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Gaussian Integer Solutions of the Diophantine Equation $x^4 + y^4 = z^3$ for $x \neq y$

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Abstract:

The investigation of determining solutions for the Diophantine equation $x^4 + y^4 = z^3$ over the Gaussian integer ring for the specific case of $x \neq y$ is discussed. The discussion includes various preliminary results later used to build the resolvent theory of the Diophantine equation studied. Our findings show the existence of infinitely many solutions. Since the analytical method used here is based on simple algebraic properties, it can be easily generalized to study the behavior and the conditions for the existence of solutions to other Diophantine equations, allowing a deeper understanding, even when no general solution is known.

Keywords: Algebraic properties, Diophantine equation, Gaussian integer, quartic equation, nontrivial solutions, symmetrical solutions.

Introduction:

The field of Diophantine equations (DEs) is ancient and vast, where no general method exists to decide whether a given DE has any solution or how many. Many studies were conducted in the past on solving equations in the ring of Gaussian integers. For example, Szabó[1] investigated some fourth-degree DEs in Gaussian integers, stating that for certain choices of the coefficients a, b, c, the solutions of the equation $ax^4 + by^4 = cz^2$ in Gaussian integers satisfy $xy = 0$. Apart from that, Najman[2] showed that the equation $x^4 \pm y^4 = iz^2$ has only trivial solutions in Gaussian integers. Then, Emory[3] showed that nontrivial quadratic solutions exist for $x^4 + y^4 = d^2z^4$ when either $d = 1$ or d is a congruent number. Apart from that, Ismail and Mohd Atan[4] investigated the integral solutions of $x^4 + y^4 = z^3$ and discovered the existence of infinitely many solutions to this type of DE in the ring of integers for both cases, $x = y$ and $x \neq y$. Moreover, Izadi et al.[5] examined solutions in the Gaussian integers for different choices of a, b and c for the Diophantine equation $ax^4 + by^4 = cz^2$. Similarly, Izadi et al.[6] examined a class of fourth-power DEs of the form $x^4 + kx^2y^2 + y^4 = z^2$ and $ax^4 + by^4 = cz^2$ in the Gaussian integers, where a and b are prime integers. In recent years, Söderlund[7] discovered that the only primitive non-zero integer solutions to the Fermat quartic $34x^4 + y^4 = z^4$ are $(x, y, z) = (\pm 2, \pm 3, \pm 5)$. The proofs are based on a previous complete solution given to another Fermat quartic, namely $x^4 + y^4 = 17z^4$. Moreover, Jakimczuk[8] investigated the equation $x^4 - y^4 = z^s$, and showed that if s is an odd prime, then the equation has infinitely many solutions (x, y, z) where $x > y > 0$ and $z > 0$. Besides, Ismail et al[9] determined the gaussian integer zeroes of

$F(x,z) = 2x^4 - z^3$ and show the existence of infinitely many non-trivial zeroes for $F(x,z) = 2x^4 - z^3$ under the general form $x = (1+i)\gamma^3$ and $c = -2\gamma^4$ for $\gamma \in Z[i]$. In recent years, Li[10] studied the Diophantine equation $x^4 + 2^n y^4 = 1$ in quadratic number fields. The author showed that nontrivial quadratic solutions to this equation arise from integer solutions to the equations $X^4 \pm 2^n Y^4 = Z^2$ investigated in 1853 by Lebesgue. Apart from that, Somanath et al. [11] studied the quadratic Diophantine equation with two unknowns $65J^2 + 225K^2 - 230JK = 1600$ and determined its non-zero separate solutions in $Z[i]$. The authors gained a few formulae and recurrence relations on the Gaussian integer solutions (J_n, K_n) of the DE. Moreover, Ahmadi and Janfada[12] showed that the quartic Diophantine equations $ax^4 + by^4 = cz^2$ has only trivial solution in the Gaussian integers for some particular choices of a, b and c, using a method based on elliptic curves. In fact, the authors exhibit two null-rank related families of elliptic curves over the Gaussian field as well as determine the torsion groups of both families. Moreover, Tho[13] showed that if the equation $x^4 + 2^n y^4 = z^4$, for n a positive integer, has a solution (x,y,z) in a cubic number field K with $xyz \neq 0$, then the Galois group of the field K is the symmetric group S_3. In addition, for every positive integer $d > 1$, there exists a number field K_d of degree d such that this equation has a solution (x,y,z) in K_d with $xyz \neq 0$. Finally, Tho (2022)[14] investigated the solutions to $x^4 + py^4 = z^4$ in cubic number fields and show that if p is a prime congruent to $11 \ mod \ 16$, the DE only has solutions $x = \pm z, y = 0$ in any cyclic cubic number field.

In this paper, an investigation is performed to determine solutions for the DE $x^4 + y^4 = z^3$ over the Gaussian integer ring for the specific case of $x \neq y$, which has remained unsolved. Note that the case $x = y$ has been solved by Ismail et al.[9].

Results and Discussion:

In this section, elementary algebraic methods are used to study the behavior of the Diophantine equation $x^4 + y^4 = z^3$ when $x \neq y$. Our interest is to determine which conditions give rise to nontrivial solutions and which ones produce no solutions or only trivial ones.

The following analysis supports the ensuing discussion. Suppose that (a,b,c) is a solution of

$$x^4 + y^4 = z^3 \quad 1$$

such that $a \neq b$, and $a, b, c \in \mathbb{Z}[i]$. Let

$$a = r + si, \quad b = t + mi, \quad \text{and} \quad c = g + hi, \quad 2$$

where $r, s, t, m, g, h \in \mathbb{Z}$, and $r \neq t$ or $s \neq m$. Then, replacing Eq.1 yields

$$(r^4 + t^4 - 6(r^2s^2 + t^2m^2) + s^4 + m^4) + 4(r^3s - rs^3 + t^3m - tm^3)i = (g^3 - 3gh^2) + (3g^2h - h^3)i,$$

which in turn implies that

$$r^4 + t^4 - 6(r^2s^2 + t^2m^2) + s^4 + m^4 = g^3 - 3gh^2, \quad 3$$

$$4(r^3s - rs^3 + t^3m - tm^3) = 3g^2h - h^3. \quad 4$$

Starting from Eq.3 and Eq.4, the paper is divided into four main cases based on possible values for r and s. Each of these cases will then be subdivided into four subcases based on possible values for t and m. Finally, it will be further subdivided into four possibilities based on values for g and h.

Case 1. Table. 1 shows all possible combinations of values studied under this case.

Table 1. All possible combinations of values under Case 1.

Case 1: $r = 0, s = 0$	**Case 1.1:** $t = 0, m = 0$	**Case 1.1.1:** $g = 0, h = 0$
		Case 1.1.2: $g = 0, h \neq 0$
		Case 1.1.3: $g \neq 0, h = 0$
		Case 1.1.4: $g \neq 0, h \neq 0$
	Case 1.2: $t = 0, m \neq 0$	**Case 1.2.1:** $g = 0, h = 0$
		Case 1.2.2: $g = 0, h \neq 0$
		Case 1.2.3: $g \neq 0, h = 0$
		Case 1.2.4: $g \neq 0, h \neq 0$
	Case 1.3: $t \neq 0, m = 0$	**Case 1.3.1:** $g = 0, h = 0$
		Case 1.3.2: $g = 0, h \neq 0$
		Case 1.3.3: $g \neq 0, h = 0$
		Case 1.3.4: $g \neq 0, h \neq 0$
	Case 1.4: $t \neq 0, m \neq 0$	**Case 1.4.1:** $g = 0, h = 0$
		Case 1.4.2: $g = 0, h \neq 0$
		Case 1.4.3: $g \neq 0, h = 0$
		Case 1.4.4: $g \neq 0, h \neq 0$

Case 1.1.1: ($r = 0,\ s = 0,\ t = 0,\ m = 0,\ g = 0,\ h = 0$)

Under these conditions, $a = b$, which is outside of our current study. (Notice that these conditions trivially satisfy the equation.) Therefore, the equation under this case is not considered.

