dots, N$. Where N is the influence domain nodes number. At the node (s_i, t_i) u_i assumed to be the Nodal Function Value.

Using radial basis function $A_i(s,t)$, $i=1,2,3,\dots, N$ and polynomial basis function $B_j(s,t)$, $j=1,2,\dots, M$, Radial basis (PIM) constructs the approximation function $u(s,t)$ to access the node points (8).

$$u(s,t) = \sum_{i=1}^N a_i A_i(s,t) + \sum_{j=1}^M b_j B_j(s,t) = a A^T(s,t) + b B^T(s,t) \quad (4)$$

Where a_i are the coefficient of $A_i(s,t)$ and b_i the coefficient of $B_j(s,t)$ (usually, $N > M$) the vectors are defined a

$$\left. \begin{aligned} a^T &= (a_1 a_2 \dots a_N) \\ b^T &= (b_1 b_2 \dots b_M) \\ A^T(s,t) &= (A_1(s,t) \quad A_2(s,t) \quad \dots \quad A_N(s,t)) \\ B^T(s,t) &= (B_1(s,t) \quad B_2(s,t) \quad \dots \quad B_M(s,t)) \end{aligned} \right\} \quad (5)$$

The general form of a multiquadrics (MQ) radial basis functions is

$$A_i(s,t) = A_i(d_i) = \sqrt{d_i^2 + c^2}, \quad 0 \leq c \leq 1 \quad (6)$$

Where d_i is the distance between the node (s_i, t_i) and the interpolating point (s, t) in the Euclidean 2 - dimensional space d_i could be described as

$$d_i = [(s - s_i)^2 + (t - t_i)^2]^{\frac{1}{2}}$$

The terms of polynomial basis function are as following:

$$B^T = (1 \quad s \quad t \quad s^2 \quad st \quad t^2 \quad \dots) \quad (7)$$

The coefficients a_i and b_i in Eq. (5) are found by taking the interpolation of the scattered nodal points N in the domain. Therefore, k^{th} point, the interpolation is:

$$u_k = u(s_k, t_k) = \sum_{i=1}^N a_i A_i(s_k, t_k) + \sum_{j=1}^M b_j B_j(s_k, t_k), \quad k = 1, \dots, N \quad (8)$$

The polynomial terms are so important to obtain the optimum approximation. To fulfill that, consider the following conditions:

$$\sum_{i=1}^n a_i B_j(s_i, t_i) = 0, \quad j = 1, 2, \dots, M \quad (9)$$

The matrix form is expressed as follows:

$$\begin{pmatrix} A_0 & B_0 \\ B_0^T & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} u^e \\ 0 \end{pmatrix} \quad \text{or} \quad \mathcal{Q} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} u^e \\ 0 \end{pmatrix} \quad (10)$$

Where the vector u^e , matrixes A_0 and B_0 are defined as:

$$u^e = (u_1, u_2, \dots, u_N)^T$$

$$A_0 = \begin{pmatrix} A_1(s_1, t_1) & A_2(s_1, t_1) & \dots & A_N(s_1, t_1) \\ A_1(s_2, t_2) & A_2(s_2, t_2) & \dots & A_N(s_2, t_2) \\ \dots & \dots & \dots & \dots \\ A_1(s_N, t_N) & A_2(s_N, t_N) & \dots & A_N(s_N, t_N) \end{pmatrix}_{N \times N} \quad (11)$$

$$B_0 = \begin{pmatrix} B_1(s_1, t_1) & B_2(s_1, t_1) & \dots & B_M(s_1, t_1) \\ B_1(s_2, t_2) & B_2(s_2, t_2) & \dots & B_M(s_2, t_2) \\ \dots & \dots & \dots & \dots \\ B_1(s_N, t_N) & B_2(s_N, t_N) & \dots & B_M(s_N, t_N) \end{pmatrix}_{N \times M}$$

The matrix A_0 is symmetric because the distance is directionless, therefore

$$A_k(s_i, t_i) = A_i(s_k, t_k).$$

If \mathcal{Q} invertible matrix, a unique solution obtained

$$\begin{pmatrix} a \\ b \end{pmatrix} = \mathcal{Q}^{-1} \begin{pmatrix} u^e \\ 0 \end{pmatrix} = \Lambda \quad (12)$$

The interpolation finally expressed as

$$u(s,t) = (A^T(s,t) \quad B^T(s,t)) \mathcal{Q}^{-1} \begin{pmatrix} u^e \\ 0 \end{pmatrix} = \Phi(s,t) \Lambda \quad (13)$$

Where shape functions $\Phi(s,t)$ is defined by

$$\Phi = (\Phi_1(s,t) \quad \Phi_2(s,t) \quad \dots \quad \Phi_N(s,t)) \quad (14)$$

in which

$$\Phi_k(s,t) = (A_k^T(s,t) \quad B_k^T(s,t)) \quad (15)$$

The shape functions would depend only on the scattered nodes position when the Radial Basis Functions been calculated, only when the inverse of matrix \mathcal{Q} is obtained.

