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Semihollow-Lifting Modules and Projectivity

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Abstract:

Throughout this paper, T is a ring with identity and F is a unitary left module over T. This paper study the relation between semihollow-lifting modules and semiprojective covers. proposition 5 shows that If T is semihollow-lifting, then every semilocal T-module has semiprojective cover. Also, give a condition under which a quotient of a semihollow-lifting module having a semiprojective cover. proposition 2 shows that if K is a projective module. K is semihollow-lifting if and only if For every submodule A of K with $\frac{K}{A}$ is hollow, then $\frac{K}{A}$ has a semiprojective cover.

Keywords: Projective cover**,** Projective modules, Semihollow lifting modules.

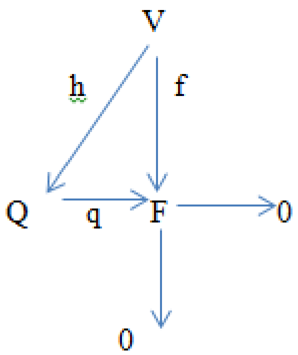
Introduction:

Let T be a ring with identity and F a unitary left module over T. A submodule E from a T-module F is called small on F ($E \ll F$) if whenever a submodule S of J with $F = E + S$ implies $S = F$ [1]. A submodule E of an T-module F is called semismall in F ($E \ll_S F$) if $E = 0$ or $E/V \ll F/V$ for every nonzero submodule V of E [2]. Let F be an T-module and let V,N be submodule of F such that $V \subset N \subset F$. K is called semicoessential submodule of N in F ($V \subseteq_{sce} N$ in F) if $\frac{N}{V} \ll_S \frac{F}{V}$ [3]. A non-zero T-module F is called a hollow module if every proper submodule of F is a small submodule of F [4]. An T-module F is called semihollow-lifting if for every submodule H` of F with $\frac{F}{H}$ hollow, there exists a submodule K of F such that $F = K \oplus K^*$ and $K \subseteq_{sce} H$ in F [5]. An T-module P is projective if and only if, for S, B are any two T-module and $Z:S \to B$ is epimorphism and for any homomorphism $L:P \to B$, ∃ a homomorphism $h:P \to S$ such that $Z \circ h = L$ [6].

A T-module U is called semiprojective cover of a T-module F if, U is projective and ∃ an epimorphism $\varphi: U \to F$ with $\ker\varphi \ll_S U$.

Lemma 1 Suppose that F is a T-module such that F has a semiprojective cover. If V is projective with an epimorphism $f: V \to F$, then V has a decomposition $V = V_1 \oplus V_2$ such that $V_1 \subseteq \ker f$ and $f|V_2 : V_2 \to F$ is a semiprojective cover of F.

Proof: Since V is projective, there is a commutative diagram



with exact row and column, as q is a small epimorphism and $qh = f$, since Q is projective, thus h splits, i.e. there is a monomorphism $g:Q \to V$ such that $hg = I_Q$ then $V = Img \oplus Kerh$. Now, put $V_1 = Img$ and $V_2 = kerh$. But $qh = f$, fg thus $V_1 \subseteq kerf$. Since $f(V_1) = f(V) = F$ then $V_1 \to F \to 0$ is exact, so is a projective cover from $fg = qhg = q$, it follows that $\ker(f|V_1) = g(\ker q)$, a small submodules of $g(Q) = V_1$; then $f|V_2 : V_2 \to F$ is a semiprojective cover of F.

Proposition 2 Let K is a projective module. Then the following statements are equivalent:

1. K is semihollow-lifting.
2. For every submodule A of K with $\frac{K}{A}$ is hollow, then $\frac{K}{A}$ has a semiprojective cover.

Proof: 1⇒2 Assume that K is a semihollow-lifting module. Let A be a submodule of K with $\frac{K}{A}$ is hollow. Thus there is a submodule A^* of A such that $K = A^* \oplus A^{**}$ and $A \cap A^{**} \ll_s A^{**}$. But K is projective, then by [6,Th.5.3.4], A^{**} is projective. Let $\pi: K \rightarrow \frac{K}{A} \rightarrow 0$ be the natural epimorphism, thus, $\pi|A^{**}: A^{**} \rightarrow \frac{K}{A} \rightarrow 0$ is an epimorphism, to see this, let $x+A \in \frac{K}{A}$. It is clear that $\pi(x) = x+A$. But $x \in K$ and $K = A^* \oplus A^{**}$, this implies that $x = a^*+a^{**}$, where $a^* \in A^*$, $a^{**} \in A^{**}$. Now, $\pi(x) = \pi(a^*+a^{**}) = \pi(a^*) + \pi(a^{**}) = \pi(a^{**})$, thus $\pi(a^{**}) = x + A$. Since $\ker(\pi| A^{**}) = A \cap A^{**}$ and A^{**} is projective, then $\pi| A^{**}: A^{**} \rightarrow \frac{K}{A} \rightarrow 0$ is a semiprojective cover of $\frac{K}{A}$.