Remark 1*:* Due to the same reason, **Case 1.1.2**, **Case 1.1.3**, and **Case 1.1.4** are discarded. Moreover, these lead to inconsistencies.

Case 1.2.1: ($r = 0,\ s = 0,\ t = 0,\ m \neq 0,\ g = 0,\ h = 0$)

From Eq.3, $m^4 = 0$, which is a contradiction since $m^4 > 0$ provided that $m \neq 0$. (Notice that Eq.4 is automatically satisfied under this case.) Therefore, the equation, in this case, is not considered.

Remark 2*:* A similar inconsistency arises in **Case 1.3.1**, **Case 2.1.1**, and **Case 3.1.1**.

Remark 3*:* A similar inconsistency arises in **Case 1.2.2**, **Case 1.3.2**, **Case 2.1.2**, and **Case 3.1.2**. Moreover, from Eq.4, these cases will lead to $h^3 = 0$, which is also a contradiction since $h^3 > 0$.

Case 1.2.3: ($r = 0,\ s = 0,\ t = 0,\ m \neq 0,\ g \neq 0,\ h = 0$**)**

From Eq.3, $m^4 = g^3$. (Notice that Eq.4 is automatically satisfied under this case.) It follows that $|m| = g^{\frac{3}{4}}$, with m an integer. This implies that $g = u^4$ for some integer u. Thus, $|m| = |u|^3$, or equivalently, $m = u^3$. Hence, $(m, g) = (u^3, u^4)$. By letting $u = \pm 1, \pm 2, \pm 3, \dots, \pm k, \dots$, where k is an integer, infinitely many solutions for (m, g) are obtained. In turn, this leads to infinitely many solutions for (a, b, c) of the form

$$\boxed{(a, b, c) = (0, n^3 i, n^4),}$$

where $n \in \mathbb{Z}$.

Remark 4*:* **Case 1.3.3** yields symmetrical solutions with $(t, g) = (u^3, u^4)$ for $u \in \mathbb{Z}$. This leads to

$$\boxed{(a, b, c) = (0, n^3, n^4).}$$

Remark 5*:* **Case 2.1.3** yields symmetrical solutions with $(s, g) = (u^3, u^4)$ for $u \in \mathbb{Z}$. This leads to

$$\boxed{(a, b, c) = (n^3 i, 0, n^4).}$$

Remark 6*:* **Case 3.1.3** yields symmetrical solutions with $(r, g) = (u^3, u^4)$ for $u \in \mathbb{Z}$. This leads to

$$\boxed{(a, b, c) = (n^3, 0, n^4).}$$

Case 1.2.4: ($r = 0,\ s = 0,\ t = 0,\ m \neq 0,\ g \neq 0,\ h \neq 0$**)**

An inconsistency arises under this case as follows. From Eq.4, $3g^2 h - h^3 = 0$, from which $h(3g^2 - h^2) = 0$. Since $h \neq 0$, then $\pm\sqrt{3} = \frac{h}{g}$, which gives rise to a contradiction since $\frac{h}{g}$ is a rational while $\sqrt{3}$ is not. Therefore, the equation under this case is not considered.

Remark 7*:* A similar inconsistency arises in **Case 2.2.4**, **Case 2.3.4**, and **Case 3.3.4.**

Remark 8*:* A similar inconsistency arises in **Case 1.3.4**, **Case 2.1.4**, and **Case 3.1.4**, the only difference being that Eq.3 is not automatically satisfied.

Case 1.4.1: ($r = 0,\ s = 0,\ t \neq 0,\ m \neq 0,\ g = 0,\ h = 0$**)**

From Eq.3, $t^4 - 6t^2m^2 + m^4 = 0$, which can be rewritten as $(t^2 - m^2)^2 = 4t^2m^2$. Thus, $|t^2 - m^2| = 2|tm|$, which implies $t^2 - m^2 = \pm 2tm$. Dividing both sides by t^2 yields

$$\left(\frac{m}{t}\right)^2 \pm 2\left(\frac{m}{t}\right) - 1 = 0,$$

which represents two quadratic equations on $\frac{m}{t}$. Upon solving them, $\frac{m}{t} = \pm\sqrt{2} \pm 1$ or $\frac{m}{t} = \pm\sqrt{2} \mp 1$, both of which represent a contradiction since $\frac{m}{t}$ is rational while $\pm\sqrt{2} \pm 1$ and $\pm\sqrt{2} \mp 1$ are not.

Remark 9*:* A similar inconsistency arises in **Case 1.4.2**, **Case 4.1.1**, and **Case 4.1.2**.

Case 1.4.3: ($r = 0,\ s = 0,\ t \neq 0,\ m \neq 0,\ g \neq 0,\ h = 0$**)**

From Eq.4, $4(t^3 m - tm^3) = 0$, which can be rewritten as $4tm(t^2 - m^2) = 0$. Since $t, m \neq 0$, then $t^2 - m^2 = 0$, which implies $|t| = |m|$ or, equivalently, $t = \pm m$. Upon replacing on Eq.3 yields

$$-4m^4 = g^3. \qquad 5$$

It is obvious that $g < 0$ and $2 \mid g$. Then, let

$$g = -2^{\alpha} v, \qquad 6$$

where $\gcd(v, 2) = 1$. Replacing on Eq.5 yields $-4m^4 = -2^{3\alpha} v^3$, which implies

$$|m| = 2^{\frac{3\alpha - 2}{4}} v^{\frac{3}{4}}. \qquad 7$$

Since m is an integer, then $3\alpha \equiv 2 \pmod 4$, which is an equation whose only solutions are of the form $\alpha = 4k + 2$ for $k \in \mathbb{Z}$. Moreover, once again, due to m being an integer, there must exist an integer u such that $v = u^4$. Then, replacing on Eq.6 yields $g = -2^{4k+2} u^4$, and replacing on Eq.7 gives $|m| = 2^{3k+1} |u|^3$. Therefore, this case leads to

$$(t, m, g) = (2^{3k+1} u^3, \pm 2^{3k+1} u^3, -2^{4k+2} u^4).$$

In turn, this leads to

$$\boxed{(a, b, c) = \left(0, 2^{3k+1} u^3 (1 \pm i), -2^{4k+2} u^4\right),}$$

for $k \geq 0$ and $u \in \mathbb{Z}$.

Remark 10*:* **Case 4.1.3** yields symmetrical solutions with $(r, s, g) = (\pm 2^{3k+1} u^3, \pm 2^{3k+1} u^3, -2^{4k+2} u^4)$ and $(r, s, g) = (\pm 2^{3k+1} u^3, \mp 2^{3k+1} u^3, -2^{4k+2} u^4)$. This leads to

$$\boxed{(a, b, c) = (2^{3k+1} u^3 (1 \pm i), 0, -2^{4k+2} u^4),}$$

for $k \geq 0$ and $u \in \mathbb{Z}$.

