Open Access **Baghdad Science Journal** **P-ISSN: 2078-8665**
Published Online First: December 2020 **E-ISSN: 2411-7986**

DOI: http://dx.doi.org/10.21123/bsj.2021.18.1.0156

T-Essentially Coretractable and Weakly T-Essentially Coretractable Modules

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Received 24/2/2019, Accepted 25/4/2020, Published Online First 6/12/2020, Published 1/3/2021

Abstract:

A new generalizations of coretractable modules are introduced where a module $\mathcal{M}$ is called t-essentially (weakly t-essentially) coretractable if for all proper submodule K of $\mathcal{M}$, there exists $f \in \text{End}(\mathcal{M})$, $f(K)=0$ and $\text{Im} f \leq_{tes} \mathcal{M}$ ($\text{Im}\, f + K \leq_{tes} \mathcal{M}$). Some basic properties are studied and many relationships between these classes and other related one are presented.

Key words: Coretractable module, Essentially coretractable module, T-essentially coretractable module, Weakly t-essentially coretractable module, Weakly essentially coretractable module.

Introduction:

In this work, all rings have identity and all modules are unitary left R-modules. A coretractable module appeared in(1). However, Amini(2), studied this class of modules, where "$\mathcal{M}$ is called coretractable if for all proper submodule K of $\mathcal{M}$, there exists $0 \neq f \in \text{Hom}(\mathcal{M}/K, \mathcal{M})$" (1). Next, Hadi and Al-Aeashi defined strongly coretractable module where" a module $\mathcal{M}$ is called strongly coretractable if for each a proper submodule K of $\mathcal{M}$, there exists a nonzero homomorphism $f: \mathcal{M}/K \rightarrow \mathcal{M}$ such that $\text{Im} f + K = \mathcal{M}$"(3). "A submodule K of $\mathcal{M}$ is called essential in $\mathcal{M}$ ($K \leq_{ess} \mathcal{M}$), if $K \cap S=(0)$, $S \leq \mathcal{M}$ implies $S=(0)$" (4), and K is t-essential submodule ($K \leq_{tes} \mathcal{M}$) if for every submodule S of $\mathcal{M}$, $S \cap K \subseteq Z_2(\mathcal{M})$ implies that $S \subseteq Z_2(\mathcal{M})$, where $Z_2(\mathcal{M})$ is called the second singular submodule and is defined by $Z(\frac{\mathcal{M}}{Z(\mathcal{M})}) = \frac{Z_2(\mathcal{M})}{Z(\mathcal{M})}$. Clear that every essential submodule is t-essential, but not conversely. However they are coincide in the class of nonsingular module (5)".

In(6), Hadi and Al-Aeashi introduced two classes related coretractable modules which are essentially coretractable and weakly essentially coretractable modules, where each of these classes is contained in the class of coretractable modules. "A module $\mathcal{M}$ is called essentially coretractable if for each proper submodule K of $\mathcal{M}$, there exists $0 \neq f: \mathcal{M}/K \rightarrow \mathcal{M}$ such that $\text{Im} f \leq_{ess} \mathcal{M}$ and $\mathcal{M}$ is weakly essentially coretractable if $\text{Im} f + K \leq_{ess} \mathcal{M}$" (6). In §$_2$ The notion t-essentially coretractable was studied, a module $\mathcal{M}$ is called t-essentially coretractable if for each proper submodule K of $\mathcal{M}$, there exists $0 \neq f: \mathcal{M}/K \rightarrow \mathcal{M}$ such that $\text{Im} f \leq_{tes} \mathcal{M}$. Also give some connections between it and other related classes of modules. In§$_3$, the notion weakly t-essentially coretractable modules are introduced and studied, as a generalization of weakly essentially coretractable module, $\mathcal{M}$ is called weakly t-essentially coretractable if for each proper submodule K of $\mathcal{M}$, there exists $0 \neq f: \mathcal{M}/K \rightarrow \mathcal{M}$ such that $f(\mathcal{M}/K) + K \leq_{tes} \mathcal{M}$. Many other connections between these classes and other related are given. Recall that " a module $\mathcal{M}$ is called epi-coretractable if for each proper submodule K of $\mathcal{M}$, there exists an epimomorphism $f \in \text{Hom}(\mathcal{M}/K, \mathcal{M})$" (7). "A module $\mathcal{M}$ is called hopfian if each epimomorphsim $f \in \text{End}(\mathcal{M})$, then f is monomorphism. And $\mathcal{M}$ is antihopfian module if $\mathcal{M}/K \cong \mathcal{M}$ for all proper submodule K of $\mathcal{M}$" (8), Clearly any antihopfian module is epi-coretractable. "An R-module M is called quasi-Dedekind if for each proper submodule K of $\mathcal{M}$, $\text{Hom}(\mathcal{M}/K, \mathcal{M})=0$ (9)" and "$\mathcal{M}$ is coquasi-Dedekind module if for each $0 \neq f \in \text{End}(\mathcal{M})$, f is an epimorphism"(10). "A module $\mathcal{M}$ is called C-coretractable (Y-coretractable) if for all proper closed (y-closed) submodule K of $\mathcal{M}$, there exists $0\neq f:\mathcal{M}/K\rightarrow\mathcal{M}$" (11), (7), where a submodule N of $\mathcal{M}$ is called y-closed if $\mathcal{M}/N$ is nonsingular module"(4). Note that every y-closed submodule of $\mathcal{M}$ is closed but the converse may not be true. They are equivalent if $\mathcal{M}$ is nonsingular (4).

T-Essentially Coretractable Modules

The concept of t-essentially coretractable modules are introduced with some of its properties.

Definition 1: A module $\mathcal{M}$ is called t-essentially coretractable if for all proper submodule K of $\mathcal{M}$, there exists $0\neq f:\mathcal{M}/K\rightarrow\mathcal{M}$ such that $f(\mathcal{M}/K)\leq_{tes}\mathcal{M}$. A ring $\mathcal{R}$ is called t-essentially coretractable if $\mathcal{R}$ is a t-essentially coretractable $\mathcal{R}$-module.