Solution of (MVFIEs) by (RPIM)

When considering the following 2 - dimension Mixed Volterra - Fredholm integral equation (MVFIE):

$$u(s,t) = f(s,t) + \int_0^t \int_{\Omega} K(s,t,x,y,u(x,y)) dx dy \quad (s,t) \in I = \Omega \times [0, T]$$

In our method, the form Point Interpolation Method (PIM) approximating function (Eq.13) is firstly obtained from a set of points. Then it calculated in a straight forward by differentiating such a closed form (RBF). That can be explained as below:

$$u(s,t) \approx u_N(s,t) = \sum_{i=1}^N a_i A_i(s,t) + \sum_{j=1}^M b_j B_j(s,t) = \Phi^T(s,t) \Lambda, \quad (16)$$

Substituting Eq. (16) into (1), we have

$$\Phi^T(s,t) \Lambda = f(s,t) + \int_0^t \int_{\Omega} K(s,t,x,y, \Phi^T(x,y) \Lambda) dx dy \quad (17)$$

Herein, we collocate Eq. (17) at point $\{(s_i, t_i)\}_{i=1}^N$ as

$$\Phi^T(s_i, t_i) \Lambda = f(s_i, t_i) + \int_0^{t_i} \int_{\Omega} K(s_i, t_i, x_i, y_i, \Phi^T(x_i, y_i) \Lambda) dx dy \quad (18)$$

Then (Eq. 18) residual function $Res(s_i, t_i)$

$$Res(s_i, t_i) = -\phi^T(s_i, t_i)A + f(s_i, t_i) + \int_0^{t_i} \int_{\Omega} K(s_i, t_i, x_i, y_i, \phi^T(x_i, y_i)A) dx dy \quad (19)$$

The coefficients $\{\Lambda_i\}_{i=1}^N$ are obtained by equalizing Eq. (19) to zero at N interpolate nodes. By applying the mean square error law to study behavior of the RPIM to get...

$$MSE = \sqrt{\frac{\sum_{i=1}^N (u(s_i, t_i) - u_N(s_i, t_i))^2}{N}}$$

Algorithm

The algorithm is implemented as the following steps:

1. Chooses N points $\{(s_i, t_i)\}_{i=1}^N$ from the domain set $(a, b) \times (a, b)$.
2. Approximate $u(s, t)$ as $u_N(s, t) = \phi^T(s, t)A$.
3. Create residual function $Res(s, t)$ from Substituting $u_N(s, t)$ into the main problem.
4. Create the N equations from Substitute points $\{(s_i, t_i)\}_{i=1}^N$ into the $Res(s, t)$.

5. Find coefficients of members of A numerical solution solving the N equations with N unknown by inverse matrix method (eq.12)
6. Applied the mean square error (MSE) law to study behavior of the RPIM.

Numerical Examples

Consider the following examples to investigate the reliability of radial point interpolation method (RPIM) to solve (MVFIE) in addition to justify the efficiency and the accuracy of our proposed method. In all examples, use (MQ.) (RBF).

Example 1:

Consider the following (MVFIE).

$$u(s, t) = s^2 + t^3 + \frac{s^2 t^3}{3} + \frac{s^2 t^7}{4} - \int_0^t \int_0^1 s^2 t^3 u(x, y) dx dy \quad (20)$$

With the exact solution below

$$u(s, t) = s^2 + t^3$$

Table (1) demonstrates MSE obtained from applying our method on Equation (20) for different values of M.

Table 1. MSE between exact & (RPIM) solution with $c = 0.1$ in Example (1)

s	t	N=8 M=3		N=8 M=5		N=8 M=7	
		Exact-RPIM		Exact-RPIM		Exact-RPIM	
0.980	0.00	1.11e-16		0.00e-00		0.00e-00	
0.898	0.05	1.12e-06		1.31e-06		1.12e-06	
0.763	0.10	1.56e-05		1.68e-05		1.56e-05	
0.592	0.15	5.43e-05		5.59e-05		5.43e-05	
0.408	0.20	8.74e-05		8.88e-05		8.75e-05	
0.237	0.25	7.39e-05		7.49e-05		7.39e-05	
0.102	0.3	2.84e-05		2.87e-05		2.84e-05	
0.199	0.4	2.03e-06		2.04e-05		2.03e-06	
MSE		2.25 e-009		2.22e-009		2.14e-009	

Example 2:

Let the following (MVFIE)

$$u(s, t) = \sin(st) - \frac{1}{2}t^2 \cos(s) + \cos(s) - \cos(s) \cos(t)$$

$$+ \int_0^t \int_0^1 t^2 \cos(s) u(x, y) dx dy \quad (21)$$

With the below exact solution:

$$u(s, t) = \sin(st)$$

Also, Table (2) demonstrates MSE obtained from applying our method on Eq. (21) for different values of M.