Case 1.4.4: ($r = 0,\ s = 0,\ t \neq 0,\ m \neq 0,\ g \neq 0,\ h \neq 0$**)**

From Eq.3 and Eq.4, the following system of equations is obtained:

$t^4 - 6t^2m^2 + m^4 = g^3 - 3gh^2$ 8

$4(t^3m - tm^3) = 3g^2h - h^3.$ 9

These equations now yield

$$(t^4 - 6t^2m^2 + m^4)^2 + (4t^3m - 4tm^3)^2 = (g^3 - 3gh^2)^2 + (3g^2h - h^3)^2,$$

which, after simplification, becomes

$(m^2 + t^2)^4 = (g^2 + h^2)^3.$ 10

By means of a similar method used for **Case 1.2.3**, there must exist an integer α such that

$t^2 + m^2 = \alpha^3,$ 11

$g^2 + h^2 = \alpha^4.$ 12

Let us consider Eq.11. From Cohen[9], there exist integers u and v, with $\gcd(u,v) = 1$, such that the solutions to this equation have the form

$(t, m, \alpha) = (u(u^2 - 3v^2), v(3u^2 - v^2), u^2 + v^2),$ 13

up to the exchange of variables t and m. Replacing Eq.13 in Eq.8 and Eq.9 and solving for g and h yields

$(g, h) = (u^4 - 6u^2v^2 + v^4, 4u^3v - 4uv^3).$ 14

Moreover, notice that the exchange of t and m can be absorbed by replacing h with $-h$. Therefore, Eq.13 and Eq.14 yield

$$\boxed{(a,b,c) = (0, (u^3 - 3uv^2) + (3vu^2 - v^3)i, (u^4 - 6u^2v^2 + v^4) + (4u^3v - 4uv^3)i)}$$

and

$$\boxed{(a,b,c) = (0, (3u^2v - v^3) + (u^3 - 3uv^2)i, (u^4 - 6u^2v^2 + v^4) - (4u^3v - 4uv^3)i),}$$

where $\gcd(u,v) = 1$.

Remark 11: Considering Eq.12 and solving for (g, h) (see Cohen[9]) yields the same results as before, after excluding non-integer solutions.

Remark 12*:* Notice that the conditions of this case convert the original equation, Eq.1, into

$b^4 = c^3,$ 15

where $b, c \in \mathbb{Z}[i]$. This is equivalent to the system Eq.8–Eq.9. Moreover, taking the absolute value on both sides of Eq.15 yields Eq.10.

Remark 13*:* Eq.15 can be solved—and therefore this case—using purely complex number techniques. Indeed, let the complex prime decomposition of b and c be

$b = u_1(1+i)^{\alpha_0} \prod_{j=1}^{l} p_j^{\alpha_j}$ 16

and

$c = u_2(1+i)^{\beta_0} \prod_{k=1}^{m} q_k^{\beta_k},$ 17

respectively, where $u_1, u_2 \in \{+1, -1, +i, -i\}$; $\alpha_0, \dots, \alpha_l, \beta_0 \dots, \beta_m$ are non-negative integers, and p_j and q_j are complex prime numbers. Replacing Eq.15 yields

$$u_1^4(1+i)^{4\alpha_0} \prod_{j=1}^{l} p_j^{4\alpha_j} = u_2^3(1+i)^{3\beta_0} \prod_{k=1}^{m} q_k^{3\beta_k}.$$

The uniqueness of the prime power decomposition in $\mathbb{Z}[i]$ yields $u_1^4 = u_2^3 = 1$, which implies $u_2 = 1$. Also, $4\alpha_0 = 3\beta_0$, from which there must exist an integer γ such that $\alpha_0 = 3\gamma$ and $\beta_0 = 4\gamma$. Moreover, $l = m$ and, after an adequate reordering, $p_j = q_k$ and $4\alpha_j = 3\beta_k$. This last equality, in turn, implies that there exist integers γ_i such that $\alpha_j = 3\gamma_j$ and $\beta_k = 4\gamma_k$. Thus, replacing on Eq.16 and Eq.17 yields

$$b = u_1(1+i)^{3\gamma} \prod_{j=1}^{m} p_j^{3\gamma_j} = u_1 \left((1+i)^{\gamma} \prod_{j=1}^{m} p_j^{\gamma_j} \right)^3$$

and

$$c = (1+i)^{4\gamma} \prod_{k=1}^{m} p_k^{4\gamma_k} = \left((1+i)^{\gamma} \prod_{k=1}^{m} p_k^{\gamma_k} \right)^4.$$

Let $n = (1+i)^{\gamma} \prod_{k=1}^{m} p_k^{\gamma^k} \in \mathbb{Z}[i]$. Therefore, $\boxed{(a,b,c) = (0, un^3, n^4),}$ where $u \in \{+1, -1, +i, -i\}$ and $n \in \mathbb{Z}[i]$, which is an equivalent solution for this case.

Remark 14*:* **Case 4.1.4** yields symmetrical solutions with

$$\boxed{(a,b,c) = ((u^3 - 3uv^2) + (3vu^2 - v^3)i, 0, (u^4 - 6u^2v^2 + v^4) + (4u^3v - 4uv^3)i)}$$

and

$$\boxed{(a,b,c) = ((3u^2v - v^3) + (u^3 - 3uv^2)i, 0, (u^4 - 6u^2v^2 + v^4) - (4u^3v - 4uv^3)i),}$$

where $\gcd(u,v) = 1$. On the other hand, since the respective conditions convert our original equation into

$$a^4 = c^3,$$

the solutions can also be written in the form

$$\boxed{(a,b,c) = (un^3, 0, n^4),}$$

where $u \in \{+1, -1, +i, -i\}$ and $n \in \mathbb{Z}[i]$.

Case 2. Table. 2 shows all possible combinations of values studied under this case.

Table 2. All possible combinations of values under Case 2.

Case 2: $r = 0, s \neq 0$	**Case 2.1:** $t = 0, m = 0$	**Case 2.1.1:** $g = 0, h = 0$
		Case 2.1.2: $g = 0, h \neq 0$
		Case 2.1.3: $g \neq 0, h = 0$
		Case 2.1.4: $g \neq 0, h \neq 0$
	Case 2.2: $t = 0, m \neq 0$	**Case 2.2.1:** $g = 0, h = 0$
		Case 2.2.2: $g = 0, h \neq 0$
		Case 2.2.3: $g \neq 0, h = 0$
		Case 2.2.4: $g \neq 0, h \neq 0$
	Case 2.3: $t \neq 0, m = 0$	**Case 2.3.1:** $g = 0, h = 0$
		Case 2.3.2: $g = 0, h \neq 0$
		Case 2.3.3: $g \neq 0, h = 0$
		Case 2.3.4: $g \neq 0, h \neq 0$
	Case 2.4: $t \neq 0, m \neq 0$	**Case 2.4.1:** $g = 0, h = 0$
		Case 2.4.2: $g = 0, h \neq 0$
		Case 2.4.3: $g \neq 0, h = 0$
		Case 2.4.4: $g \neq 0, h \neq 0$

Case 2.2.1: ($r = 0,\ s \neq 0,\ t = 0,\ m \neq 0,\ g = 0,\ h = 0$)

From Eq.3, $s^4 + m^4 = 0$, which is a contradiction since $s^4 + m^4 > 0$. (Notice that Eq.4 is automatically satisfied under this case.) Therefore, the equation under this case is not considered.

Remark 15*:* A similar inconsistency arises in **Case 2.3.1**, **Case 3.2.1**, and **Case 3.3.1**.

Remark 16*:* A similar inconsistency arises in **Case 2.2.2**, **Case 2.3.2**, **Case 3.2.2**, and **Case 3.3.2**. Moreover, from Eq.4, these cases will lead to $h^3 = 0$, which is also an inconsistency since $h^3 > 0$.

Case 2.2.3: ($r = 0,\ s \neq 0,\ t = 0,\ m \neq 0,\ g \neq 0,\ h = 0$**)**

From Eq.3, $s^4 + m^4 = g^3$. (Under these conditions, Eq.4 is automatically satisfied.) Since s, m and g are all integers, from Theorem 1.2 and Theorem 1.3 in Ismail and Mohd Atan[4], the triplet $(x, y, z) = (s, m, g)$ is a solution to the equation $x^4 + y^4 = z^3$ if and only if $s = m = 4n^3$ and $g = 8n^4$ (which contradicts the hypothesis that $a \neq b$), or $s = un^{3k-1}$, $m = vn^{3k-1}$ and $g = n^{4k-1}$, where $n = u^4 + v^4$, and for any integer k. It follows from Eq.2 that

$$\boxed{(a, b, c) = (un^{3k-1} i, vn^{3k-1} i, n^{4k-1})},$$

where $u \neq v$.