Examples and Remarks 1:

(1) It is clear that a module $\mathcal{M}$ is t-essentially coretractable if and only if $\forall K<\mathcal{M}$, there exists $f\in End(\mathcal{M})$, $f(K)=0$ and $Imf\leq_{tes}\mathcal{M}$.

(2) For every $\mathcal{R}$-module, the following implications are hold:
essentially coretractable module ⇒ t-essentially coretractable module ⇒ coretractable module.
The converse of each implication may be not hold, as the following examples show:
The Z-module Z_6 is not essentially coretractable see (6, Example(2.2(3))), but Z_6 is coretractable module. Also Z_6 as Z-module t-essentially coretractable since for each $N\leq Z_6$, $\exists f\in End(Z_6)$ and $f(N)=0$ and $Imf\leq_{tes}Z_6$ (because $Imf+Z_2(Z_6)=Imf+Z_6=Z_6\leq_{ess}Z_6$ and by (5, proposition(1.1)), $Imf\leq_{tes}Z_6$). Beside these Z_6 as Z_6-module is coretractable module, however it is not t-essentially coretractable since for each $0\neq f\in End(Z_6)$. $Imf+Z_2(Z_6)=Imf+(0)=Imf\not\leq_{ess}Z_6$; that is Imf is not t-essential in Z_6 by (5, Proposition (1.1)).

(3) The two concepts t-essentially coretractable module and semisimple are independent, see the following examples: The Z-module Z_4 is t-essentially coretractable since it is essentially coretractable by (6, Example (2.2(4)), but Z_4 is not semisimple.
The Z_6-module Z_6 is semisimple but it is not t-essentially coretractable see Rem. & Exa. (2(3)). Also $\mathcal{M}=Z_2\oplus Z_2$ as Z_2-module is semisimple module but it is not t-essentially coretractable

(4) Clearly every antihopfian module is t-essentially coretractable (since every antihopfian is essentially coretractable (6) which implies t-essentially coretractable).

(5) Every t-essentially coretractable module is C-coretractable, Y-coretractable module. The converse is not true, see Z as Z-module is C-coretractable and it is not t-essentially coretractable.

(6) A module $\mathcal{M}$ is t-essentially coretractable if and only if $\mathcal{M}$ is t-essentially coretractable $\bar{\mathcal{R}}$-module ($\bar{\mathcal{R}}=\mathcal{R}/ann\,\mathcal{M}$).

(7) If $\mathcal{R}$ is t-essentially coretractable ring, $\mathcal{M}$ is faithful cyclic $\mathcal{R}$-module. Then $\mathcal{M}$ is t-essentially coretractable

(8) A ring $\mathcal{R}$ is t-essentially coretractable if and only if for each proper ideal I of $\mathcal{R}$, there exists $r\in\mathcal{R}$, $r\neq 0$ such that $r\in annI$ and $\langle r\rangle\leq_{tes}\mathcal{R}$.

Proposition 1: Let $\mathcal{M}$ be a nonsingular module. Then $\mathcal{M}$ is essentially coretractable if and only if it is t-essentially coretractable.

Proof: (⇒) It is obvious since every essential submodule is t-essential..

(⇐) Let $K<\mathcal{M}$, since $\mathcal{M}$ is t-essentially coretractable, so $\exists 0\neq f:\mathcal{M}/K\rightarrow\mathcal{M}$ and $Imf\leq_{tes}\mathcal{M}$. But $\mathcal{M}$ is nonsingular hence $Imf\leq_{ess}\mathcal{M}$, therefore $\mathcal{M}$ is essentially coretractable.

Proposition 2: Let $\mathcal{M}$ be a uniform module. If $\mathcal{M}$ is coretractable, then $\mathcal{M}$ is essentially coretractable and hence t-essentially coretractable.

Proof:
Let $\mathcal{M}$ be a coretractable module and $K<\mathcal{M}$, so $\exists 0\neq f:\mathcal{M}/K\rightarrow\mathcal{M}$ that means $Imf\neq 0$. But $\mathcal{M}$ is uniform hence $Imf\leq_{ess}\mathcal{M}$, therefore $\mathcal{M}$ is essentially coretractable.

"A module $\mathcal{M}$ is called d-Rickart if for each $f\in End(\mathcal{M})$, $Imf<^{\oplus}\mathcal{M}$" (9), so see the following:

Proposition 3: Every d-Rickart essentially coretractable module is epi-coretractable module.

Proof: Let $K<\mathcal{M}$, since $\mathcal{M}$ is essentially coretractable, so $\exists f:\mathcal{M}\rightarrow\mathcal{M}$ and $Imf\leq_{ess}\mathcal{M}$. But $\mathcal{M}$ is d-Rickart, so Imf is a direct summand in $\mathcal{M}$, therefore $Imf=\mathcal{M}$. Thus $\mathcal{M}$ is epi-coretractable module.

Proposition 4: Let $\mathcal{M}$ be Z_2-torsion module (that is $Z_2(\mathcal{M})=\mathcal{M}$). Then $\mathcal{M}$ is coretractable if and only if $\mathcal{M}$ is t-essentially coretractable.

Proof: Since $\mathcal{M}$ is coretractable, so for all $K<\mathcal{M}$, so $\exists 0\neq f:\mathcal{M}/K\rightarrow\mathcal{M}$. Then $Imf\leq\mathcal{M}$. But $Imf+Z_2(\mathcal{M})=Imf+\mathcal{M}=\mathcal{M}\leq_{ess}\mathcal{M}$ so by (5,Proposition1.1) $Imf\leq_{tes}\mathcal{M}$. Thus $\mathcal{M}$ is t-essentially coretractable module. The converse is clear.