2⇒1Let A be a submodule of K such that $\frac{K}{A}$ is hollow and let $\varphi: K \rightarrow \frac{K}{A}$ be the natural epimorphism. By(2),$\frac{K}{A}$ has a semiprojective cover.Thus by Lemma1, there exists a decomposition $K = K_1 \oplus K_2$ such that $\varphi|K_2: K_2 \rightarrow \frac{K}{A} \rightarrow 0$ is a semiprojective cover and $K_1 \subseteq \mathrm{Ker}\varphi$,this implies that $K_1 \subseteq A$ and $\ker(\varphi| K_2) = A \cap K_2 \ll_s K_2$. Then K is a semihollow-lifting module.

The following is an immediate corollary from prop. 2

Corollary 3 The following statements are equivalent for any ring T.

1. T is semihollow-lifting.
2. For every ideal J of T such that $\frac{T}{J}$ is hollow, then $\frac{T}{J}$ has a semiprojective cover.

A T-module F is called semisimple if every submodule of F is a direct summand of F [6].

Let N and L be submodules of a module F, N is called semisupplement of L in F if F = N + L and $N \cap L \ll_s N$ [7].

Now, give new definitions:

A T-module F is called semimaximal submodule if and only if the quotient F/D is a semisimple module.

A T-module F is called semilocal if F has a unique semimaximal submodule N which contents all proper submodule of F. For example, every semisupplement of a semimaximal submodule in a module is a semilocal module.

Proposition 4 Assume that F is a nonzero T-module, the following statements are equivalent:
1) F is semihollow and Rad(F) ≠F
2) F is semihollow and cyclic
3) F is semilocal

Proof: 1⇒2 Since $\mathrm{Rad}(F) \ll_s F$ and F/ Rad(F) semisimple then F is cyclic.

2⇒3⇒1 Clear.

Proposition 5 Assume that T is a ring. If T is semihollow-lifting, then every semilocal T-module F has semiprojective cover.

Proof: Suppose that T is semihollow-lifting. Let F be a semilocal T-module, thus by prop. 4, F = Ta for some $a \in F$. Define $\varphi: T \rightarrow Ta$, by $\varphi(t) = ta$, $\forall\, t \in T$. It is clear that φ is an epimorphism. Then by the first isomorphism theorem,$\frac{T}{\ker\varphi} \cong Ta$. It is clear that $\ker\varphi = \mathrm{Ann}(a)$. Thus $\frac{T}{\mathrm{Ann}(a)} \cong Ta$. This implies that $\frac{T}{\mathrm{Ann}(a)}$ is semilocal. Now, put A = Ann(a) so, $\frac{T}{A}$ is semihollow. But T is semihollow-lifting, thus ∃ an ideal A^* of T such that $A^* \subseteq A$, $T = A^* \oplus A^{**}$ and $A \cap A^{**} \ll_s A^{**}$. Let $\pi: T \rightarrow \frac{T}{A}$ be the natural epimorphism, thus, $\pi|A^{**}: A^{**} \rightarrow \frac{T}{A} \rightarrow 0$ is an epimorphism. It is clear that, $\ker(\pi| A^{**}) = A \cap A^{**}$, hence $\ker(\pi| A^{**}) \ll_s A^{**}$. Then $\pi| A^{**}: A^{**} \rightarrow \frac{T}{A}$ is a semiprojective cover of $\frac{T}{A}$. Hence F has a semiprojective cover.

Let F be an T-module. Let K and N be submodules of F. K is a strong semisupplement of N in F if K is a semisupplement of N in F and $K \cap N$ is a direct summand of N [8].

Theorem 6 Assume that F is a T-module, then F is semihollow-lifting if and only if for every submodule V of F with $\frac{F}{V}$ is hollow has a strong semisupplement in F.