Remark 17*:* **Case 2.3.3** leads to symmetrical solutions with $(s, t, g) = (4n^3, 4n^3, 8n^4)$ *and* $(s, t, g) = (un^{3k-1}, vn^{3k-1}, n^{4k-1})$, where $n = u^4 + v^4$, and for any integer k. These yields, respectively,

$$\boxed{(a, b, c) = (4n^3 i, 4n^3, 8n^4)}$$

and

$$\boxed{(a, b, c) = \left(un^{3k-1} i, vn^{3k-1}, n^{4k-1}\right)}.$$

Remark 18*:* **Case 3.2.3** leads to symmetrical solutions with $(r, m, g) = (4n^3, 4n^3, 8n^4)$ and $(r, m, g) = (un^{3k-1}, vn^{3k-1}, n^{4k-1})$, where $n = u^4 + v^4$, and for any integer k. These yields, respectively,

$$\boxed{(a, b, c) = (4n^3, 4n^3 i, 8n^4)}$$

and

$$\boxed{(a, b, c) = \left(un^{3k-1}, vn^{3k-1} i, n^{4k-1}\right)}.$$

Remark 19*:* **Case 3.3.3** leads to symmetrical solutions with $(r, t, g) = (4n^3, 4n^3, 8n^4)$ (which contradicts the hypothesis $a \neq b$) and $(r, t, g) = (un^{3k-1}, vn^{3k-1}, n^{4k-1})$, where $n = u^4 + v^4$, and for any integer k. This yields

$$\boxed{(a, b, c) = (un^{3k-1}, vn^{3k-1}, n^{4k-1})},$$

where $u \neq v$.

Case 2.4.1: ($r = 0,\ s \neq 0,\ t \neq 0,\ m \neq 0,\ g = 0,\ h = 0$**)**

An inconsistency arises under this case as follows. From Eq.4, $4(t^3 m - tm^3) = 0$, from which $4tm(t^2 - m^2) = 0$. Since $t,\ m \neq 0$, then $t^2 - m^2 = 0$, or equivalently, $|t| = |m|$. Upon replacing on Eq.3, $-4t^4 + s^4 = 0$. This leads us to $\frac{s}{t} = \pm\sqrt{2}$, which is a contradiction since $\frac{s}{t}$ is a rational while $\sqrt{2}$ is not. Therefore, the equation under this case is not considered.

Remark 20*:* A similar inconsistency arises in **Case 3.4.1**, **Case 4.2.1,** and **Case 4.3.1**.

Case 2.4.2: ($r = 0,\ s \neq 0,\ t \neq 0,\ m \neq 0,\ g = 0,\ h \neq 0$**)**

An inconsistency arises under this case as follows. From Eq.3, $t^4 - 6t^2 m^2 + m^4 + s^4 = 0$. Rearranging this equation yields

$$\left(\frac{s^2}{m^2}\right)^2 + \left(\frac{t^2 - 3m^2}{m^2}\right)^2 = 8.$$

Also, $8 = 2^2 + 2^2$. These two equations imply that

$$\frac{s^2}{m^2} = 2 \quad \text{and} \quad \frac{t^2 - 3m^2}{m^2} = \pm 2,$$

where the first equality gives rise to a contradiction since $\frac{s}{m} = \pm\sqrt{2}$. Therefore, the equation under this case is not considered.

Remark 21*:* A similar inconsistency arises in **Case 3.4.2**, **Case 4.2.2** and **Case 4.3.2**.

Case 2.4.3: ($r = 0,\ s \neq 0,\ t \neq 0,\ m \neq 0,\ g \neq 0,\ h = 0$)

Eq.3 and Eq.4 yield

$$t^4 - 6t^2m^2 + m^4 + s^4 = g^3, \quad 18$$

$$4t^3m - 4tm^3 = 0, \quad 19$$

respectively. Here, Eq.19 can be rewritten as $4tm(t^2 - m^2) = 0$. Since $t, m \neq 0$, then $|t| = |m|$. Substituting in Eq.18 yields

$$s^4 - 4m^4 = g^3. \quad 20$$

There are two possibilities that can be considered here:

(i) $|s| = |m|$,
(ii) $|s| \neq |m|$.

Under (i), the following theorem is obtained, which states the form of solutions to Eq.20 when $|s| = |m|$.

Theorem 1: *The solutions to the equation* $x^4 - 4y^4 = z^3$*, when* $|x| = |y|$*, are given by* $x = s$*,* $y = m$ *and* $z = g$*, where*

$$(s, m, g) = (9n^3, \pm 9n^3, -27n^4)$$

.

Proof: Let $(x, y, z) = (s, m, g)$ be a solution to $x^4 - 4y^4 = z^3$ with $|s| = |m|$. Then,

$$-3m^4 = g^3 \quad 21$$

This clearly implies that $3 \mid g$ and g is negative. Let $g = -3^e u$, where $\gcd(3, u) = 1$ and $e > 1$. Thus, from Eq.21, the equation

$$-3m^4 = -3^{3e}u^3$$

is obtained, which yields

$$m = \pm 3^{\frac{3e-1}{4}} u^{\frac{3}{4}}. \quad 22$$

Since m is an integer, then $\frac{3e-1}{4}$ is an integer and there exists an integer v such that $u = v^4$. Thus, $3e - 1 \equiv 0 \pmod 4$, which on simplifying gives $e = 3 + 4j$ for some integer j. It follows from Eq.22 that

$$m = \pm 3^{2+3j}v^3. \quad 23$$

By Eq.21 and Eq.23, $g^3 = -3(3^{2+3j}v^3)^4 = -3^3(3^j v)^4$. Let $n = 3^j v$. Then, $g = -27n^4$, from which Eq.23 gives $m = \pm 9n^3$. Therefore, $s = \pm 9n^3$. Hence, considering that $|s| = |m|$ (or $s = \pm m$) yields

$$(s, m, g) = (9n^3, \pm 9n^3, -27n^4),$$

as asserted. ■

Now, remembering that $|t| = |m|$, the solutions for the system Eq.18–Eq.19 under the condition $|s| = |m|$ are given by

$$(s, t, m, g) = (9n^3, 9n^3, \pm 9n^3, -27n^4)$$

and

$$(s, t, m, g) = (9n^3, -9n^3, \pm 9n^3, -27n^4).$$

This, in turn, gives us the solutions to the original Eq.1, as

$$\boxed{(a, b, c) = (9n^3 i, 9n^3(1 \pm i), -27n^4)}$$

and

$$\boxed{(a, b, c) = (9n^3 i, -9n^3(1 \pm i), -27n^4).}$$

Next, under (ii), Eq.20 has no solutions when $|s| \neq |m|$. First, the following result is stated.

Lemma 1: *Let* u *and* v *be integers such that* $\gcd(u, v) = 1$*, and let* $\gcd(u^2 - 2v^2, u^2 + 2v^2) = d$*. Then,* $d = 1$ *if* u *is odd and* $d = 2$ *if* u *is even.*

Proof: Let $\gcd(u^2 - 2v^2, u^2 + 2v^2) = d$. There exist s and t such that

$$u^2 - 2v^2 = ds \quad \text{and} \quad u^2 + 2v^2 = dt.$$

Suppose first that u is odd. Then, d is odd since both $u^2 - 2v^2$ and $u^2 + 2v^2$ are odd. Also,

$$2u^2 = d(s + t) \quad \text{and} \quad 4v^2 = d(t - s).$$

Since $\gcd(d, 2) = 1$, then $d \mid u^2$ and $d \mid v^2$, which implies that $d = 1$ since $\gcd(u, v) = 1$.