By applying Proposition(4), see the following: Let $\mathcal{M}=Z_n\oplus Z_m$ as Z-module for each $n, m\in Z_+$. $\mathcal{M}$ is Z_2-torsion module and coretractable, hence $\mathcal{M}$ is t-essentially coretractable.

" A module $\mathcal{M}$ is ***t***-semisimple if for every submodule N of $\mathcal{M}$ there exists a direct summand K of $\mathcal{M}$ such that K is t-essential submodule of N" (5).

Proposition 5: Let $\mathcal{M}$ be a t-semisimple module. Then $\mathcal{M}$ is t-essentially coretractable if and only if for each $K<\mathcal{M}$, ∃ f∈Hom($\mathcal{M}/K$,$\mathcal{M}$), Imf+$Z_2(\mathcal{M})$= $\mathcal{M}$.

Proof: (⇒) If $\mathcal{M}$ is t-essentially coretractable, so for all $K<\mathcal{M}$, so ∃0≠f: $\mathcal{M}/K\to\mathcal{M}$, Im$f\leq_{tes}\mathcal{M}$. Hence Im$f$+$Z_2(\mathcal{M})\leq_{ess}\mathcal{M}$ by (5, Proposition1.1) which implies Imf+$Z_2(\mathcal{M})\leq_{tes}\mathcal{M}$. Thus Im$f$+$Z_2(\mathcal{M})$= $\mathcal{M}$ by ((5, Corollary2.7)).

(⇐) By hypothesis for each K< $\mathcal{M}$, ∃f∈Hom($\mathcal{M}/K$, $\mathcal{M}$), Im f + $Z_2(\mathcal{M})$= $\mathcal{M}$, hence Imf+$Z_2(\mathcal{M})\leq_{ess}\mathcal{M}$ and by (5, proposition 1.1), Im$f\leq_{tes}\mathcal{M}$.

It is to be noted that a direct summand of coretractable module need not be coretractable (12). Also it is to be noted that any direct summand of essentially coretractable module is essentially coretractable see (6, corollary(2.7)), however if $\mathcal{M}$=C⊕N is a Z_2-torsion R-module and C is a cogenerator (where an R-module $\mathcal{M}$ is called a cogenerator if for any R-module N and 0≠x∈N, there exists f: N→ $\mathcal{M}$ such that f(x)≠0. (12)) and N is any R-module, then $\mathcal{M}$ is coretractable module by (12, proposition(1.5)) and so by proposition(6) M is t-essentially coretractable but N need not be coretractable and hence N need not be t-essentially coretractable. Recall that "A module $\mathcal{M}$ is compressible if it can be embedded in any nonzero its submodule"(13).

Proposition 6: Let $\mathcal{M}$ be a compressible t-essentially coretractable module and D be a direct summand of $\mathcal{M}$, then D is t-essentially coretractable.

Proof: Since $D\leq^{\oplus}\mathcal{M}$, so $\mathcal{M}$=D⊕W for some W≤ $\mathcal{M}$. Let K< D, hence K⊕W≤ $\mathcal{M}$ and so ∃0≠ f:$\frac{\mathcal{M}}{K\oplus W}\to\mathcal{M}$ with Im$f\leq_{tes}\mathcal{M}$. Now since $\frac{\mathcal{M}}{K\oplus W}\cong\frac{D}{K}$ and $\mathcal{M}$ is compressible module, so ∃g: $\mathcal{M}\to$D, g is monomorphism. Then g∘f: $\frac{D}{K}\to$D and g∘f($\frac{D}{K}$)= g(f($\frac{D}{K}$)). But Im$f\leq_{tes}\mathcal{M}$ and g is monomorphism, so g(f($\frac{D}{K}$)) $\leq_{tes}$D by (14, proposition(1.1.23)). Also g∘f≠0, because if g∘f=0 that is g∘f($\frac{D}{K}$)=0 so f($\frac{D}{K}$)=0 which is contradiction.

Proposition 7: Let Y be a y-closed submodule of t-essentially coretractable module $\mathcal{M}$. Then $\frac{\mathcal{M}}{Y}$ is t-essentially coretractable module.

Proof: Let W/Y < $\mathcal{M}$/Y, hence W<M and so ∃ f:$\mathcal{M}\to\mathcal{M}$, f(W)=0 and Im$f\leq_{tes}\mathcal{M}$. Define g: $\frac{\mathcal{M}}{Y}\to\frac{\mathcal{M}}{Y}$ by g(m+Y)= f (m)+Y, g is well defined and homomorphism with g($\frac{W}{Y}$)=0. Img= $\frac{Imf+Y}{Y}$, but Im$f\leq_{tes}$ M which implies Im f +Y$\leq_{tes}\mathcal{M}$ so Im f + Y+$Z_2(\mathcal{M})\leq_{ess}\mathcal{M}$ by (5, Proposition1.1). Beside this Y is closed submodule of $\mathcal{M}$ (since it is y-closed) and hence $\frac{Imf+Y+Z_2(\mathcal{M})}{Y}\leq_{ess}\frac{\mathcal{M}}{Y}$ by (4). This implies that $\frac{Imf+Y}{Y}+\frac{Y+Z_2(\mathcal{M})}{Y}\leq_{ess}\frac{\mathcal{M}}{Y}$, and so that $\frac{Imf+Y}{Y}+Z_2(\frac{\mathcal{M}}{Y})\leq_{ess}\frac{\mathcal{M}}{Y}$. Thus Img$\leq_{tes}\frac{\mathcal{M}}{Y}$ by (5, Proposition(1.1)).

Corollary 1: Let f: $\mathcal{M}\to\mathcal{M}'$ be an epimorphism and ker$f$ be y-closed submodule of $\mathcal{M}$. Then $\mathcal{M}'$ is t-essentially coretractable module whenever $\mathcal{M}$ is t-essentially coretractable.