Proof: Assume that F is a semihollow-lifting module and V is a submodule of F such that $\frac{F}{V}$ is hollow. Hence ∃ a submodule K of V such that $K \subseteq_{sce} V$ in F and $F = K \oplus K^*$, for some $K^* \subseteq F$. By modular law, $V = V \cap (K \oplus K^*) = K \oplus (V \cap K^*)$. It is easy to show that $F = V + K^*$. To show $V \cap K^* \ll_s K^*$. Let $(V \cap K^*) + D = K^*$, where $D \subseteq K^*$. So $F = K + K^* = K + (V \cap K^*) + D$. This implies that $F = V + D$ and $\frac{F}{K} = \frac{V+D}{K} = \frac{V}{K} + \frac{D+K}{K}$. But $K \subseteq_{sce} V$ in F, thus $F = D + K$. Since $F = K \oplus K^*$ and $D \subseteq K^*$, then $D = K^*$ and hence $V \cap K^* \ll_s K^*$. Thus V has a strong semisupplement K^* in F.

Conversely, Take V to be a submodule of F such that $\frac{F}{V}$ is hollow. Thus by our assumption there is a strong semisupplement K of V in F, then $F = V + K$, $N \cap K \ll_s K$ and $V = (V \cap K) \oplus L$, where $L \subseteq F$. Now, $F = V + K = (V \cap K) + L + K = L + K$. It is clear that $L \cap K = 0$, so $F = L \oplus K$. To show that $L \subseteq_{sce} V$ in F. Let $\frac{N}{L} + \frac{D}{L} = \frac{F}{L}$, where $D \subseteq F$ containing L, thus $V + D = F$. Hence $F = (V \cap K) + L + D$. Since $V \cap K \ll_s F$, then $F = L + D$. But $L \subseteq D$, thus $F = D$. Then F is semihollow-lifting.

A T-module F is said to have the (finite) exchange property if for any (finite) index set I, whenever $F \oplus N = \oplus_{i \in I} Ai$, for modules N and Ai, then $F \oplus N = F \oplus (\oplus_{i \in I} Bi)$ for submodules $Bi \subseteq Ai$ [9].

Lemma 7 Let F_0 be a direct summand of a module F such that F_0 has a finite exchange property. If $M_0 \subseteq V \subseteq F$ and V has a strong semisupplement in F, then $\frac{V}{F_0}$ has a strong semisupplement in $\frac{F}{F_0}$.

Proof: Let K be a strong semisupplement of V in F, then $F = V + K$, $V \cap K \ll_s K$ and $V = (V \cap K) \oplus L$, where $L \leq F$. So $F = V + K = (V \cap K) + L + K = L + K$ and hence $F = L \oplus K$. Since F_0 be a direct summand of F, thus $F = F_0 \oplus F_1$, for some $F_1 \leq F$. But F_0 has a finite exchange property, thus $F_0 \oplus F_1 = F_0 \oplus L^* \oplus K^*$, where $L^* \leq L$ and $K^* \leq K$. Hence $F = F_0 + V + K^*$. But $F_0 \leq V \leq F$, thus $F = V + K^*$. Since $F = V + K$, then by minimality of K, $K = K^*$. Now, put $L_1 = F_0 \oplus L^*$. But $F = F_0 \oplus L^* \oplus K^*$, thus $F = F_0 \oplus L^* \oplus K$ and $F = L_1 \oplus K$. Now, $\frac{F}{F_0} = \frac{L_1 \oplus K}{F_0} = \frac{L_1}{F_0} \oplus \frac{K+F_0}{F_0}$ and get, $\frac{F}{F_0} = \frac{F_0 \oplus L^*}{F_0} \oplus \frac{K+F_0}{F_0}$, but $F_0 \leq V$ and $L^* \leq V$ thus $F_0 + L^* \leq V$ and hence, $\frac{F}{F_0} = \frac{V}{F_0} + \frac{K+F_0}{F_0}$. But $\frac{V}{F_0} = \frac{V}{F_0} \cap \frac{F}{F_0} = \frac{V}{F_0} \cap \left(\frac{F_0 \oplus L^*}{F_0} \oplus \frac{K+F_0}{F_0}\right) = \frac{F_0 \oplus L^*}{F_0} \oplus \frac{(V \cap K)+F_0}{F_0}$, then $\frac{V}{F_0} \cap \frac{K+F_0}{F_0}$ is a direct summand of $\frac{V}{F_0}$. To show that $\frac{V}{F_0} \cap \frac{K+F_0}{F_0} \ll_s \frac{K+F_0}{F_0}$, define $g: K \to \frac{K+F_0}{F_0}$ as follows, $g(x) = x + F_0$, for all $x \in K$. clearly g is an epimorphism. But $V \cap K \ll_s K$, thus $g(V \cap K) = \frac{(V \cap K)+F_0}{F_0} \ll_s \frac{K+F_0}{F_0}$, and hence $\frac{V}{F_0} \cap \frac{K+F_0}{F_0} \ll_s \frac{K+F_0}{F_0}$. Then $\frac{K+F_0}{F_0}$ is a strong semisupplement of $\frac{V}{F_0}$ in $\frac{F}{F_0}$.