Suppose next that u is even. Let $u = 2^e w$, where e is a positive integer and $\gcd(2, w) = 1$. Then,

$$u^2 - 2v^2 = (2^e w)^2 - 2v^2 \quad \text{and} \quad u^2 + 2v^2 = (2^e w)^2 + 2v^2,$$

from which

$$u^2 - 2v^2 = 2(2^{2e-1}w^2 - v^2) \quad \text{and} \quad u^2 + 2v^2 = 2(2^{2e-1}w^2 + v^2).$$

Now, since $\gcd(u, v) = 1$, it follows that v is odd and $\gcd(w, v) = 1$. Thus, a similar procedure as the above yields

$$\gcd(2^{2e-1}w^2 - v^2, 2^{2e-1}w^2 + v^2) = 1,$$

which implies

$$\gcd(u^2 - 2v^2, u^2 + 2v^2) = \gcd(2(2^{2e-1}w^2 - v^2), 2(2^{2e-1}w^2 + v^2)) = 2.$$

Therefore, $\gcd(u^2 - 2v^2, u^2 + 2v^2) = 1$ when u is odd, and $\gcd(u^2 - 2v^2, u^2 + 2v^2) = 2$ when u is even, as asserted. ■

The following lemma states the nonexistence of solutions for Eq.20 under certain conditions.

Lemma 2: *There are no integer solutions to* $x^4 - 4y^4 = z^3$ *such that* $\gcd(x, y) = 1$*,* x *is odd, and* $y \neq 0$*.*

Proof: Suppose there exist integers u, v and g such that $u^4 - 4v^4 = g^3$, with $\gcd(u, v) = 1$, u odd, and $v \neq 0$. Then,

$$(u^2 - 2v^2)(u^2 + 2v^2) = g^3.$$

Since u is odd, by Lemma 1, $\gcd(u^2-2v^2, u^2+2v^2)=1$, so (u^2+2v^2) and (u^2-2v^2) are coprime factors of g^3. Let $g=ab$ such that $u^2+2v^2=a^3$ and $u^2-2v^2=b^3$. Then, $\gcd(a,b)=1$. Moreover, it is readily seen that

$$a^3+b^3=2u^2, \quad 24$$

$$a^3-b^3=4v^2. \quad 25$$

From Cohen[9], Eq.24 has disjoint parameterized solutions according to the following cases (up to the exchange of u and v).

(a) For $s,t\in\mathbb{Z}$ such that $\gcd(s,t)=1$, s is odd and $s\not\equiv t \pmod 3$,

$$\begin{cases} a=(s^2+2t^2)(5s^2+8ts+2t^2) \\ b=-(s^2+4ts-2t^2)(3s^2+4ts+2t^2) \\ u=\pm(s^2-2ts-2t^2) \\ (7s^4+20ts^3+24t^2s^2+8t^3s+4t^4) \end{cases}$$

Replacing in Eq.25 yields

$$v^2=2s(19s^4-4s^3t+8st^3+4t^4)(s^4+4s^3t+16s^2t^2+24st^3+12t^4)(s^2+st+t^2)(s+2t). \quad 26$$

Since v is an integer, at least one of the parameterized factors in Eq.26 must be even. It is proven, in turn, that none of them is even, which leads to a contradiction. It can readily be seen that it is enough to prove that s^2+st+t^2 is odd, so let us suppose it is even. Then, there exists an integer k such that $s^2+st+t^2=2k$. Upon rewriting, $s^2+t(s+t)=2k$, which implies that s and $t(s+t)$ have the same parity. Thus, $t(s+t)$ should be odd, implying that t and $t+s$ are odd. However, this is a contradiction since $t+s$ would then be the sum of two odd numbers. Thus, none of the parameterized factors in Eq.26 is even.

(b) For $s,t\in\mathbb{Z}$ such that $\gcd(s,t)=1$, $s\not\equiv t \pmod 2$ and $3\nmid t$,

$$\begin{cases} a=(3s^2+2ts+t^2)(3s^2+6ts+t^2) \\ b=(3s^2-6ts+t^2)(3s^2+2ts+2t^2) \\ u=\pm(3s^2-t^2)(9s^4+18t^2s^2+t^4) \end{cases}$$

Replacing in Eq.25 yields

$$v^2=2st(81s^4-6s^2t^2+t^4)(3s^4-2s^2t^2+3t^4)(3s^2+t^2). \quad 27$$

Note that all the parameterized factors of Eq.27 must be coprime. Indeed, it is known that $\gcd(s,t)=1$ and it is evident that s does not divide any of the remaining factors, nor does t. Then, only the following cases need to be considered:

(b.1) Let $d=\gcd(81s^4-6s^2t^2+t^4, 3s^4-2s^2t^2+3t^4)$ and suppose $d\neq 1$. Then, there exist integers α and β such that

$$81s^4-6s^2t^2+t^4=d\alpha, \quad 28$$

$$3s^4-2s^2t^2+3t^4=d\beta, \quad 29$$

Subtracting Eq.27 times Eq.29 from Eq.28 yields $16t^2(3s^2-5t^2)=d(\alpha-27\beta)$. This implies $d\mid 16$ or $d\mid t^2$ or $d\mid(3s^2-5t^2)$.

- If $d\mid 16$, then there is a contradiction. Indeed, since $d\neq 1$, then d must be even, implying that the left-hand-side of Eq.28 is also even, which is not possible by the hypothesis $s\not\equiv t \pmod 2$.
- If $d\mid t^2$, then Eq.29 yields $d\mid s^4$ or $d=3$. It is obvious that $d\mid s^4$ is not possible since $\gcd(s,t)=1$. On the other hand, if $d=3$, then Eq.28 yields $3\mid t$, which is a contradiction.
- If $d\mid(3s^2-5t^2)$, then there exists an integer γ such that

$$3s^2-5t^2=d\gamma. \quad 30$$

Multiplying Eq.28 by 3 and subtracting Eq.29 yields $16s^2(15s^2-t^2)=d(3\alpha-\beta)$. Since it is already known that $d\nmid 16$, then $d\mid s^2$ or $d\mid(15s^2-t^2)$. If $d\mid s^2$, then Eq.28 yields $d\mid t^4$, which is not possible because $\gcd(s,t)=1$. Then, there must exist an integer δ such that

$$15s^2-t^2=d\delta. \quad 31$$

From Eq.30 and Eq.31, $24t^2=d(-5\gamma+\delta)$, implying that $d\mid 24$ or $d\mid t^2$, both of which lead to a contradiction as seen before.

(b.2) Let $d=\gcd(81s^4-6s^2t^2+t^4, 3s^2+t^2)$ and suppose $d\neq 1$. Then, there exist integers α and β such that

$$81s^4-6s^2t^2+t^4=d\alpha, \quad 32$$

$$3s^2+t^2=d\beta. \quad 33$$

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Multiplying Eq.33 by t^2 and subtracting it from Eq.32 yields
$$9s^2(9s^2 - t^2) = d(\alpha - t^2\beta).$$
By similar arguments as in case (b.1), it is readily seen that $d \nmid 9$ and $d \nmid s^2$, which implies that $d \mid (9s^2 - t^2)$. Then, there exists an integer γ such that

$$9s^2 - t^2 = d\gamma \tag{34}$$

From Eq.33 and Eq.34, $12s^2 = d(\beta + \gamma)$, implying $d \mid 12$ or $d \mid s^2$, both of which lead to a contradiction.

(b.3) Let $d = \gcd(3s^4 - 2s^2t^2 + 3t^4, 3s^2 + t^2)$ and suppose $d \neq 1$. Then, there exist integers α and β such that

$$3s^4 - 2s^2t^2 + 3t^4 = d\alpha, \tag{35}$$

$$3s^2 + t^2 = d\beta. \tag{36}$$

Multiplying Eq.36 by $3t^2$ and subtracting it from Eq.35 yields
$$s^2(3s^2 - 11t^2) = d(\alpha - t^2\beta).$$
Similar arguments as in case (b.1) show that $d \nmid s^2$, thus $d \mid (3s^2 - 11t^2)$. Then, there exists an integer γ such that

$$3s^2 - 11t^2 = d\gamma. \tag{37}$$

From Eq.36 and Eq.37, $12t^2 = d(\beta - \gamma)$, implying $d \mid 12$ or $d \mid t^2$, both of which lead to contradictions.