Proof: By the 1st fundamental Theorem $\frac{\mathcal{M}}{kerf}\cong\mathcal{M}'$. But $\frac{\mathcal{M}}{kerf}$ is t-essentially coretractable module by Proposition7. Thus $\mathcal{M}'$ is t-essentially coretractable module.

Corollary 2: Let $\mathcal{M}$ be a t-essentially coretractable module. Then $\frac{\mathcal{M}}{Z_2(\mathcal{M})}$ is t-essentially coretractable.

Proof: Since $Z_2(\mathcal{M})$ is y-closed, then the required condition hold by Proposition(7).

Corollary 3: Let $\mathcal{M}$=D⊕W be a t-essentially coretractable module such that $Z_2(\mathcal{M})\subseteq$W. Then D is t-essentially coretractable module.

Proof: Let $\mathcal{M}$=D⊕W . Since W is direct summand, so W is closed, but $Z_2(\mathcal{M})\subseteq$W by hypothesis and hence W is y-closed submodule. Hence by Proposition(7), $\frac{\mathcal{M}}{W}$ is t-essentially coretractable module but $\frac{\mathcal{M}}{W}\cong D$. Thus D is t-essentially coretractable module.

Note that" a finite direct sum of coretractable modules is coretractable module", see(2, Proposition(2.6)).

Remark 2: The direct sum of t-essentially coretractable modules need not be t-essentially coretractable module, for example: Let $\mathcal{M}$=$Z_2\oplus Z_2$ as Z_2-module. $\mathcal{M}$ is not t-essentially coretractable but Z_2 is t-essentially coretractable Z_2-module.

Recall that" a module $\mathcal{M}$ is called duo if every submodule of $\mathcal{M}$ is fully invariant, where a submodule N of $\mathcal{M}$ is called fully invariant if for each $f\in$ End($\mathcal{M}$), f(N)⊆N"(15). "A submodule N of an R-module $\mathcal{M}$ is called stable if for each homomorphism f:N →$\mathcal{M}$, f(N) ≤ N"(16).

Proposition 8: Let $\mathcal{M}=\mathcal{M}_1\oplus\mathcal{M}_2$ be a duo module. Then $\mathcal{M}$ is t-essentially coretractable if and only if $\mathcal{M}_1$ and $\mathcal{M}_2$ are t-essentially coretractable modules, provided that for each $K<\mathcal{M}$, $K\cap\mathcal{M}_1<\mathcal{M}_1$ and $K\cap\mathcal{M}_2<\mathcal{M}_2$.

Proof: (⇐) Let K< $\mathcal{M}$. Since $\mathcal{M}$ is duo module, $K=(K\cap\mathcal{M}_1)\oplus(K\cap\mathcal{M}_2)$ (15). Put $K_1=K\cap\mathcal{M}_1$ and $K_2=K\cap\mathcal{M}_2$. Hence by hypothesis $K_1<\mathcal{M}_1$ and $K_2<\mathcal{M}_2$. As $\mathcal{M}_1$ and $\mathcal{M}_2$ are t-essentially coretractable modules, so ∃ f: $\frac{\mathcal{M}_1}{K_1}\to\mathcal{M}_1$ and g:$\frac{\mathcal{M}_2}{K_2}\to\mathcal{M}_2$ with Imf and Img are t-essential submodules of $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively. Define h: $(\frac{\mathcal{M}_1}{K_1}\oplus\frac{\mathcal{M}_2}{K_2})\to\mathcal{M}$ by h(x+K_1, y+K_2)=(f(x+K_1),

g(y+K_2)), h is a homomorphism. Imh= Imf⊕Img$\leq_{tes}$ $\mathcal{M}_1 \oplus \mathcal{M}_2$= $\mathcal{M}$, but $\frac{\mathcal{M}}{K} \cong (\frac{\mathcal{M}_1}{K_1} \oplus \frac{\mathcal{M}_2}{K_2})$ see Kasch hence there exists an isomorphism α, where α: $\frac{\mathcal{M}}{K} \longrightarrow \frac{\mathcal{M}_1}{K_1} \oplus \frac{\mathcal{M}_2}{K_2}$ and so α∘h: $\frac{\mathcal{M}}{K} \longrightarrow \mathcal{M}$. As Im(α∘h)= α(Imh), Imh$\leq_{tes}$ $\mathcal{M}$ and α is isomorphism so that α(Imh) $\leq_{tes}\mathcal{M}$ that is Im(α∘h) $\leq_{tes}\mathcal{M}= \mathcal{M}_1\mathcal{M}_2$. Thus $\mathcal{M}$ is t-essentially coretractable.

(⇒) To prove $\mathcal{M}_1$ is t-essentially coretractable module. Let $K< \mathcal{M}_1$.Then K⊕ $\mathcal{M}_2< \mathcal{M}$ and since $\mathcal{M}$ is t-essentially coretractable, ∃0≠f∈End($\mathcal{M}$) and f(K⊕ $\mathcal{M}_2$)=0 and Im$f$$\leq_{tes}\mathcal{M}$. Now f(K⊕ $\mathcal{M}_2$)=0 implies to f(K)=0 and f($\mathcal{M}_2$)=0. Let g=$f|_{\mathcal{M}_1}$: $\mathcal{M}_1 \rightarrow \mathcal{M}$. Since $\mathcal{M}_1$ is a fully invariant direct summand of $\mathcal{M}$, $\mathcal{M}_1$ is stable; that is g($\mathcal{M}_1$) ⊆ $\mathcal{M}_1$ and hence f($\mathcal{M}_1$)⊆ $\mathcal{M}_1$. It follows that g($\mathcal{M}_1$)=f($\mathcal{M}_1$)=f($\mathcal{M}_1$⊕ $\mathcal{M}_2$)=f($\mathcal{M}$)$\leq_{tes}\mathcal{M}= \mathcal{M}_1 \oplus \mathcal{M}_2$. Thus g($\mathcal{M}_1$) ⊕(0) $\leq_{tes}\mathcal{M}_1 \oplus \mathcal{M}_2$ which implies to g($\mathcal{M}_1$)$\leq_{tes}\mathcal{M}_1$. Also g(K)=f(K)=0. Thus $\mathcal{M}_1$ is t-essentially coretractable module.

