Exponential Function of a bounded Linear Operator on a Hilbert Space.

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Received 28, May, 2013
Accepted 18, September, 2013

Abstract:

In this paper, we introduce an exponential of an operator defined on a Hilbert space H, and we study its properties and find some of properties of T inherited to exponential operator, so we study the spectrum of exponential operator e^T according to the operator T.

Key words: Self-adjoint operator, positive operator, normal operator, quasinormal operator, binormal operator hyponormal operator and compact operator.

Introduction:

Let B(H) be a space of all bounded linear operator on a Hilbert space H (real or complex).

We introduced a new bounded linear operator defined on H, as a limit of sequence or power series of linear operator T. Giaquinta; Modica in [1] gave a definition of an exponential operator $e^T$of a bounded linear operator T as the sum of power series of T, and it started the properties of exponential operator of bounded linear operator T. In this paper we study the inherited properties of T into the operator e^T, and the spectrum of exponential operator e^T according to the operator T. Such properties of T can be found in [2], [3], [4], [5], [6], [7] and [8].

Preminilaries:

Definition:

Let $T \in B(H)$ then $e^T: H \to H$ defines as $e^T x = \sum_{n=0}^{\infty} \frac{1}{n!} T^n x$. So, we write $e^T = \sum_{n=0}^{\infty} \frac{1}{n!} T^n$.

We need to check the definition of exponential operator is well-define, i.e. The power series is convergent for each x∈ H, by following proposition in [1] .

Proposition:

Let H be a Hilbert space and $T \in B(H)$.

1. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence R > 0 and $\|T\| \leq R$. Then the series $\sum_{n=0}^{\infty} a_n T^n$ convergence in B(H) and define a linear continuous operator.
2. The series $\sum_{k=0}^{\infty} \frac{1}{k!} T^k$ converges in B(H) and define the linear continuous operator $e^T = \sum_{k=0}^{\infty} \frac{1}{k!} T^k$.

Examples:

1 . $e^0 = I$, Where 0 is a zero operator and I is an identity operator defined on H.

2. $e^I = \sum_{n=0}^{\infty} \frac{1}{n!} I^n = \sum_{n=0}^{\infty} \frac{1}{n!} I = eI$.

3. If T is a nilpotent of degree $n \in N$, i.e. $T^n = 0$ in [2], then $e^T = \sum_{k=0}^{n-1} \frac{1}{k!} T^k$,

$$e^T = \sum_{n=0}^{\infty} \frac{1}{n!} T^n = I + T + \frac{1}{2!} T^2 + \cdots + \frac{1}{(n-1)!} T^{n-1}.$$

This paper consists of three sections. In section one we study some

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properties of an exponential operator on H. While, in section two we study some properties operator T on H inherited to operator e^T. In section three, we study the spectrum of exponential operator e^T according to the operator T.

1. Some properties of an Exponential operator on a Hilbert space H:

In [1] Mariano gave some properties of e^T without proof. In this section we present its proofs.

Proposition (1.1)

Let $T, S \in B(H)$ we have the following properties of e^T:

1. If T S= S T then $e^{T+S} = e^T e^S = e^S e^T$.
2. $e^T e^{-T} = I$, and hence the inverse of e^T is e^{-T}, i.e. $(e^T)^{-1} = e^{-T}$.
3. $e^{(\alpha+\beta)T} = e^{\alpha T} e^{\beta T}$, for any α, β scalar.
4. $\|e^T\| \leq e^{\|T\|}$.
5. $(e^T)^* = e^{T^*}$.

Proof:

1. By using of multiplication of absolutely convergent series we get :

$$e^T e^S = \sum_{n=o}^{\infty} \frac{1}{n!} T^n \sum_{m=0}^{\infty} \frac{1}{m!} S^m = \sum_{n=0}^{\infty} \sum_{k=o}^{n} \frac{1}{k!(n-k)!} T^k S^{n-k} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} T^k S^{n-k} = \sum_{n=0}^{\infty} \frac{1}{n!} (T+S)^n$$

$$= e^{T+S} .$$

2. $e^T e^{-T} = e^{T+(-T)} = e^0 = I$, by part (1) .
3. The result following by part one of this proposition .
4. We have $\left\|\sum_{k=0}^{n} T^k\right\| \leq \sum_{k=0}^{n} \left\|T^k\right\| \leq \sum_{k=0}^{n} \|T\|^k$.
And $\left\|\sum_{k=0}^{n} \frac{1}{k!} T^k\right\|$ converges to $\|e^T\|$ and $\sum_{k=0}^{n} \frac{1}{k!} \|T\|^k$ to $e^{\|T\|}$. So ,we have $\|e^T\| \leq e^{\|T\|}$.
5. $(e^T)^* = \left(\sum_{n=0}^{\infty} \frac{1}{n!} T^n \right)^* = \sum_{n=0}^{\infty} \frac{1}{n!} (T^n)^* = \sum_{n=0}^{\infty} \frac{1}{n!} (T^*)^n = e^{T^*}$.