It is proven that all parameterized factors on Eq.27 are coprime, concluding that all those factors are squares, except for the one that is even (either s or t, which must be of the form $2^{2k-1}\alpha^2$ for some positive integers k and α). In particular,

$$3s^4 - 2s^2t^2 + 3t^4 = r^2, \tag{38}$$

for some integer r. Since $s \not\equiv t \pmod 2$, there exists an integer k such that $s - t = 2k + 1$ or, equivalently, $s = 2k + t + 1$. Replacing on Eq.38 yields

$$\begin{aligned} r^2 = 48k^4 + 96k^3t + 64k^2t^2 + 16kt^3 \\ + 4t^4 + 96k^3 + 144k^2t \\ + 64kt^2 + 8t^3 + 72k^2 \\ + 72kt + 16t^2 + 24k \\ + 12t + 3. \end{aligned}$$

It can be seen that the left-hand-side of this equation has the form $4n + 3$ for some integer n, i.e., $4n + 3 = r^2$. However, $r^2 \not\equiv 3 \pmod 4$ for all $r \in \mathbb{Z}$, which leads to a contradiction.

(c) For $s, t \in \mathbb{Z}$ such that $\gcd(s, t) = 1$, s is odd and $3 \nmid t$,

$$\begin{cases} a = -3s^4 + 12t^2s^2 + 4t^4, \\ b = 3s^4 + 12t^2s^2 - 4t^4, \\ u = 6ts(3s^4 + 4t^4). \end{cases}$$

Since u is odd by hypothesis, this is a contradiction. Hence, this case does not need to be considered.

(d) For $s, t \in \mathbb{Z}$ such that $\gcd(s, t) = 1$, t is odd and $3 \nmid t$,

$$\begin{cases} a = -12^4 + 12t^2s^2 + t^4, \\ b = 12s^4 + 12t^2s^2 - t^4, \\ u = 6ts(12s^4 + t^4). \end{cases}$$

Since u is odd by hypothesis, this is a contradiction. Hence, this case does not need to be considered.

Therefore, there are no integer solutions to $x^4 - 4y^4 = z^3$ with $\gcd(x, y) = 1$, x odd, and $y \neq 0$. ■

The following result states the nonexistence of solutions to Eq.20 when $\gcd(x, y) = 1$ and x is even. Notice that these conditions automatically imply that $y \neq 0$. Thus, this result is "complementary" to the previous lemma considering exactly the same hypotheses, except for the fact that x is now even.

Lemma 3: *There are no integer solutions to* $x^4 - 4y^4 = z^3$ *with* $\gcd(x, y) = 1$ *and* x *even.*

Proof: Suppose $x = u$, $y = v$ and $z = g$ satisfy the equation $x^4 - 4y^4 = z^3$, with $\gcd(u, v)$=1 and u an even integer. Let $u = 2^e w$, with $e \geq 1$ and $\gcd(2, w) =$. Then,
$$(2^e w)^4 - 4v^4 = g^3,$$
from which
$$4(2^{4e-1}w^4 - v^4) = g^3.$$
It can be clearly seen that g is even. Hence, let $g = 2^f m$, with $f \geq 1$ and $\gcd(2, m) = 1$. Then,
$$4(2^{4e-2}w^4 - v^4) = 2^{3f}m^3,$$
or equivalently,

$$2^{4e-2}w^4 - v^4 = 2^{3f-2}m^3. \tag{39}$$

Since $f \geq 1$, then $3f - 2 \geq 1$, and the right-hand-side of Eq.39 is even. However, since $4e - 2 > 0$ and v is odd, the left-hand-side of Eq.39 must be odd. Therefore, there is a contradiction. As a conclusion, there are no integer solutions $x = u$, $y = v$ and $z = g$ to the equation $x^4 - 4y^4 = z^3$ such that $\gcd(u, v) = 1$ and u is even. ■

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The following result shows the nonexistence of nontrivial solutions to Eq.20 such that s and m are coprime.

Lemma 4: *There exist no integer solutions to the equation* $x^4 - 4y^4 = z^3$ *with* $\gcd(x, y) = 1$ *and* $y \neq 0$.

Proof: The direct consequence of Lemma 2 and Lemma 3. ■

Finally, the following theorem states the nonexistence of nontrivial solutions to Eq.20 when $|x| \neq |y|$, i.e.,the main result for (ii).

Theorem 2: *The equation* $x^4 - 4y^4 = z^3$ *has no integer solutions with* $|x| \neq |y|$ and $x, y \neq 0$.

Proof: The method of contradiction is employed. Suppose there exists a solution $x = s$, $y = m$ and $z = g$ to this equation with $|x| \neq |y|$ and $x, y \neq 0$. Then, $s^4 - 4m^4 = g^3$ with $|s| \neq |m|$. Let $d = \gcd(s, m)$, $u = \frac{s}{d}$ and $v = \frac{m}{d}$. Then, $\gcd(u, v) = 1$ and $v \neq 0$. Since $d \mid s$ and $d \mid m$, it yields $d^4 \mid g^3$. That is,

$$u^4 - 4v^4 = \frac{g^3}{d^4}, \quad 40$$

where $\frac{g^3}{d^4}$ is an integer. Let $= \frac{g^3}{d^4}$. Then, $wd^4 = g^3$, and thus $= w^{\frac{1}{3}} d^{\frac{4}{3}}$. Since g is an integer, there exist integers h and k such that $w = h^3$ and $d = k^3$. Replacing in Eq.40 yields $u^4 - 4v^4 = h^3$. Thus, (u, v, h) is a solution to the equation $x^4 - 4y^4 = z^3$ with $\gcd(u, v) = 1$. This contradicts Lemma 4. Therefore, there are no integer solutions $x = s, y = m$ and $z = g$ to the equation $x^4 - 4y^4 = z^3$ with $|s| \neq |m|$ and $x, y \neq 0$. ■

Corollary 1: *The only integer solutions to the equation* $x^4 - 4y^4 = z^3$ *with* $|x| \neq |y|$ *are* $(x, y, z) = (0, \pm 2^{3k+1}u^3, -2^{4k+2}u^4)$, *for* $k \geq 0$ *and* $u \in \mathbb{Z}$, *and* $(x, y, z) = (n^3, 0, n^4)$, *for* $n \in \mathbb{Z}$.

Proof: Notice that, given the assertion of the previous theorem, it is enough to prove that there existsolutions such that $|x| \neq |y|$ when $x = 0$ or $y = 0$. Indeed, suppose $x = 0$. Then, $-4y^4 = z^3$, which is the same as Eq.5 with $y = m$ and $z = g$. The solutions to this equation are given as $(x, y, z) = (0, \pm 2^{3k+1}u^3, -2^{4k+2}u^4)$, *for* $k \geq 0$ *and* $u \in \mathbb{Z}$.

On the other hand, suppose $y = 0$. Then,

$$x = z^{\frac{3}{4}}. \quad 41$$

Since x is an integer, there exists an integer n such that $z = n^4$. Replacing in Eq.41 yields $x = n^3$. ■

Remark 22*:* Although the previous corollary shows there exist solutions for Eq.20 with $|s| \neq |m|$, it does not need to be considered under the context of the case currently studied (i.e., **Case 2.4.3**) because one of the corresponding conditions is $m \neq 0$.