Recall that" an $\mathcal{R}$-module $\mathcal{M}$ is called a multiplication module if for each submodule N of $\mathcal{M}$, there exists a right ideal in R such that $\mathcal{M}$I=N"(17).

Proposition 9: Let $\mathcal{M}$be a finitely generated faithful multiplication module. Then $\mathcal{M}$ is t-essentially coretractable if and only if $\mathcal{R}$ is t-essentially coretractable ring, where $\mathcal{R}$ is a commutative ring.

Proof: (⇒) Let I < $\mathcal{R}$. Then N= $\mathcal{M}I < \mathcal{M}$ since $\mathcal{M}$ is a finitely generated faithful multiplication module. As $\mathcal{M}$ is t-essentially coretractable, ∃ f∈End($\mathcal{M}$), f(N)=0 and Im$f \leq_{tes} \mathcal{M}$. But $\mathcal{M}$is finitely generated multiplication module, so ∃0 ≠r ∈ $\mathcal{R}$ such that f(m)=mr ∀m∈ $\mathcal{M}$. Define g: $\mathcal{R} \rightarrow \mathcal{R}$ by g(a)=ar ∀a∈ $\mathcal{R}$. Clearly g is an $\mathcal{R}$-homomorphism and g(I)=Ir and g($\mathcal{R}$)=$\mathcal{R}r$. Since f(N)=f($\mathcal{M}$I)=f($\mathcal{M}$)I=$\mathcal{M}$rI=$\mathcal{M}$Ir=0, then Ir⊆ann$\mathcal{M}$=0 and so Ir=0, that is g(I)=0. Also Imf=$\mathcal{M}$<r>$\leq_{tes}\mathcal{M}$ implies that <r>$\leq_{tes}$ $\mathcal{R}$ by (14, Lemma(1.1.24)). Thus g($\mathcal{R}$) $\leq_{tes}\mathcal{R}$. Hence $\mathcal{R}$ is t-essentially coretractable.

(⇐) Let N<$\mathcal{M}$. Since $\mathcal{M}$ is a multiplication module, then N=$\mathcal{M}$I for some I≤$\mathcal{R}$. But $\mathcal{M}$ is a finitely generated faithful multiplication module, so I<$\mathcal{R}$. As $\mathcal{R}$ is t-essentially coretractable, so by Rem. & Exa. (1(9)), ∃r∈ $\mathcal{R}$, r≠0 such that r∈annI and <r>$\leq_{tes}\mathcal{R}$. Now, define f: $\mathcal{M} \rightarrow \mathcal{M}$ by f(m)=mr ∀m∈ $\mathcal{M}$. It is clear that f is an $\mathcal{R}$-homomorphism, f(N)=f($\mathcal{M}I$)=f($\mathcal{M}$)I=$\mathcal{M}$rI=$\mathcal{M}$Ir=0 and f($\mathcal{M}$)= $\mathcal{M}r$= $\mathcal{M}$<r> since <r>$\leq_{tes}$ $\mathcal{R}$, then by (14, Lemma(1.1.24)), $\mathcal{M}r \leq_{tes}\mathcal{M}$; that is f($\mathcal{M}$) $\leq_{tes}\mathcal{M}$. Thus $\mathcal{M}$ is t-essentially coretractable.

Corollary 4: Let $\mathcal{M}$ be a finitely generated faithful multiplication module over a commutative ring $\mathcal{R}$. Then the following are equivalent:

(1) $\mathcal{M}$ is t-essentially coretractable module;
(2) $\mathcal{R}$ is t-essentially coretractable ring;
(3) $End(\mathcal{M})$ is t-essentially coretractable ring.

Proof: (1)⟺(2) It holds by proposition(9).

(2)⟺(3) Since $\mathcal{M}$ is finitely generated faithful multiplication module, ∀ f∈End($\mathcal{M}$) ∃0 ≠r ∈ $\mathcal{R}$ and f(m)=mr ∀m ∈ $\mathcal{M}$. Define φ: $\mathcal{R} \rightarrow$ End($\mathcal{M}$) by φ(r)=f if f(m)=mr ∀m∈ $\mathcal{M}$. φ is well-defined and epimorphism, but ker$f$=ann$\mathcal{M}$=0. Thus End($\mathcal{M}$)≅ $\mathcal{R}$/ann$\mathcal{M}$ ≅ $\mathcal{R}$, therefore End($\mathcal{M}$) is t-essentially coretractable.

Weakly t-Essentially Coretractable Modules

Definition 2: A module $\mathcal{M}$ is called weakly t-essentially coretractable if ∀ W< $\mathcal{M}$, ∃ 0 ≠f: $\mathcal{M}$/W→$\mathcal{M}$ such that f($\mathcal{M}$ /W)+W $\leq_{tes}$ $\mathcal{M}$. A ring $\mathcal{R}$ is called weakly t-essentially coretractable if $\mathcal{R}$ is a weakly t-essentially coretractable $\mathcal{R}$-module.

Examples and Remarks 3:

(1) A module $\mathcal{M}$ is weakly t-essentially coretractable if and only if ∀ W< $\mathcal{M}$, ∃ f∈End($\mathcal{M}$), f(W)=0 and Imf+W$\leq_{tes}\mathcal{M}$.

Proof: Clear.

(2) It is clear that any essentially coretractable module is weakly t-essentially coretractable.
The converse of this part is not true, consider the following: Z_{12} as Z-module is weakly t-essentially coretractable and not essentially coretractable see (6).

