Open Access
Baghdad Science Journal Vol.13(4)2016

DOI: http://dx.doi.org/10.21123/bsj.2016.13.4.0853

π-Armendariz Rings and Related Concepts

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Received 16/ 11/2015
Accepted 5/ 4/2016

Abstract:

In this paper we investigated some new properties of π-Armendariz rings and studied the relationships between π-Armendariz rings and central Armendariz rings, nil-Armendariz rings, semicommutative rings, skew Armendariz rings, α-compatible rings and others. We proved that if R is a central Armendariz, then R is π-Armendariz ring. Also we explained how skew Armendariz rings can be π-Armendariz, for that we proved that if R is a skew Armendarizα-compatible ring, then R is π-Armendariz. Examples are given to illustrate the relations between concepts.

Key words: Armendariz ring, π-Armendariz ring, central Armendariz ring, α-compatible ring, semicommutative ring.

Introduction:

Throughout this paper R is an associative ring with identity, unless otherwise stated.The polynomial ring with an indeterminate x over R is denoted by $R[x]$ in which elements are polynomials in x with coefficients in R. For a ring R, $P(R)$ is the prime radical (i.e., the intersection of all prime ideals of R), and $N(R)$ is the set of all nilpotent elements of R. Following Rege et. al. [1] a ring R is said to be Armendariz if whenever two polynomials $f(x) = s_0 + s_1x + \cdots + s_nx^n$ and $g(x) = t_0 + t_1x + \cdots + t_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $s_it_j = 0$ for all i, j. There are many relationships between the concept of Armendariz rings and many kinds of other rings or some generalizations of Armendariz rings. Due to Agayev et. al. [2] a ring R is central Armendariz if whenever $f(x) = s_0 + s_1x + \cdots + s_nx^n$ and $g(x) = t_0 + t_1x + \cdots + t_mx^m \in R[x]$, $f(x)g(x) = 0$ implies $s_it_j \in C(R)$ for each i and j. All commutative rings, reduced rings (a ring R is called a reduced ring if it has no non-zero nilpotent elements), Armendariz rings and subrings of central Armendariz rings are central Armendariz. In [2] proved that central Armendariz rings are abelian rings (The ring R is called abelian if every idempotent is central, that is, $ae = ea$ for any $e^2 = e$, $a \in R$) and there exists an abelian ring but not central Armendariz. Therefore the class of central Armendariz rings lies strictly