Remark 23*:* **Case 3.4.3** yields symmetrical solutions with

$$(r, t, m, g) = (9n^3, 9n^3, \pm 9n^3, -27n^4),$$
$$(r, t, m, g) = (9n^3, -9n^3, \pm 9n^3, -27n^4).$$

This leads to $\boxed{(a, b, c) = (9n^3, 9n^3(1 \pm i), -27n^4)}$ and $\boxed{(a, b, c) = (9n^3, -9n^3(1 \pm i), -27n^4).}$

Remark 24*:* **Case 4.2.3** yields symmetrical solutions with

$$(r, s, m, g) = (9n^3, \pm 9n^3, 9n^3, -27n^4),$$
$$(r, s, m, g) = (-9n^3, \pm 9n^3, 9n^3, -27n^4).$$

This leads to $\boxed{(a, b, c) = (9n^3(1 \pm i), 9n^3 i, -27n^4)}$ and $\boxed{(a, b, c) = (-9n^3(1 \pm i), 9n^3 i, -27n^4).}$

Remark 25*:* **Case 4.3.3** yields symmetrical solutions with

$$(r, s, t, g) = (9n^3, \pm 9n^3, 9n^3, -27n^4),$$
$$(r, s, t, g) = (-9n^3, \pm 9n^3, 9n^3, -27n^4).$$

This leads to $\boxed{(a, b, c) = (9n^3(1 \pm i), 9n^3, -27n^4)}$ and $\boxed{(a, b, c) = (-9n^3(1 \pm i), 9n^3, -27n^4).}$

Case 2.4.4: ($r = 0$, $s \neq 0$, $t \neq 0$, $m \neq 0$, $g \neq 0$, $h \neq 0$**)**

This case will not be considered as it requires a more in-depth analysis than what is intended in the current discussion.

Remark 26: **Case 3.4.4**, **Case 4.2.4** and **Case 4.3.4** are symmetrical and should have the same solutions.

Remark 27*:* **Case 4.4.2** and **Case 4.4.3** also fall beyond the scope of our current discussion; hence they will not be analyzed here.

Case 3. Table. 3, shows all possible combinations of values studied under this case. Notice that all these possibilities are symmetrical to previous subcases and have already been solved.

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Table 3. All possible combinations of values under Case 3.

Case 3: $r \neq 0, s = 0$	**Case 3.1:** $t = 0, m = 0$	**Case 3.1.1:** $g = 0, h = 0$
		Case 3.1.2: $g = 0, h \neq 0$
		Case 3.1.3: $g \neq 0, h = 0$
		Case 3.1.4: $g \neq 0, h \neq 0$
	Case 3.2: $t = 0, m \neq 0$	**Case 3.2.1:** $g = 0, h = 0$
		Case 3.2.2: $g = 0, h \neq 0$
		Case 3.2.3: $g \neq 0, h = 0$
		Case 3.2.4: $g \neq 0, h \neq 0$
	Case 3.3: $t \neq 0, m = 0$	**Case 3.3.1:** $g = 0, h = 0$
		Case 3.3.2: $g = 0, h \neq 0$
		Case 3.3.3: $g \neq 0, h = 0$
		Case 3.3.4: $g \neq 0, h \neq 0$
	Case 3.4: $t \neq 0, m \neq 0$	**Case 3.4.1:** $g = 0, h = 0$
		Case 3.4.2: $g = 0, h \neq 0$
		Case 3.4.3: $g \neq 0, h = 0$
		Case 3.4.4: $g \neq 0, h \neq 0$

Case 4. Table. 4 shows all possible combinations of values studied under this case.

Table 4. All possible combinations of values under Case 4.

Case 4: $r \neq 0, s \neq 0$	**Case 4.1:** $t = 0, m = 0$	**Case 4.1.1:** $g = 0, h = 0$
		Case 4.1.2: $g = 0, h \neq 0$
		Case 4.1.3: $g \neq 0, h = 0$
		Case 4.1.4: $g \neq 0, h \neq 0$
	Case 4.2: $t = 0, m \neq 0$	**Case 4.2.1:** $g = 0, h = 0$
		Case 4.2.2: $g = 0, h \neq 0$
		Case 4.2.3: $g \neq 0, h = 0$
		Case 4.2.4: $g \neq 0, h \neq 0$
	Case 4.3: $t \neq 0, m = 0$	**Case 4.3.1:** $g = 0, h = 0$
		Case 4.3.2: $g = 0, h \neq 0$
		Case 4.3.3: $g \neq 0, h = 0$
		Case 4.3.4: $g \neq 0, h \neq 0$
	Case 4.4: $t \neq 0, m \neq 0$	**Case 4.4.1:** $g = 0, h = 0$
		Case 4.4.2: $g = 0, h \neq 0$
		Case 4.4.3: $g \neq 0, h = 0$
		Case 4.4.4: $g \neq 0, h \neq 0$

Case 4.4.1: ($r \neq 0,\ s \neq 0,\ t \neq 0,\ m \neq 0,\ g = 0,\ h = 0$)

From Eq.3 and Eq.4, the following system of equations is obtained:

$$r^4 + t^4 - 6(r^2s^2 + t^2m^2) + s^4 + m^4 = 0$$
$$4(r^3s - rs^3 + t^3m - tm^3) = 0,$$

which, after reordering and factorization, can be written as

$$(r^2 - s^2)^2 - 4r^2s^2 + (t^2 - m^2)^2 - 4t^2m^2 = 0$$
$$4rs(r^2 - s^2) + 4tm(t^2 - m^2) = 0.$$

By letting $\alpha = r^2 - s^2$, $\beta = 2rs$, $\gamma = t^2 - m^2$, and $\delta = 2tm$, this can be further rewritten as

$$\alpha^2 - \beta^2 + \gamma^2 - \delta^2 = 0 \qquad 42$$
$$2\alpha\beta + 2\gamma\delta = 0. \qquad 43$$

Since $r, s \neq 0$, it is evident that $\beta \neq 0$. Thus, dividing both sides of Eq.43 by β yields

$$a = -\frac{\lambda\delta}{\beta}. \qquad 44$$

Replacing Eq.44 into Eq.42 and solving for β^2 yields $\beta^2 = \pm\gamma^2$, implying that $\beta = \gamma$ or $\beta = -\gamma$.

Suppose $\beta = \gamma$. Replacing on Eq.44 results in $\alpha = -\delta$. Then, the following system of equations is obtained given by

$$t^2 - m^2 = 2rs$$
$$r^2 - s^2 = -2tm.$$

Solving in a similar manner as the previous system, this results in $s^2 = \frac{(t+m)^2}{2}$ or $s^2 = -\frac{(t+m)^2}{2}$, both of which lead to a contradiction because s, t and m are integers.

Suppose now that $\beta = -\gamma$. Replacing on Eq.44 results in $\alpha = \delta$. Then, the following system of equations

$$t^2 - m^2 = -2rs$$
$$r^2 - s^2 = 2tm$$

is obtained. Observe that this system is the same as the previous one, but with t and m exchanged. Thus, no solutions exist. Therefore, the equation under this case is not considered.

Case 4.4.4: ($r \neq 0,\ s \neq 0,\ t \neq 0,\ m \neq 0,\ g \neq 0,\ h \neq 0$)

Observe that solving this case is equivalent to obtaining a general solution for Eq.1. The next section is dedicated exclusively to this endeavour.

The general form of solutions

In this section, the general form of the solutions to equation Eq.1 is studied. For this purpose, the knowledge gathered in the previous section is employed. The following conjecture states a reasonable property proved by Cohen[9] for the case of rational integers. Moreover, Theorem 3.1 of Ismail et al.[8] implies this assumption to be true for the case $x = y$ in Gaussian integers, and the results of our previous discussion seem to support it for $x \neq y$.

Conjecture 1: *Let* $(x, y, z) = (a, b, c)$ *be a solution to* $x^4 + y^4 = z^3$ *with* $xyz \neq 0$ *in Gaussian integers. Then,* $\gcd(a, b, c) \neq 1$. (Notice that this implies that $\gcd(a,b,c) \notin \{+1,-1,+i,-i\}$, i.e., the GCD is not a unit.)

Theorem 3: *The triplet $(x,y,z)=(a,b,c)$ is a solution to $x^4+y^4=z^3$ with $x \neq y$ in Gaussian integers if and only if there exist $\alpha,n,u,v \in \mathbb{Z}[i]$, with $u \neq v$ and $n=u^4+v^4$, such that*
$a = \alpha^3 n^{3k-1} u$, $b = \alpha^3 n^{3k-1} u$, *and* $c = \alpha^4 n^{4k}$
for some integer $k>0$.