(3) Every weakly t-essentially coretractable module is coretractable module.

(4) By Rem. & Exa. (1(3)), the semisimple and t-essentially coretractable modules are independent. However a semisimple module is weakly t-essentially coretractable.

Proof: Let W<$\mathcal{M}$. $\mathcal{M}$is semisimple module, so ∃ D$\leq^{\oplus}$ $\mathcal{M}$, W⊕D=$\mathcal{M}$ hence $\mathcal{M}$/W≅D. Let f=i∘g, where i is the inclusion map from D into M and g is an isomorphism between $\mathcal{M}$/W and D. It is clear that f≠0 and Im f +W$\leq_{tes}\mathcal{M}$. Therefore $\mathcal{M}$is weakly t-essentially coretractable module.

By applying (4), consider $Z_2 \oplus Z_2$ as Z_2-module is weakly t-essentially coretractable, but not t-essentially coretractable module. Also the converse may not be true, for example see Z_{12} as Z-module is weakly t-essentially coretractable but not semisimple module.

(5) It is clear that any module over semisimple ring $\mathcal{R}$ is semisimple and hence it is weakly t-essentially coretractable module by Rem. & Exa. (3(4)).

(6) A module $\mathcal{M}$ is weakly t-essentially coretractable $\mathcal{R}$-module if and only if $\mathcal{M}$ is weakly t-essentially coretractable $\bar{\mathcal{R}}$-module (where, $\bar{\mathcal{R}}=\mathcal{R}/\mathrm{ann}\mathcal{M}$).

(7) Let M be a nonsingular R-module. Then $\mathcal{M}$ is weakly essentially coretractable if and only if weakly t-essentially coretractable.

Proof: It follows directly since essential and t-essential concepts are coincide in nonsingular modules(14, corollary(1.1.19)).

(8) Every strongly coretractable (respectively, epi-coretractable) module is weakly t-essentially coretractable, since for each $W<\mathcal{M}$, $\exists f\in \mathrm{End}(\mathcal{M})$ with $\mathrm{Im}f+W=\mathcal{M}$ and so $\mathrm{Im}f+W\leq_{tes}\mathcal{M}$.
One can see that Z_4 as Z-module does not satisfy the converse.

(9) A ring $\mathcal{R}$ is weakly t-essentially coretractable if and only if for each proper ideal I of $\mathcal{R}$, there exists $r\in\mathcal{R}$, $r\neq 0$ such that $r\in \mathrm{ann}I$ and $<r>+I\leq_{tes}\mathcal{R}$.

Proposition 10: Let $\mathcal{M}$ be a nonsingular uniform module. Then the following statements are equivalent:

(1) $\mathcal{M}$ is essentially coretractable;
(2) $\mathcal{M}$ is t-essentially coretractable;
(3) $\mathcal{M}$ is weakly t-essentially coretractable;
(4) $\mathcal{M}$ is weakly essentially coretractable.

Proof: (1)⟺(2) It follows by Proposition (1).

(2)⇒(3) Let $W<\mathcal{M}$. Since $\mathcal{M}$ is t-essentially coretractable, so $\exists f\in\mathrm{End}(\mathcal{M})$, $f(W)=0$, $\mathrm{Im}f\leq_{tes}\mathcal{M}$ and hence $\mathrm{Im}f+W\leq_{tes}\mathcal{M}$. Thus $\mathcal{M}$ is weakly t-essentially coretractable.

(3)⟺(4) Let $K<\mathcal{M}$, since $\mathcal{M}$ is weakly t-essentially coretractable, so $\exists f:\mathcal{M}/K\to\mathcal{M}$ and $\mathrm{Im}f+K\leq_{tes}\mathcal{M}$. But $\mathcal{M}$ is nonsingular hence $\mathrm{Im}f+K\leq_{ess}\mathcal{M}$, therefore $\mathcal{M}$ is weakly essentially coretractable

(4)⟺(1) Since $\mathcal{M}$ is nonsingular uniform, then $\mathcal{M}$ is quasi-Dedekind by (18, Proposition(1.5)), and hence (4)⟺(1) by using (6, Proposition(3.9)).

Proposition 11: Let $\mathcal{M}$ be Z_2-torsion R-module. Then the following are equivalent:

(1) $\mathcal{M}$ is coretractable;
(2) $\mathcal{M}$ is t-essentially coretractable;
(3) $\mathcal{M}$ is weakly t-essentially coretractable

Proof: (1) ⟺ (2) It follows by proposition(4).
(2) ⇒ (3) It is clear.
(3) ⇒(1) It follows by Rem. & Exa. (3(3)).

Proposition 12: Let $\mathcal{M}$ be a weakly t-essentially coretractable module and D be a fully invariant direct summand of $\mathcal{M}$, then D is weakly t-essentially coretractable module.

Proof: Since $D\leq^{\oplus}\mathcal{M}$, so $\mathcal{M}=D\oplus W$ for some $W\leq\mathcal{M}$. Let K<D, hence $K\oplus W<D\oplus W=\mathcal{M}$ since $\mathcal{M}$ is weakly t-essentially coretractable, then $\exists f\in \mathrm{End}_R(\mathcal{M})$, $f(K\oplus W)=0$ and $f(\mathcal{M})+(K\oplus W)\leq_{tes}\mathcal{M}$. Let $g=f|_D:D\longrightarrow M$. Since $D\leq^{\oplus}\mathcal{M}$, D is fully invariant, so D is stable in $\mathcal{M}$ by (19, Lemma(2.1.6)), hence $g\in\mathrm{End}(D)$.