between classes of Armendariz rings and abelian rings. Mohammadi et. al. in [3] introduced the concept of nil-semicommutative rings such that nil-semicommutative rings are 2-primal (nil-Armendariz, weak Armendariz respectively). Recall that a ring R is nil-semicommutative if $a, b \in N(R)$ satisfy $ab = 0$ then $arb = 0$ for any $r \in R$, a ring R is 2-primal if $P(R) = N(R)$, also R is nil-Armendariz if whenever two polynomials $f(x) = s_0 + s_1x + \cdots + s_nx^n$ and $g(x) = t_0 + t_1x + \cdots + t_mx^m \in R[x]$ such that $f(x)g(x) \in N(R)[x]$ implies $s_it_j \in N(R)$ for each i, j, and finally a ring R is weak Armendariz $f(x)g(x) = 0$ implies $s_it_j \in N(R)$ for every two polynomials $f(x) = s_0 + s_1x + \cdots + s_nx^n$ and $g(x) = t_0 + t_1x + \cdots + t_mx^m \in R[x]$.
Abduldaim and Chen in [4] studied and investigated some properties and relationships between different generalizations of Armendariz rings and the concept of π-McCoy rings. The concept of π-McCoy rings introduced as a generalization of McCoy rings [5]. A ring R is called π-McCoy if $f(x)g(x) \in N(R[x])$ implies $rf(x) \in N(R[x])$ for some nonzero $r \in R$, where $f(x)$ and $g(x)$ are nonzero polynomials in $R[x]$. Huh et. al. [6] introduced the notion of π-Armendariz rings. A ring R is called π-Armendariz if whenever $f(x) = s_0 + s_1x + \cdots + s_nx^n$ and $g(x) = t_0 + t_1x + \cdots + t_mx^m \in R[x]$, $f(x)g(x) \in N(R[x])$ implies that $s_it_j \in N(R)$ for each i and j. It is clear that every Armendariz ring is π-Armendariz, but the converse may not be true in general. It was proved that 2-primal rings are π-Armendariz. But the converse need not be true. Also many properties of π-Armendariz rings were studied.
Motivated by all the above in this paper we have been studied and investigated many relationships between π-Armendariz rings and other kinds of rings like central Armendariz rings, nil-Armendariz rings, semicommutative rings, skew Armendariz rings, α-compatible rings (Moussavi [1], a ring R is α-compatible if for each $a, b \in R$ we have that $ab = 0$ if and only if $a\alpha(b) = 0$. Moreover, R is said to be δ-compatible if for each $a, b \in R$ we have that $ab = 0$ implies that $a\delta(b) = 0$. If R is both α-compatible and δ-compatible, we say that R is (α, δ)-compatible), 2-primal, $p.p.$-rings (a ring R is called a left $p.p.$-ring if each principal left ideal of R is projective, or equivalently, if the left annihilator of each element of R is generated by an idempotent) and others. We proved that (1) For an endomorphism α of a ring R. If R is a skew Armendariz α-compatible ring, then R is π-Armendariz, (2) If R is a central Armendariz $p.p$-ring, then R is π-Armendariz ring, (3) If R is a 2-primal ring, then R is nil-Armendariz, (4) Every semicommutative ring is π-Armendariz.
Finally, we mentioned that skew polynomial rings play an important role and have applications in several domains like coding theory, Galois representations theory in positive characteristic, cryptography, control theory, and solving ordinary differential equations.

1. Main Results

In this section we study some new properties of π-Armendariz rings and investigate the relationships between these rings and several known concepts like central Armendariz rings, nil-Armendariz rings, *s*kew Armendariz rings, 2-primal rings and others.

Proposition 1.1: If R is a central Armendariz $p.p$-ring, then R is π-Armendariz ring.

Proof: Assume that R is a central Armendariz $p.p$-ring. To prove that R is $\pi-$Armendariz, suppose that $f(x)g(x) \in N(R[x])$ where $f(x) =$

$s_0 + s_1x + \cdots + s_mx^m$ and $g(x) = t_0 + t_1x + \cdots + t_nx^n \in R[x]$. We claim that $s_it_j \in N(R)$ for each i and j. Since R is a central Armendariz ring then

$$\begin{aligned} 0 &= f(x)g(x) \\ &= \left(\sum_{i=0}^{m} s_ix^i\right)\left(\sum_{j=0}^{n} t_jx^j\right) \\ &= (s_0 + s_1x + \cdots + s_mx^m)(t_0 + t_1x + \cdots + t_nx^n) \\ &= s_0(t_0 + t_1x + \cdots + t_nx^n) + s_1x(t_0 + t_1x + \cdots + t_nx^n) + \cdots\cdots \\ &\quad + s_mx^m(t_0 + t_1x + \cdots + t_nx^n) \\ &= s_0t_0 + s_0t_1x + \cdots + s_0t_nx^n + s_1xt_0 + s_1xt_1x + \cdots + s_1xt_nx^n + \cdots\cdots + s_mx^mt_0 \\ &\quad + s_mx^mt_1x + \cdots + s_mx^mt_nx^n \\ &= s_0t_0 + (s_0t_1 + s_1t_0)x + (s_0t_2 + s_1t_1 + s_2t_0)x^2 + \cdots\cdots + s_mt_nx^{m+n} \end{aligned}$$

which implies that

$$s_0t_0 = 0 \qquad (1)$$
$$s_0t_1 + s_1t_0 = 0 \qquad (2)$$
$$s_0t_2 + s_1t_1 + s_2t_0 = 0 \qquad (3)$$
$$\vdots$$