Table 2. MSE between Exact & (RPIM) solution with $c = 0.1$ in Example (2)

s	t	N=5 M=2		N=5 M=3		N=5 M=5	
		Exact-RPIM		Exact-RPIM		Exact-RPIM	
0.953	0.00	2.95e-17		0.00e-00		3.69e-18	
0.769	0.05	5.96e-07		6.73e-06		4.33e-07	
0.500	0.10	1.76e-06		1.29e-05		1.73e-06	
0.231	0.15	8.10e-06		3.67e-05		1.38e-05	
0.469	0.20	5.81e-05		8.80e-05		6.55e-05	
MSE		6.88e-10		1.86e-09		8.96 e-10	

Example 3:

Mixed nonlinear Volterra - Fredholm integral equation is following

$$u(s, t) - \int_0^t \int_0^1 (s -$$

$$t)u^2(x, y) dx dy = st - \frac{st^3}{9} + \frac{t^4}{9} \quad (22)$$

With exact solution:

$$u(s, t) = st$$

method on Equation. (22) for different values of M.

Table (3) demonstrates MSE obtained from applying our

Table 3. MSE between Exact & (RPIM) solution with $c = 0.1$ in Example (3)

s	t	N=20 & M=8		N=20 & M=6		N=20 & M=5	
		Exact-RPIM		Exact-RPIM		Exact-RPIM	
0.003	0.00	1.56e-17		2.78e-17		3.47e-17	
0.127	0.20	6.46e-05		6.48e-05		6.48e-05	
0.386	0.40	9.23e-05		9.39e-05		9.31e-05	
0.687	0.60	1.47e-03		1.52e-03		1.41e-03	
0.920	0.80	2.19e-03		2.42e-03		1.54e-03	
0.996	1.00	5.84e-04		7.59e-04		7.77e-05	
MSE		1.39e-006		1.64e-006		8.97e-007	

Conclusion:

In this work, we have studied a collocation method to solve linear and non-linear mixed Volterra – Fredholm integral equations (MVFIEs). Our approach is built on combining the radial and polynomial basis functions in a point interpolation collocation method. Zeros of shifts Legendre are used as collocation points. The possible singularity that could associate in collocation methods is overcome by involving the radial basis functions. Many examples are solved by our method for different numbers of term N and M. It can be seen from the results in all the Tables, that it is clear that the approximate solution is in high agreement with the exact solution and the accuracy of RPIM results is in positive relationship with the numbers term (N) of RBF and (M) of polynomial (P). The efficient of RPIM method is excellent and of high precision. In general the accuracy and efficiency of our method (especially, for linear MVFIEs) are great and this method is easy to compute. Also, it can be easily extended and applied to nonlinear systems.

Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Anbar.

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الحل العددي لمعادلة فولتيرا - فريدهولم التكاملية المختلطة بطريقة التجميع

امين شامان امين

ناهض سليم محمد سعيد

الرياضيات التطبيقية، كلية العلوم، جامعة الانبار، الانبار، العراق

الخلاصة :

معادلات فولتيرا- فريدهولم التكاملية المختلطة (MVFIES) لديها اهتمام كبير من قبل الباحثين مؤخرًا. الطريقة العددية التي اقترحت لحل هذا النوع من المعادلات تستعمل نقاط التجميع وتقريب الحل بواسطة الدالة اساس الشعاعي (radial basis function) و متعددة حدود من الدرجة الثانية واندرج النقطة من دون استخدام الشبكة، ولسهولة الحل تم استخدام اصفار متعددة حدود ليجندر كنقاط تجمع. الغرض الرئيسي من استخدام دالة اساس الشعاعي ومتعدد الحدود هو التغلب على التفرد الذي قد يرتبط بأساليب التجميع. علاوة على ذلك، فإن وظيفة الاستيفاء التي تم الحصول عليها تمر عبر كل النقاط المنتشرة في مجال ما ، وبالتالي فإن وظائف الشكل هي من خصائص خاصة دلتا. تمت مقارنة الحل الدقيق للحلول الانتقائية بالنتائج التي تم الحصول عليها من التجارب العددية من أجل التحقق من دقة وكفاءة طريقتنا.

الكلمات المفتاحية: متعددة حدود ليجندر، معادلة فولتيرا – فريدهولم المختلطة، دالة اساس الشعاعي .