There is another equivalent definition of an exponential operator of a bounded linear on a Hilbert space H, as a limit of sequence of some bounded operators [1].

Theorem (1.2)

Let T be a bounded linear operator defined on a Hilbert space H , then $\left(I + \frac{1}{n} T\right)^n \to e^T$.

The proof of this theorem can be found in [1]

2.Main Results:

In this section, we are going to give some properties of linear operators defined on a Hilbert space H, that inherited an exponential operator many them: self-ajoint, positive, normral, quasinormal , hyponormal and compact.

Lemma(2.1)[2]

1. If T is a self-adjoint operator. Then αT is a self-adjoint, for all real numberα.
2. If T,S are self-adjoint linear operators on H. Then T+S is a self-adjoint .
3. If T, S are self-adjoint linear operators on H. Then TS is a self - adjoinet if and only if $TS = ST$.
4. If T is a self-adjoinet operator. Then T^n is a self-adjoint , too for any positive integer n ≥ 2.
5. If (T_n) is a sequence of bounded self-adjoint linear operators on H, and T_n converges to a linear operator T . Then the operator T is also self-adjoint.

Proposition (2.2)

If T is a self-adjoint operator on a Hilbert space H, then so is e^T.

Proof:

If T is self-adjoint operator and n any positive integer, we have that lemma (2.1) parts (1), (2) and (4) $(I+\frac{1}{n}T)^n$ is a self-adjoint . But $(I+\frac{1}{n}T)^n$ convergent to e^T , then by lemma (2.1) part (5) e^T is self-adjoint .

Definition (2.3)[2]

Let $T \in B(H)$ be a self-adjoint operator, it is said a positive operator if $T \geq 0$, i.e. $\langle Tx, x \rangle \geq 0$, for all x in H.

Lemma (2.4)[2]

1. If T is a positive operator. Then αT is a positive, each non negative scalar α.
2. If T,S are positive linear operators. Then $T+S$ is positive.
3. If T,S are positive linear operators and $TS = ST$. Then TS is positive.
4. If T is a positive operator . Then T^n is positive , too for any positive integer $n \geq 2$.
5. The limit of a sequence of positive linear operators on H, is a positive operator.

Proposition (2.5)

If T is a positive operator on a Hilbert space H, then so is e^T

Proof :

If T is a positive operator and n any positive integer, then by lemma (2.4) parts (1),(2) and (4) we have $(I+\frac{1}{n}T)^n$ is a positive operator. But$(I+\frac{1}{n}T)^n$ converges to e^T, then by lemma (2.4) part (5) we have $e^T$is positive.

Remark (2.6)

If T is a skew-self-adjoint operator , i.e. $T^* = -T$ in [2] , then e^T may not be a skew, to see this, we have the following example:

Let T= 2iI be a linear operator on a complex Hilbert space H. We have $T^* = (2iI)^* = -2iI = -T$, hence T is a skew-self-adjoint operator . But $(e^T)^* = \left(e^{2iI}\right)^* = e^{(2iI)^*} = e^{-2iI}$, i.e. e^T is not a skew-self-adjoint.

Proposition (2.7)

If T is a normal operator on H , then e^T is also normal .

Proof :

T is a normal operator $TT^* = TT^*$ in this implies by (1.1) part (1), we have:

$e^Te^{T^*} = e^{T+T^*} = e^{T^*+T} = e^{T^*}e^T$,

hence $e^T$is normal.

Definition (2.8)[3]

Let T be a bounded linear operator on H. It is called a quasinormal if T commutes with T^*T, i.e. $T(T^*T) = (T^*T)T$

Lemma (2.9)

Let $T, S \in B(H)$ be quasinormal operators then :

1. αT is a quasinormal , α for any scalar.
2. T+S is a quasinormal with property that each commute with the adjoint of the other.
3. ST is a quasinormal if the following conditions are satisfied:
   (i) $ST = TS$ (ii) $ST^*=T^*S$
4. The limit of a sequence of quasinormal linear operators on H, is a quasinormal operator.