Since R is central Armendariz, then R is an abelian ring [2, Proposition 2.1] and R is $p.p.$ring, hence there exist idempotent elements $e_i \in R$ such that $ann(s_i) = e_iR$ for each i. Therefore $t_0 = e_0t_0$ and $s_0e_0 = 0$. By multiplying equation (2) by e_0 we get

$$0 = s_0t_1e_0 + s_1t_0e_0 = s_0e_0t_1 + s_1t_0e_0 = s_1t_0$$

Consequently, equation (2) gives $s_0t_1 = 0$ which implies that $t_1 = e_0t_1$. In the same way we multiply equation (3) by e_0 we have

$$0 = s_0t_2e_0 + s_1t_1e_0 + s_2t_0e_0 = s_1t_1 + s_2t_0$$

multiply the last equation by e_1, we have $0 = s_1t_1e_1 + s_2t_0e_1 = s_2t_0$. Keep on doing the same multiplication process for all equations, we get $s_it_j = 0 \in N(R)$ for all i and j which means that $R$is π-Armrndariz.

Next we show that the converse of Proposition 1.1 is not true in general. The following example illustrates that π-Armendariz rings may not be central Armendariz rings.

Example 1.2: Let S be a reduced ring and let

$$R_4 = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} \middle| a, a_{ij} \in S \right\}.$$

R_4 is π-Armendariz ring by [6, Theorem 2.4], but R_4 is not central Armendariz ring for if $f(x), g(x) \in R_4[x]$ such that

$$f(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x$$

And

$$g(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} x$$

then

$$\begin{aligned} f(x)g(x) &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} x \\ &+ \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} x \end{aligned}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x^2$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x^2 = 0$$

but

$$a_0 b_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$b_1 a_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$a_0b_1 \neq b_1a_0$which means that neither $a_0b_1 \notin C(R)$ nor $b_1a_0 \notin C(R)$. Therefore R_4 is not central Armendariz.

Since reduced rings need not be $p.p.$-rings [7], so that we have examples of Armendariz rings but not $p.p.$-rings.

Remark 1.3: Let R be a ring and M be an (R,R)-bimodule. Recall that the trivial extension of R by M is defined to be the set $T = T(R,M)$ of all pairs (r,m) where $r \in R$and $m \in M$, that is:

$$T = T(R,M) = R \oplus M = \{(r,m) | r \in R, m \in M\}$$

with addition defined componentwise as

$$(r_1,m_1) + (r_2,m_2) = (r_1 + r_2, m_1 + m_2)$$

and multiplication defined according to the rule

$$(r_1,m_1)(r_2,m_2) = (r_1r_2, r_1m_2 + m_1r_2)$$

for all $r_1,r_2 \in R$ and $m_1,m_2 \in M$. Clearly $T = T(R,M)$ forms a ring and it is commutative if and only if R is commutative.

Now we show that the condition "$p.p.$-rings" in Proposition 1.1 is not unnecessary.

Example 1.4: The ring $R = T(\mathbb{Z}_8, \mathbb{Z}_8)$ is commutative, so that R is central Armendariz. But R is not $p.p.$-rings [2].

Next we give another condition such that central Armendariz rings implies π-Armendariz rings.

Proposition 2.5: If R is a central Armendariz ring without zero divisor, then R is π-Armendariz ring.

Proof: Suppose that R is a central Armendariz ring without zero divisor. To prove that R is π-Armendariz, let $f(x)g(x) \in N(R[x])$ where $f(x) = s_0 + s_1x + \cdots + s_mx^m$ and $g(x) = t_0 + t_1x + \cdots + t_nx^n \in R[x]$. We claim that $s_it_j \in N(R)$for each i and j. Since R is a central Armendariz ring then