Claim $g(D)+K\leq_{tes}D$. Since $f(K+W)=0$, so $f(W)=0$ and $f(K)=0$. Thus $g(K)=0$. $f(M)=f(D\oplus W)=f(D)\subseteq D$, but $f(\mathcal{M})+(K\oplus W)\leq_{tes}\mathcal{M}$ implies $f(D)+(K\oplus W)\leq_{tes}\mathcal{M}$. Hence $(f(D)+K)\oplus W\leq_{tes}M=D\oplus W$, so that $f(D)+K\leq_{tes}D$ (11). It follows that $g(D)+K\leq_{tes}D$ and so D is weakly t-essentially coretractable module.

Corollary 5: Let $\mathcal{M}=\mathcal{M}_1\oplus\mathcal{M}_2$ and $\mathcal{M}$ be duo module. Then $\mathcal{M}$ is weakly t-essentially coretractable if and only if $\mathcal{M}_1$ and $\mathcal{M}_2$ are weakly t-essentially coretractable.

Proof: Since $\mathcal{M}$ is duo, $\mathcal{M}_1$ and $\mathcal{M}_2$ are fully invariant submodules, but $\mathcal{M}_1$ and $\mathcal{M}_2$ are direct summand, then by Proposition(12), $\mathcal{M}_1$ and $\mathcal{M}_2$ are weakly t-essentially coretractable modules.

Corollary 6: Let $\mathcal{M}=\mathcal{M}_1\oplus\mathcal{M}_2$ with $\mathrm{ann}\mathcal{M}_1+\mathrm{ann}\mathcal{M}_2=\mathcal{R}$. If $\mathcal{M}$ is weakly t-essentially coretractable module, then $\mathcal{M}_1$ and $\mathcal{M}_2$ are weakly t-essentially coretractable.

Proof: Since $\mathrm{ann}\mathcal{M}_1+\mathrm{ann}\mathcal{M}_2=R$, then $\mathcal{M}_1=\mathcal{M}_1\mathrm{ann}\mathcal{M}_2$ and $\mathcal{M}_2=\mathcal{M}_2\mathrm{ann}\mathcal{M}_1$. Hence for each $f:\mathcal{M}\longrightarrow\mathcal{M}$, $f(\mathcal{M}_1)=f(\mathcal{M}_1)\mathrm{ann}\mathcal{M}_2\subseteq\mathcal{M}\mathrm{ann}\mathcal{M}_2=\mathcal{M}_1\mathrm{ann}\mathcal{M}_2=\mathcal{M}_1$ hence $\mathcal{M}_1$ is a fully invariant. Similarity $\mathcal{M}_2$ is a fully invariant. But $\mathcal{M}_1$ and $\mathcal{M}_2$ are direct summands. Thus they are weakly t-essentially coretractable modules by Proposition (12).

Recall that " an $\mathcal{R}$-module $\mathcal{M}$ is called a polyform if for any submodule $L\subseteq\mathcal{M}$ and for any $0\neq\varphi:L\to\mathcal{M}$, Ker φ is not essential in L. Equivalently, if for any submodule $L\subseteq\mathcal{M}$ and for any $\varphi: L\to\mathcal{M}$, $\mathrm{Ker}\varphi\leq_e L$ implies $\varphi=0$ "(9).

Proposition 13: Let $\mathcal{M}$ be a polyform module. If $\bar{\mathcal{M}}$ (where $\bar{\mathcal{M}}$ is the quasi-injective hull of $\mathcal{M}$) is weakly t-essentially coretractable module, then $\mathcal{M}$ is weakly t-essentially coretractable

Proof: Since $\mathcal{M}$ is polyform, so $\mathrm{End}(\bar{\mathcal{M}})$ is regular ring by (9, Theorem(2.1)). But $\bar{\mathcal{M}}$ is coretractable hence by (3, Proposition(2.1)), $\bar{\mathcal{M}}$ is semisimple and hence $\mathcal{M}$ is semisimple. Thus $\mathcal{M}$ is weakly t-essentially coretractable module.

It is known every nonsingular module is polyform, so one can get the following directly:

Corollary 7: Let $\mathcal{M}$ be a nonsingular module. If $\bar{\mathcal{M}}$ is weakly t-essentially coretractable module, then $\mathcal{M}$ is weakly t-essentially coretractable.

Proposition 14: Let $\mathcal{M}$ be a nonsingular $\mathcal{R}$-module. Then the following statements are equivalent:

(1) $\mathcal{M}$ is coretractable;
(2) $\mathcal{M}$ is semisimple;
(3) $\mathcal{M}$ is weakly t-essentially coretractable;
(4) $\mathcal{M}$ is weakly essentially coretractable.

Proof: (1) ⇒ (2) By (12, proposition 1.3).
(2) ⇒ (3) By Rem. & Exa. (3(3)).
(3) ⇒(4) It follows by Rem. & Exa. (3(7)).
(4) ⇒(1) It is clear.

Proposition 15: Let $\mathcal{M}$be a finitely generated faithful multiplication module. Then $\mathcal{M}$ is weakly t-essentially coretractable if and only if $\mathcal{R}$ is weakly t-essentially coretractable ring, where $\mathcal{R}$ is a commutative ring.

Proof: It is similar to the proof of proposition(9).

Corollary 8: Let $\mathcal{M}$be a finitely generated faithful multiplication module over a commutative ring $\mathcal{R}$. Then the following are equivalent:

(1) $\mathcal{M}$ is weakly t-essentially coretractable module;
(2) $\mathcal{R}$ is weakly t-essentially coretractable ring;
(3) $End(\mathcal{M})$ is weakly t-essentially coretractable ring.

Proposition 16: Let $\mathcal{M}$ be a weakly t-essentially coretractable and t-semisimple $\mathcal{R}$-module such that N⊆ $Z_2(\mathcal{M})$, ∀N< $\mathcal{M}$. Then $\mathcal{M}$is t-essentially coretractable.