The proof of this lemma can be found in [2] ,[3]

Remark (2.10)

By using mathematical induction and lemma (2.9) part (3) , we have T^n is quasionormal operator on a Hilbert space H, for $n \geq 2$.

Proposition (2.11)

Let T be a quasinormal operator on H, then e^T is also quasinormal .

Proof :

If T is a quasinormal operator and n any positive integer, then by lemma (2.9) parts (1),(2) and (2.10) we have $(I+\frac{1}{n}T)^n$ is a quasinormal operator. But$(I+\frac{1}{n}T)^n$ converges to e^T , then by lemma (2.9) part (4), we have $e^T$is quasinormal.

Definition (2.12) [4]

An operator T on H is said a binormal if TT^* commutes with T^*T, i.e. $[TT^*, T^*T] = 0$

Remark (2.13)

If T is binormal operator on H. Then e^T may not be binormal , so we are going to

example to show this :

Let $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ can easily verify that T be binormal and $T^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, we have :

$$e^T = \sum_{n=0}^{\infty} \frac{1}{n!} T^n x = I + T$$

$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $e^{T^*} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, that's why

$e^T e^{T^*} e^{T^*} e^T = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$ and

$e^{T^*} e^T e^T e^{T^*} = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$.

Which are not equal, therefore e^T is not binormal operator.

Definition (2.14)[4]

An operator T on a Hilbert space H. It is said a hyponormal if $T^*T - T^*T \geq 0$, i.e. $\langle (T^*T - T^*T)\, x, x \rangle \geq 0$, for every $x \in H$.

Lemma (2.15)

Let T,S be hyponormal operators on H, then:

1. αT is a hyponormal, for each $\alpha \in \mathbb{C}$
2. If T,S are hyponormal operators with the property either commute with adjoint of the other. Then T+S is hyponormal.
3. If $T_n: H \to H$ (n= 1,2,…) is a sequence of hyponormal operator and $T_n \to T$ then T hyponormal.

The proof of this lemma can be found in [2] **.**

Remark (2.16)

In [5], P.R Halmos gave example of a hyponormal operator T such that T^2 is not hyponormal implies that T^n may not be a hyponormal for some $n \geq 2$.

proposition (2.17)

If T is a hyponormal and a binormal operator, then T^n is a hyponormal for $n \geq 1$.

We can find the proof in [4].

We are going to proof that if T is hyponormal and binormal then e^T is hyponormal.

Proposition (2.18)

If T is a hyponormal and a binormal operator then e^T is hyponormal .

Proof:

If T is hyponormal and binormal operator and n any positive integer, we have that lemma (2.15) parts (1),(2) and proposition (2.17) $(I + \frac{1}{n}T)^n$ is a hyponormal . But $(I + \frac{1}{n}T)^n$ convergent to e^T by lemma(2.15) part (3) e^T is hyponormal.

Definition (2.19)[2]

An operator T on a Hilbert space H, is said to be compact if for each bounded sequence (x_n) in H, the sequence (Tx_n) contains and convergent subsequence **.**

Lemma (2.20)[2]

1. If T, S , U $\in B(H)$ are compact operators on H, and $\alpha \in \mathbb{C}$, then αT , T+S and U T, T U are compact operators.
2. If T is a compact operator on H, then T^n is a compact for any positive integer $n \geq 2$.
3. If (T_n) is a sequence of compact linear operators on H. Suppose that T_n converges to linear operator T , then the operator T is compact .

Theorem (2.21)

If $T \in B(H)$ is a compact operator. Then:

1. e^T is compact if H is finite dimension.
2. $e^T - I$ is compact if H is infinite dimension.

Proof :

1. I is a compact operator since H is finite dimensional Hilbert space in[2] ,

hence $S_n = \sum_{k=0}^{n} \frac{1}{k!} T^k$ is compact operator by lemma (2.20) parts (1) and (2), Therefore S_n convergent to the compact operator by (2.20) part (3), i.e. e^T is compact.

2. I is not a compact operator, if H is infinite dimensional Hilbert space[2] . But $S_n^* = \sum_{k=1}^{n} \frac{1}{k!} T^k$ is compact operator if T is compact by lemma (2.20) parts (1)and (2), therefore $e^T - I$ is compact .