$$\begin{aligned} 0 &= f(x)g(x) \\ &= \left(\sum_{i=0}^{m} s_i x^i\right)\left(\sum_{j=0}^{n} t_j x^j\right) \\ &= (s_0 + s_1x + \cdots + s_mx^m)(t_0 + t_1x + \cdots + t_nx^n) \\ &= s_0t_0 + (s_0t_1 + s_1t_0)x + (s_0t_2 + s_1t_1 + s_2t_0)x^2 + \cdots + (s_0t_n + s_1t_{n-1} + \cdots + s_mt_0)x^m \end{aligned}$$

thus

$$s_0t_0 = 0 \qquad \cdots (1)$$
$$s_0t_1 + s_1t_0 = 0 \qquad \cdots (2)$$
$$s_0t_2 + s_1t_1 + s_2t_0 = 0 \qquad \cdots (3)$$
$$s_0t_3 + s_1t_2 + s_2t_1 + s_3t_0 = 0 \qquad \cdots (4)$$

$$s_0t_4 + s_1t_3 + s_2t_2 + s_3t_1 + s_4t_0 = 0 \quad \cdots(5)$$

$$s_0t_5 + s_1t_4 + s_2t_3 + s_3t_2 + s_4t_1 + s_5t_0 = 0 \quad \cdots(6)$$

$$s_0t_6 + s_1t_5 + s_2t_4 + s_3t_3 + s_4t_2 + s_5t_1 + s_6t_0 = 0 \quad \cdots(7)$$

$$s_0t_7 + s_1t_6 + s_2t_5 + s_3t_4 + s_4t_4 + s_5t_3 + s_6t_2 + s_7t_0 = 0 \quad \cdots(8)$$

$$s_0t_8 + s_1t_7 + s_2t_6 + s_3t_5 + s_4t_4 + s_5t_3 + s_6t_2 + s_7t_1 + s_8t_0 = 0 \quad \cdots(9)$$

$$s_0t_9 + s_1t_8 + s_2t_7 + s_3t_6 + s_4t_5 + s_5t_4 + s_6t_3 + s_7t_2 + s_8t_1 + s_9b_0 = 0 \quad \cdots(10)$$

$$\vdots$$

Since the ring R without zero devisor, then equation (1) gives $s_0 = 0$ or $t_0 = 0$. Take $s_0 = 0$, hence equation (2) gives $s_0t_1 + s_1t_0 = s_1t_0 = 0$. Again if $s_1 = 0$ or $t_0 = 0$, choose $s_1 = 0$ and equation (3) gives $s_0t_2 + s_1t_1 + s_2t_0 = s_2t_0 = 0$. By continuing apply the same steps we get $s_it_j = 0 \in N(R)$ for all i and j and therefore R is π-Armrndariz.

Corollary 1.6: Every central Armendariz domain is π-Armendariz ring. It is known that every 2-primal ring is nil-Armendariz [3], next we prove the same result using the relationship between π-Armrndariz rings and nil-Armendariz rings. First, we recall that every 2-primal ring is π-Armendariz [6], depending on this result we have the following:

Proposition 1.7: If R is a 2-primal ring, then R is nil-Armendariz.

Proof: Assume that R is a 2-primal ring. To prove that R is nil-Armendariz, suppose that $f(x)g(x) \in N(R)[x]$ where $f(x) = s_0 + s_1x + \cdots + s_mx^m$ and $g(x) = t_0 + t_1x + \cdots + t_nx^n \in R[x]$. We claim that $s_it_j \in N(R)$for each i and j. Since $R$is 2-primal, then $N(R[x]) = N(R)[x]$ [8, Lemma 3.8] and R is π-Armendariz [6, Proposition 1.3]. Hence$f(x)g(x) \in N(R[x]) = N(R)[x]$ implies $s_it_j \in N(R)$. Therefore R is a nil-Armendariz ring.

Corollary 1.8: Let R be a is π-Armendariz ring such that $N(R[x]) = N(R)[x]$, then R is nil-Armendariz.