Proof: Let N< $\mathcal{M}$. Since $\mathcal{M}$ is weakly t-essentially coretractable $\mathcal{R}$-module, ∃ f∈End($\mathcal{M}$), f(N)=0 and Im$f+N\leq_{tes}\mathcal{M}$. Then by (5, proposition(1.1)), Im$f+N+Z_2(\mathcal{M})\leq_{ess}\mathcal{M}$. Since N⊆$Z_2(\mathcal{M})$, then Im$f+Z_2(\mathcal{M})\leq_{ess}\mathcal{M}$. Again by (5,proposition(1.1)), Im$f\leq_{tes}\mathcal{M}$. Thus $\mathcal{M}$ is t-essentially coretractable module

Recall that "a ring $\mathcal{R}$ is called completely coretractable ring (briefly, CC-ring) if every $\mathcal{R}$-module is coretractable"(2). See the following:

Definition 3: A ring $\mathcal{R}$ is called completely weakly t-essentially coretractable ring if every $\mathcal{R}$-module is weakly t-essentially coretractable.

Remarks and Examples 4:

(1) It clear that, every completely weakly t-essentially coretractable ring is CC-ring.
(2) By Rem. & Exa. (3(5)), every semisimple ring is completely weakly t-essentially coretractable ring.
(3) Recall that" a ring $\mathcal{R}$ is called Kasch if every simple $\mathcal{R}$-module can be embedded in $\mathcal{R}$"(2). If $\mathcal{R}$ is Kasch ring and J($\mathcal{R}$)=0, then $\mathcal{R}$ is semisimple (2) and hence by Rem. & Exa.(4(2)), $\mathcal{R}$ is completely weakly t-essentially coretractable ring.
(4) If $\mathcal{R}$ is Kasch ring and regular (in the sense of Von Neumann), then $\mathcal{R}$ is semisimple by (2) and hence by Rem. & Exa.(4(2)), $\mathcal{R}$ is completely weakly t-essentially coretractable ring.
(5) Since a dual Rickart ring is semisimple (2), then every dual Rickart ring is completely weakly t-essentially coretractable ring by Rem. & Exa. (4(2)).

Proposition 17: Let $\mathcal{R}$ be a commutative ring such that $\bar{\mathcal{R}}=\mathcal{R}/\text{ann}\mathcal{M}$ is a weakly t-essentially coretractable $\mathcal{R}$-module. Then every cyclic $\mathcal{R}$-module is weakly t-essentially coretractable

Proof: Let $\mathcal{M}$be a cyclic $\mathcal{R}$-module. Then $\mathcal{M}\cong\bar{\mathcal{R}}$ is weakly t-essentially coretractable $\mathcal{R}$-module.

Corollary 9: Let $\mathcal{R}$ be a commutative ring. Then the following are equivalent:

(1) $\mathcal{R}$ is completely weakly t-essentially coretractable ;
(2) Every cyclic $\mathcal{R}$-module is weakly t-essentially coretractable
(3) $\mathcal{R}$/ann $\mathcal{M}$ is weakly t-essentially coretractable $\mathcal{R}$-module.

Proposition 18: Let $\mathcal{R}$ be a ring with J($\mathcal{R}$)=0. Then the following are equivalent:

(1) $\mathcal{R}$ is weakly t-essentially coretractable ring;
(2) $\mathcal{R}$ is completely weakly t-essentially coretractable ring;
(3) $\mathcal{R}$ is weakly t-essentially coretractable ring;
(4) All free $\mathcal{R}$-module is weakly t-essentially coretractable;
(5) All finitely generated free $\mathcal{R}$-module are weakly t-essentially coretractable.

Proof: (1) ⇒ (3) Since $\mathcal{R}$ is weakly t-essentially coretractable ring, $\mathcal{R}$ is coretractable and hence by (2, Lemma(3.7)), $\mathcal{R}$ is semisimple.
(3) ⇒ (2) and (3) ⇒ (4) are clear by Rem. & Exa.(4(2)).
(4) ⇒ (5) It is clear.
(5) ⇒ (1) It is clear, since $\mathcal{R}$ is a finitely generated free $\mathcal{R}$-module.

Remark 5: Since every commutative regular ring $\mathcal{R}$ (in the sense of Von Neumann) satisfies J($\mathcal{R}$)=0, then proposition(18) is hold for commutative regular rings.

Corollary 11: If $\mathcal{R}$ is a commutative regular (in the sense of Von Neumann) weakly t-essentially coretractable ring, then $\mathcal{R}$ is a principal ideal ring (PIR).

Proof: By previous proposition, $\mathcal{R}$ is semisimple. Hence every ideal generated by idempotent element. Thus $\mathcal{R}$ is a principal ideal ring.

Conclusion:

In this paper, the notions of t-essentially and weakly t-essentially coretractable modules are defined as a generalization of essentially and weakly essentially coretractable module. Also, several results are given. Further the completely weakly t-essentially coretractable rings are defined and investigated.

Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Baghdad.

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المقاسات المنكمشة المضادة الجوهرية والجوهرية الضعيفة من النوع-T

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الخلاصة:

نقدم في هذا البحث اعمامات جديدة للمقاسات المنكمشة المضادة حيث اطلقنا على المقاس $\mathcal{M}$ اسم منكمش مضاد جوهري من النوع-T أو منكمش مضاد جوهري ضعيف من النوع-T إذا كان لكل مقاس جزئي K من المقاس $\mathcal{M}$ ، يوجد $f \in \text{End}(\mathcal{M})$ و $f(K)=0$ بحيث ان $\text{Im}f \leq_{tes} \mathcal{M}$ او $\text{Im}f + K \leq_{tes} \mathcal{M}$. تم دراسة بعض الخصائص الأساسية و عرض العديد من العلاقات بين هذه المقاسات ومقاسات اخرى ذات الصلة.

الكلمات المفتاحية: وحدة قابلة للتجديد ، وحدة قابلة للتجزئة بشكل أساسي، وحدة قابلة للتجديد بشكل أساسي، وحدة قابلة للتجديد بشكل ضعيف، وحدة قابلة للتجميع بشكل أساسي.