Remark (2.22)

1. If T is a compact operator on infinite dimensional Hilbert space H. Then e^T is not necessary compact, to see this, the T = 0 (zero operator) is a compact, where $e^T = e^0 = I$ which is not compact in [2].
2. If T is isometric operator on H , then $\|Tx\| = \|x\| \ \forall\, x \in H$. Then e^T may not be isometric ,to see this, we give the following example:
 If T = I , then $\|T\|$ =1, hence $\|e^T\| = \|e^I\| = e\|I\| = e$.
3. If T is a unitary operator on H, then $TT^* = T^*T = I$. therefore e^T may not be unitary to see this,
 we give the example:
 If $T = \left(\frac{\sqrt{3}}{2} - \frac{1}{2} i\right) I$ and $T^* = \left(\frac{\sqrt{3}}{2} + \frac{1}{2} i\right) I$, implies that $TT^* = T^*T = I$, i.e. T is unitary operator . We have $e^T = e^{\left(\frac{\sqrt{3}}{2} - \frac{1}{2} i\right) I} = e^{\left(\frac{\sqrt{3}}{2} - \frac{1}{2} i\right)} I$ and $e^{T^*} = e^{\left(\frac{\sqrt{3}}{2} + \frac{1}{2} i\right) I} = e^{\left(\frac{\sqrt{3}}{2} + \frac{1}{2} i\right)} I$. But $e^T e^{T^*} = e^{\sqrt{3}} I \neq I$.

3.The Spectrum of an exponential operator on a Hilbert space H:

The spectrum of a linear operator on a Hilbert space H, is a subset of the set of complex numbers λ, for which $T - \lambda I$ is not invertible , denoted by $\sigma(T)$. The complement of the spectrum of linear operator is resolvent, and it is denoted by $\rho(T)$.

Definition (3.1) [2]

Let T be a linear operator on a Hilbert space H .

1. The eigenvalue of T is a complex number λ, for which $T - \lambda I$ is not injective, i.e. There exists a non-zero vector *x* in H, such that $(T - \lambda I)(x) = 0$, the vector x is called eigenvector of T and the set of all eigenvalues of T denoted by $\sigma_P(T)$ is called the set of point spectrum of T.
2. The continuous spectrum of T, is a set of complex numbers λ, for which $T - \lambda I$ is injective and $T - \lambda I$ is not surjective, but the range of H by linear operator $T - \lambda I$ is dense in H. The continuous spectrum of T is denoted by $\sigma_c(T)$.
3. The residual spectrum of T, is the set of all complex numbers λ, for which $T - \lambda I$ is injective and the range of H dose not equal H. The residual spectrum of T denoted by $\sigma_{r(T)}$.
4. The spectral radius of linear operator T is denoted by r (T) and it is defined as follows :
 $r(T) = \sup\{ |\lambda| , \lambda \in \sigma(T) \} = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$

Proposition (3.2)

Let $T \in B(H)$ and λ be eigenvalue of T, then e^λ is eigenvalue of e^T.

Proof:

There exists a non zero vector x in H, such that $Tx = \lambda x$ (since λ is an eigenvalue of T), hence $T^n x = \lambda^n x$. But $e^T x = \sum_{n=0}^{\infty} \frac{1}{n!} T^n x = \sum_{n=0}^{\infty} \frac{\lambda^n x}{n!} = \left(\sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\right) x = e^\lambda x$.
Therefore e^λ is an eigenvalue of e^T and x is a corresponding eigenvector .

Remark (3.3)

In[2] E. Kreyszing, proved that, if H is finite dimensional Hilbert space. and $T \in B(H)$, then $\sigma(T) \neq \phi$. Furthermore $\lambda \in \sigma(T)$ if and only if λ is eigenvalue of T. Hence if H is a finite dimensional Hilbert space then $\lambda \in \sigma(e^T)$ if and only if λ is eigenvalue of e^T.

In the following example we are going to compute the spectrum of the some linear operators.

Examples (3.4)

1. $\sigma(I) = \{1\}$, so $\sigma(e^0) = \sigma(I) = \{1\}$
2. $\sigma(e^I) = \sigma(eI) = \{e\}$.
3. Let T be a nilpotent operator on a finite Hilbert space H. With order n, we have $e^T = \sum_{k=0}^{n-1} \frac{1}{k!} T^k$, and $\sigma(e^T) = \sigma\left(\sum_{K=0}^{n-1} \frac{1}{k!} T^k\right) = \left\{\sum_{k=0}^{n-1} \frac{1}{k!} \lambda^k : \lambda \in \sigma(T)\right\}$ by [2]. But $\sigma(T) = \{0\}$ by [2], hence $\sigma(e^T) = \{1\}$.