Now we investigate the relationship of π-Armendariz rings with the concept semicommutative rings and some of its kinds. Recall that (1) a ring R is semicommutative if for any $a, b \in R$, $ab = 0$ implies that $aRb = 0$. (2) a ring R is central semicommutative if $st = 0$ implies that $srt \in C(R)$ for any $s, t, r \in R$ [6].,(3) a ring R is nil-semicommutative if for every $a, b \in N(R)$, $ab = 0$ implies $aRb = 0$. Now since (1) every central semicommutative ring is 2-primal, (2) Nil-semicommutative rings are 2-primal, (3) every semicommutative ring is central semicommutative [12], then we have the following:

Corollary 1.9: Every central semicommutative ring is π-Armendariz.

Proof: Is immediate by [12, Proposition 1.3].

Corollary 1.10: Every Nil-semicommutative ring is π-Armendariz.

Proof: Is immediate by [12, Proposition 1.3].

Corollary 1.9: Every semicommutative ring is π-Armendariz.

Theorem 1.10: Let R be a ring with an endomorphism α. If R is a skew Armendariz α-compatible ring, then Risπ-Armendariz.

Proof: Suppose that R is a skew Armendariz α-compatible ring. To prove that R is a π-Armendariz ring. Let $f(x) = \sum_{i=0}^{m} s_ix^i$ and $g(x) = \sum_{j=0}^{n} t_jx^j$ in $R[x]$ such that $f(x)g(x) \in N(R[x])$. Then there exists a positive integer k such that $(f(x)g(x))^k = 0$. Now we need to prove that if $f_1(x), f_2(x), \cdots, f_n(x) \in R[x; \alpha, 0]$ such that $f_1(x)f_2(x) \cdots f_n(x) = 0$, then $s_1s_2 \cdots s_n = 0$ where $s_1, s_2, \cdots, s_n \in R$. By using the induction on n, we get the following:

Case 1: Suppose the result is true when $n = 1$.

Case 2: We show the result is true when $n = 2$. Assume that $f_1(x) = \sum_{i=0}^{m} s_i x^i$ and $f_2(x) = \sum_{j=0}^{n} t_j x^j \in R[x;\alpha,0]$ satisfy $f_1(x)f_2(x) = 0$, we have to prove that $s_i t_j = 0$ for each i, j.

$$0 = f_1(x)f_2(x)$$
$$= \left(\sum_{i=0}^{m} s_i x^i\right)\left(\sum_{j=0}^{n} t_j x^j\right)$$
$$= (s_0 + s_1 x + \cdots + s_m x^m)(t_0 + t_1 x + \cdots + t_{n-1}x^{n-1} + t_n x^n)$$

Since R is skew Armendariz, then $s_0 t_j = 0$ for $0 \le j \le n$, and so $s_0 f_g^h(t_j) = 0$ for every $0 \le j \le n$ and $0 \le s \le t$. Therefore

$$= (s_1 x + s_2 x^2 + \cdots + s_m x^m)(t_0 + t_1 x + \cdots + t_n x^n)$$
$$= (s_1 + s_2 x + \cdots + s_m x^{m-1})x(t_0 + t_1 x + \cdots + t_n x^n)$$
$$= (s_1 + s_2 x + \cdots + s_m x^m)(x(t_0) + x(t_1)x + \cdots + x(t_n)x^n)$$
$$= (s_1 + s_2 x + \cdots + s_m x^m)$$
$$\begin{pmatrix} \big(\alpha(t_0)x + \delta(t_0)\big) + \begin{pmatrix}\alpha(t_1)x^2 \\ +\delta(t_1)x\end{pmatrix} + \\ (\alpha(t_2)x^3 + \delta(t_2)x^2) + \cdots + \\ (\alpha(t_n)x^{n+1} + \delta(t_n)x^n) \end{pmatrix}$$
$$= (s_1 + s_2 x + \cdots + s_m x^{m-1})$$
$$\begin{pmatrix} \delta(t_0) + \big(\alpha(t_0) + \delta(t_1)\big)x + \\ \big(\alpha(t_1) + \delta(t_2)\big)x^2 + \cdots + \\ \big(\alpha(t_{n-1}) + \delta(t_n)\big)x^n \\ +\alpha(t_n)x^{n+1} \end{pmatrix}$$