Proof. It is trivially evident that if a,b,c have the stated form, then the triplet $(x,y,z)=(a,b,c)$ is a solution of the equation. On the other hand, suppose $(x,y,z)=(a,b,c)$ is a solution to the Diophantine equation $x^4+y^4=z^3$ with $x \neq y$ in Gaussian integers. The following proves that a,b,c have the form indicated in the theorem statement. Indeed,

$$a^4+b^4=c^3. \quad 45$$

Let $d=\gcd(a,b,c)$ and $u,v,w \in \mathbb{Z}[i]$ such that
$$a=du,\ b=dv \text{ and } c=dw. \quad 46$$
Since $a \neq b$, it is evident that $u \neq v$. Dividing both sides of Eq.45 by d^3 yields
$$d(u^4+v^4)=w^3. \quad 47$$
Let $n=u^4+v^4$. Then, Eq.47 becomes

$$dn=w^3. \quad 48$$
Suppose d is a cube, then n would also be a cube, i.e., $p^3=u^4+v^4$ for some $p \in \mathbb{Z}[i]$. But, $\gcd(u,v,p)=\gcd(u,v)=1$, which contradicts Conjecture 1. Therefore, d cannot be a cube, which yields $d=\alpha^3 n^{3k-1}$ for some $\alpha \in \mathbb{Z}[i]$ and some integer $k>0$. Then, Eq.48 implies $w=\alpha n^k$. Therefore, replacing on Eq.46 gives
$a=\alpha^3 n^{3k-1} u$, $b=\alpha^3 n^{3k-1} u$, and $c=\alpha^4 n^{4k}$
as asserted. ■

Corollary 2: *If the triplet $(x,y,z)=(a,b,c)$ is a solution to the Diophantine equation $x^4+y^4=z^3$ with $x \neq y$ in Gaussian integers, then $\gcd(a,b,c)=\alpha^3 n^{3k-1}$, where α, n and k are as in* Theorem 3.

Remark 28*:* This result applies to the equation $x^4+y^4=z^3$ with $x \neq y$ in rational integers $\mathbb{Z}$. Therefore, it generalizes the result of Theorem 1.3 in Ismail and Mohd Atan[4].

Conclusions:

In this work, the algebraic properties of the $x^4+y^4=z^3$ in Gaussian integers for $x \neq y$ have been examined. The main focus has been on studying some of the conditions that give rise to nontrivial solutions and their particular forms. Our findings show the existence of infinitely many solutions. Since the analytical method used in this study is based on simple algebraic properties, it can be easily generalized to study the behavior and conditions for the existence of solutions to other Diophantine equations, allowing a deeper understanding, even when there is no general solution is known. In the particular case of the current study, a general solution has been found to the equation $x^4+y^4=z^3$ in Gaussian integers, for $x \neq y$.

Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are ours. Besides, the Figures and images, which are not ours, have been given permission for re-publication and attached to the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at Universiti Sains Islam Malaysia.

Authors' contributions statement:

S.I., K.A.M.A and D.S. developed the presented idea. S.I. and K.A.M.A developed the theory, while D.S. performed the computations. S.I. and D.S. verified the analytical methods. K.A.M.A encouraged S.I. and D.S. to investigate unsolved cases and supervised the findings of this work. D.S. also assisted in enhancing the manuscript in terms of readability and proofreading. All authors discussed the results and contributed to the final manuscript.

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References:

1. Szabó S. Some fourth degree diophantine equations in gaussian integers. Integers Electron J Comb Number Theory. 2004; 4(A16): A16. http://emis.dsd.sztaki.hu/journals/integers/papers/e16/e16.pdf
2. Najman F. The Diophantine equation $x^4 \pm y^4 = iz^2$ in Gaussian integers. Am Math Mon. 2010; 117(7): 637–41. https://doi.org/10.4169/000298910X496769
3. Emory M. The diophantine equations $X^4 + Y^4 = D^2Z^4$ in quadratic fields. Integers Electron J Comb Number Theory. 2012; 12: A65. https://www.emis.de/journals/integers/papers/m65/m65.pdf
4. Ismail S, Mohd Atan KAM. On the Integral Solutions of the Diophantine Equation $x^4 + y^4 = z^3$. Pertanika J Sci Technol. 2013; 21(1): 119–26.

http://www.pertanika.upm.edu.my/pjst/browse/archives?article=JST-0391-2012 .
5. Izadi F, Naghdali RF, Brown PG. Some quartic diophantine equations in the gaussian integers. Bull Aust Math Soc. 2015; 92(2): 187–94. https://doi.org/10.1017/S0004972715000465
6. Izadi F, Rasool NF, Amaneh AV. Fourth power Diophantine equations in Gaussian integers. Proce Math Sci. 2018; 128(2): 1–6. https://doi.org/10.1007/s12044-018-0390-7
7. Söderlund GA. Note on the Fermat Quartic $34x^4 + y^4 = z^4$. Notes Number Theory Discrete Math. 2020; 26(4): 103–5.
8. Jakimczuk R. Generation of Infinite Sequences of Pairwise Relatively Prime Integers. Transnat J Math Anal Appl. 2021; 9(1): 9–21. https://www.researchgate.net/profile/Rafael-Jakimczuk/publication/353622249_generation_of_infinite_sequences_of_pairwise_relatively_prime_integers/links/
9. Ismail S, Mohd Atan KA, Sejas Viscarra D, Eshkuvatov Z. Determination of Gaussian Integer Zeroes of $F(x, z) = 2x^4 - z^3$. Malaysian J Math Sci. 2022; 16(2): 317–328.
10. Li A. The diophantine equations $x^4+2^ny^4=1$ in quadratic number fields. Bull Aust Math Soc. 2021; 104(1): 21–28. http://doi.org/10.1017/S0004972720001173
11. Somanath M, Raja K, Kannan J, Sangeetha V. on the Gaussian Integer solutions for an elliptic diophantine equations . Adv Appl Math Sci. 2021; 20(5): 815-822.
12. Ahmadi A, Janfada AS. On Quartic Diophantine Equations With Trivial Solutions In The Gaussian Integers. Int Electron. J Algebra. 2022; 31:134-142. https://doi.org/10.24330/ieja.964819
13. Tho NX. The equation $x^4+2^ny^4=z^4$ in algebraic number fields. Acta Math Hungar. 2022; 167(1): 309-331. https://doi.org/10.1007/s10474-022-01226-1
14. Tho NX. Solutions to $x^4 + py^4 = z^4$ in cubic number fields. Arch. Math. 2022; 119: 269–277. https://doi.org/10.1007/s00013-022-01744-y

حلول غاوسي الصحيحة لمعادلة ديوفانتين $x^4 + y^4 = z^3$ لـ $\mathbf{x \neq y}$

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الخلاصة:

تمت مناقشة حساب تحديد الحلول لمعادلة ديوفانتين $x^4 + y^4 = z^3$ على الحلقة الصحيحة الغاوسية للحالة المحددة لـ x≠ y.
تتضمن المناقشة نتائج أولية مختلفة استخدمت لاحقًا لبناء نظرية المذيب لمعادلة ديوفانتين التي تمت دراستها. تظهر النتائج التي توصلنا إليها وجود عدد لا حصر له من الحلول. نظرًا لأن الطريقة التحليلية المستخدمة هنا والتي تستند إلى خصائص جبرية بسيطة ، لذا يمكن تعميمها بسهولة لدراسة السلوك والشروط لوجود حلول لمعادلات ديوفانتين الأخرى ، مما يسمح بفهم أعمق ، حتى في حالة عدم وجود حل عام معروف.

الكلمات المفتاحية: الخصائص الجبرية، معادلة ديوفانتين، العدد الصحيح الغاوسي، المعادلة الرباعية، حلول غير بديهية، حلول متناظرة