Theorem (3.5) [2]

Let T be a bounded self-adjoint linear operator on a complex Hilbert space H. Then:

1. The spectrum $\sigma(T)$ is real.
2. The residual spectrum $\sigma_r(T)$ is empty.
3. $r(T) = \| T \|$.

Proposition (3.6)

If $T \in B(H)$ and T is a self-adjoint operator. Then :

1. $\sigma_P(e^T)$ subset of real number and $\sigma_r(e^T) = \emptyset$
2. $r(e^T) \le e^{r(T)}$

Proof:

1. T is a self-adjoint operator, then e^T is self-adjoint by proposition (2.1). Hence $\sigma_P(e^T)$ is subset of real number by theorem (3.5) and $\sigma_r(e^T) = \emptyset$.
2. By theorem (3.5), we have $r(e^T) = \|e^T\|$ and by proposition (1.1) part (4), we have $r(e^T) \le e^{\|T\|} = e^{r(T)}$.

Lemma (3.7) [2]

T is a positive self-adjoint if and only if $\sigma(T) \subseteq [0,\infty)$.

Proposition (3.8)

If T is a positive self-adjoint on a complex Hilbert space H. Then $\sigma(e^T) \subseteq [1,\infty)$.

Proof:

If T is a positive operator, then T^n is also positive (by proposition (2.4) part (4)), i.e. $< T^n x, x > \ge 0$, for x in H and n positive integer. So, we have $< e^T x, x > = < \sum_{n=0}^{\infty} \frac{1}{n!} T^n x, x > = \sum_{n=0}^{\infty} \frac{1}{n!} < T^n x, x > = \|x\|^2 + \sum_{n=1}^{\infty} \frac{1}{n!} < T^n x, x >$. Hence inf $\{< e^T x, x >: x \in H \text{ and } \|x\| = 1\} \ge 1$, then $\sigma(e^T) \subseteq [1,\infty)$.

Remarks (3.9)

1. In [6] M. Akkouch , proves that if T is a normal operator on H. Then:

1) $\rho(T) = \{ \lambda: \lambda \in \mathbb{C}, R_{T-\lambda I} = H \}$
2) $\sigma_p(T) = \{ \lambda: \lambda \in \mathbb{C}, \overline{R_{T-\lambda I}} \neq H \}$
3) $\sigma_c(T) = \{ \lambda: \lambda \in \mathbb{C}, \overline{R_{T-\lambda I}} = H \}$
4) $\sigma_r(T)$ is empty.

So, if e^T is normal operator by proposition (2.8), we have :

1) $\rho(e^T) = \{ \lambda: \lambda \in \mathbb{C}, R_{e^T-\lambda I} = H \}$
2) $\sigma_P(e^T) = \{ \lambda: \lambda \in \mathbb{C}, \overline{R_{e^T-\lambda I}} \neq H\}$
3) $\sigma_c(e^T) = \{\lambda: \lambda \in \mathbb{C}, \overline{R_{e^T-\lambda I}} = H \}$
4) $\sigma_r(e^T)$ is empty.

2. In [6], we have if T is a normal operator on a Hilbert space H, then $r(T) = \|T\|$, so $r(e^T) = \|e^T\| \le e^{r(T)}$, (because e^T is normal if T is a normal by proposition (2.8)) .
3. In [7] , we have if T is a hyponormal operator , then $\sigma(T) = \sigma_P(T^*)$. Hence $\sigma(e^T) = \sigma_P(e^{T^*})$, (because (e^T) is hyponormal if T is hyponormal and binormal by proposition (2.19)) .
4. In [8] , we have if T is a hyponormal operator on a Hilbert space H , the $r(T) = \| T \|$, therefore $r(e^T) = \| e^T\| \le e^{r(T)}$, because e^T is hyponomral if (T is hyponormal and binormal by proposition (2.19)) .

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الداله الأسيه لمؤثر خطي مقيد على فضاء هلبرت

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الخلاصة :
قدمنا في هذا البحث مؤثر القوى لمؤثر معين معرف على فضاء هلبرت مع دراسة خواص مؤثر القوى . كما تم دراسة خواص المؤثر المعين والتي تورث إلى مؤثر القوى. ودرسنا طيف مؤثر القوى الذي يمنحه المؤثر المعين.