Since R is skew Armendariz we can apply the same steps as above to get $s_1\alpha(t_n) = 0$, $s_1\big(\alpha(t_{k-1}) + \delta(t_k)\big) = 0$ for $1 \le k \le n$, and $s_1\delta(t_0) = 0$. Since R is (α,δ)-compatible, then $s_1\alpha(t_n) = 0$ [9, Lemma 2.3(3)], hence $s_1 t_n = 0$. By using $s_1\big(\alpha(t_{n-1}) + \delta(t_n)\big) = 0$ and $s_1\delta(t_n) = 0$, we attain $s_1 t_{n-1} = 0$. By repeating this procedurewe achieve that $s_1 t_j = 0$ for every $0 \le j \le n$. Now suppose $i > 2$ and

$$0 = \big(s_i x^i + s_{i+1}x^{i+1} + \cdots + s_m x^m\big)(t_0 + t_1 x + \cdots + t_n x^n)$$
$$= \big(s_i + s_{i+1}x + \cdots + s_m x^{m-i}\big)x^i(t_0 + t_1 x + \cdots + t_n x^n)$$
$$= \big(s_i + s_{i+1}x + \cdots + s_m x^{m-i}\big)\big(x^i(t_0) + x^i(t_1)x + \cdots + x^i(t_n)x^n\big)$$
$$= (s_i + s_{i+1}x + \cdots + s_m x^m)\left(f_0^i(t_0)x^0 + \sum_{g+h=1} f_g^i(t_h)x + \cdots + \sum_{g+h=n+i} f_g^i(t_h)x^{n+i}\right)$$

Where $0 \le s \le i$and $0 \le t \le n$. Because that $R$is skew Armendariz, hence:

$$s_i\big(\textstyle\sum_{g+h=k} f_g^i(t_h)\big) = 0,\ k = 0,1,2,\cdots,n+i$$

In case that $g + h = n + i$, then $g = i$ and $h = n$ which implies that

$$s_i\big(\textstyle\sum_{g+h=n+i} f_g^i(t_h)\big) = s_i f_i^i(t_n) = s_i\alpha^i(t_n) = 0.$$

But R is (α,δ)-compatible, thus $s_i t_n = 0$ and so $s_i f_g^h(t_n) = 0$ for each $0 \le g \le h$.

Now in case that $g + h = n + i - 1$, then $g = i - 1$ and $h = n$ so that

$$s_i\left(\sum_{g+h=n+i-1} f_g^i(t_h)\right) = s_i f_{i-1}^i(t_n) + s_i\alpha^i(t_{n-1}) = s_i\alpha^i(t_{n-1}) = 0$$

therefore we get $s_i t_{n-1} = 0$.

Next let p be a positive integer such that for all $j < p$, $s_i t_{n-j} = 0$, we have to prove that $s_i t_{n-p} = 0$.

Let $g + h = n = i + p$ it is also true if we take $p \ge i$, so.

$$0 = s_i\left(\sum_{g+h=n+i-p} f_g^h(t_h)\right)$$
$$= s_i\left(f_i^i\big(t_{n-p}\big) + f_{i-1}^i\big(t_{n-(p-1)}\big) + \cdots + f_0^i\big(t_{n-(p-i)}\big)\right)$$
$$= s_i\alpha^i\big(t_{n-p}\big) + s_i f_{i-1}^i\big(t_{n-(p-1)}\big) + \cdots\cdots + s_i f_0^i\big(t_{n-(p-1)}\big)$$

By assumption we have $s_i t_{n-j} = 0$ for each $0 \le j \le p$. Then $s_i f_g^h(t_{n-j}) = 0$ for each $0 \le j \le p$ and $0 \le g \le h$. Therefore $s_i t_j = 0$ for each $0 \le j \le n$.

By applying induction on i, we get $s_i t_j = 0$ for each $0 \le i \le m$ and $0 \le j \le n$.

Case 3: Assume that $n > 2$ and by considering $z(x) = f_2(x) f_3(x) \cdots f_n(x)$. Thus $f_1(x) z(x) = 0$. Since R is a skew Armendariz ring then $s_1 s_z = 0$ where $s_1 \in coef\big(f(x_1)\big)$ and $s_z \in coef\big(z(x)\big)$. This implies that for all $s_1 \in coef\big(f(x_1)\big)$, $s_1 f_2(x) \cdots f_n(x) = 0$, and by induction, since the coefficients of $s_1 f_2(x)$ are of the form $s_1 s_2$ where $s_2 \in coef\big(f_2(x)\big)$. Finally we get $s_1 s_2 \cdots s_n = 0$ where $s_1, s_2, \cdots, s_n \in R$.

At last we have to prove that R is a π-Armendariz ring. We have that $f(x)g(x) \in N(R[x])$ where $f(x) = \sum_{i=0}^{m} s_i x^i$ and $g(x) = \sum_{j=0}^{n} t_j x^j$ in $R[x]$. Therefore there exists a positive integer k such that $(f(x)g(x))^k = 0$. But since $s_1 s_2 \cdots s_n = 0$, hence $s_i t_j \in N(R)$ for every $0 \le i \le m$ and $0 \le j \le n$. Therefore R is a π-Armendariz ring.

In the following we give an example about a π-Armendariz ring but not skew Armendariz.

Example 1.11: Let R be a reduced ring and let

$$R_4 = \left\{ \left(\begin{matrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{matrix} \right) \middle| \, a, a_{ij} \in R \right\}.$$

By [6, Theorem 2.4] and [6, Lemma1.1 (3)] R_4 is a π-Armendariz ring, but R_4 is not skew Armendariz by [10, Corollary 2.3] because R_4 is not Armendariz by [11, Example 3].

Theorem 2.12: Let R be a ring with an endomorphism α. If R is an α-Armendariz α-compatible ring, then R is π-Armendariz.

Proof: Let R be an α-Armendariz ring, then for any two polynomials $f(x) = \sum_{i=0}^{m} s_i x^i$ and $g(x) = \sum_{j=0}^{n} t_j x^j \in R[x;\alpha]$ satisfies $f(x)g(x) = 0$ implies that $s_i t_j = 0$ for each i, j. Suppose that $f(x)g(x) \in N(R[x])$, we should prove that $s_i t_j \in N(R)$. Since R is α-Armendariz, then:-

$$\begin{aligned}
0 &= f(x)g(x) \\
&= \left(\sum_{i=0}^{m} s_i x^i\right)\left(\sum_{j=0}^{n} t_j x^j\right) \\
&= (s_0 + s_1 x + s_2 x^2 + \cdots + s_m x^m)(t_0 + t_1 x + t_2 x^2 + \cdots + t_n x^n) \\
&= s_0(t_0 + t_1 x + t_2 x^2 + \cdots + t_n x^n) + s_1 x(t_0 + t_1 x + t_2 x^2 + \cdots + t_n x^n) \\
&\quad + s_2 x^2 (t_0 + t_1 x + t_2 x^2 + \cdots + t_n x^n) + \cdots + s_m x^m (t_0 + t_1 x + t_2 x^2 + \cdots + t_n x^n) \\
&= (s_0 t_0 + s_0 t_1 x + s_0 t_2 x^2 + \cdots + s_0 t_n x^n) + (s_1 x t_0 + s_1 x t_1 x + s_1 x t_2 x^2 + \cdots + s_1 x t_n x^n) \\
&\quad + (s_2 x^2 t_0 + s_2 x^2 t_1 x + s_2 x^2 t_2 x^2 + \cdots s_2 x^2 t_n x^n) + \cdots \\
&\quad + (s_m x^m t_0 + s_m x^m t_1 x + s_m x^m t_2 x^2 + \cdots + s_m x^m t_n x^n) \\
&= (s_0 t_0 + s_0 t_1 x + \cdots + s_0 t_n x^n) + (s_1 \alpha(t_0) x + s_1 \alpha(t_1) x^2 + \cdots + s_1 \alpha(t_n) x^n) \\
&\quad + (s_2 \alpha^2(t_0) x^2 + s_2 \alpha^2(t_1) x^3 + \cdots + s_2 \alpha^2(t_n) x^n) + \cdots \\
&\quad + (s_m \alpha^m(t_0) x^m + s_m \alpha^m(t_1) x^{m+1} + \cdots + s_m \alpha^m(t_n) x^{m+n})
\end{aligned}$$

$$= s_0t_0 + \big(s_0t_1 + s_1\alpha(t_0)\big)x + \big(s_0t_2 + s_1\alpha(b_1) + s_2\alpha^2(t_0)\big)x^2 + \big(s_0t_3 + s_1\alpha(t_2) + s_2\alpha^2(t_1) + s_3\alpha^3(t_0)\big)x^3 + \cdots + s_m\alpha^m(b_n)x^{m+n}$$

$$s_0t_0 = 0 \qquad (0)$$

$$s_0t_1 + s_1\alpha(t_1) = 0 \qquad (1)$$

$$s_0t_2 + s_1\alpha(t_1) + s_2\alpha^2(t_0) = 0 \qquad (2)$$

$$s_0t_3 + s_1\alpha(t_2) + s_2\alpha^2(t_1) + s_3\alpha^3(t_0) = 0 \qquad (3)$$

$$\cdots$$

$$\cdots$$

$$s_m\alpha^m(t_n) = 0 \qquad (n)$$

From equation (0) we get $s_0t_0 = 0$, in the equation(1); $s_0t_1 + s_1\alpha(t_0) = 0$ and so from definition of α-Armendariz we get $s_1\alpha(t_0) = 0$ and by the condition of α-compatible, $s_1\alpha(t_0) = 0$ if and only if $s_1t_0 = 0$ [9, Lemma 2.3 (1)]. Hence $s_i\alpha^i(t_j) = 0$ if and only if $s_it_j = 0 \in N(R)$, therefore R is π-Armendariz.

In the next example we show that R is a π-Armendariz ring, but it is not α-compatible.

Example 1.13: Let S be areduced ring and let $S_2 = UTM_2(S)$ be the ring of all 2 by 2 upper triangular matrices over S.

$$S_2 = \left\{\begin{pmatrix} s & t \\ 0 & s \end{pmatrix} \middle| s, t \in S\right\}$$

S_2 is a π-Armendariz ring by [6, Theorem 2.4], but S_2 is not α-compatable ring, for if, suppose $\alpha: S_2 \to S_2$ be an endomorphism defined by

$$\alpha\left(\begin{pmatrix} s & t \\ 0 & s \end{pmatrix}\right) = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\alpha\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0,$$

but

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0.$$

Hence R is not (α, δ)-compatible.

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حلقات ارمندرايز من النمط π والمفاهيم ذات العلاقة

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الخلاصة:

في هذا البحث بحثنا بعض الصفات الجديدة لحلقات ارمندرايز من النمطπ ودرسنا العلاقة بين حلقات ارمندرايز من النمط π و حلقات ارمندرايز المركزية، حلقات ارمندرايز المعدومة، الحلقات الشبه ابدالية،حلقات ارمندرايز المتلاشية، الحلقات التوافقية على التشاكل αوغيرها. برهنا انه اذا كانت الحلقةR هي ارمندرايز مركزية فان R هي ارمندرايز من النمط π. ايضا فقد وضحنا كيف ان حلقات ارمندرايز المتلاشية من الممكن ان تكون ارمندرايز من النمط π ، لذلك برهنا انه اذا كانت الحلقة Rارمندرايز متلاشية وتوافقية على التشاكل α فان الحلقة R تكون ارمندرايز من النمط π. اعطيت امثلة لتوضيح العلاقات بين المفاهيم.

الكلمات المفتاحية: حلقة ارمندرايز، حلقة ارمندرايز من النمط π، حلقة ارمندرايز مركزية، حلقة توافقية على التشاكل α، حلقة شبه ابدالية.