Proposition 8 Let F_0 be a direct summand of an T-module F such that F_0 has the finite exchange property. If F is a semihollow-lifting module, then $\frac{F}{F_0}$ is also semihollow-lifting.

Proof: Take a submodule B of F, such that $F_0 \leq B$ and $\frac{F/F_0}{B/F_0}$ is a hollow module. From (third isomorphism theorem), $\frac{F}{B} \cong \frac{F/F_0}{B/F_0}$ and thus $\frac{F}{B}$ is a hollow module. But F is semihollow-lifting, thus by Th. 6, B has a strong semisupplement in F. Since F_0 is a direct summand of F and has the finite exchange property, then by Lemma 7, $\frac{B}{F_0}$ has a strong semisupplement in $\frac{F}{F_0}$. Then by Th. 6, $\frac{F}{F_0}$ is a semihollow-lifting module.

Proposition 9 Assume that F is a semihollow-lifting module that has a semimaximal submodule. Then F has a semilocal submodule which is a direct summand.

Proof: Let F be a semihollow-lifting module and S be a semimaximal submodule of F. Then, $\frac{F}{S}$ is a semisimple module and hence $\frac{F}{S}$ is a semihollow module. Then, S has a strong semisupplement H in F.Thus, $F = S + H$, $S \cap H \ll_s H$ and $S = (S \cap H) \oplus L$, where $L \subseteq F$. Hence $F = H \oplus L$. Then H is a direct summand of F. But H is a semisupplement of a semimaximal submodule, thus H is semilocal.

The following proposition gives a decomposition of any projective hollow-lifting module.

Proposition 10 Assume that F be T-module and let X anon zero projective module, then there exists a semimaximal submodule in X.

Proof: Assume Rad P = P. To show that every finitely generated sub module $N \subset P$ is zero. Let $\{K_\lambda\}_\Lambda$ be a family of finitely generated (cyclic) modules in $\sigma[F]$ and $h: \oplus_\Lambda K_\lambda \to P$ an epimorphism. Since P projective, there exists $g: P \to \oplus_\Lambda K_\lambda$ with gh = idp with N, and also g(N), finitely generated there is a finite subset $E \subset \Lambda$ with $g(N) \subset \oplus_E K_\lambda$ with the canonical projection $\pi: \oplus_\Lambda K_\lambda \to \oplus_E K_\lambda$ obtain an endomorphism f:

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g πh of P with nf = ng πh = ngh = n for all n ∈N. Imf is contained in the finitely generated submodule $h(\bigoplus_E K_\lambda) \subset P$ which is superfluous in P (since Rad P=P) By [10, 22.2], f ∈ $Jac(End_R(P))$,i.e.1-f is an isomorphism and N⊂Ke(1-f) = 0.

A sequences $\{A_n\}$ of events is called a stable sequence if for every event B has the limit $\lim_{n\to\infty} p(A_n B)= Q(B)$ exists [11].

Theorem 11 Assume that I is a left ideal of T which is contained uniquely in the semimaximal left ideal F of T. Suppose T satisfies the following chain condition. For every c∈ F(I), the sequence of left ideals I ⊆ (I: α) ⊆ (I: α^2) ⊆…. stable. Then End (T/I) is semilocal.

Proof: If F is a left T-module and f : F → F is an epimorphism such that the sequence of submodules ker f ⊆ ker f^2 ⊆ …. stable, thus a standard argument displays that f is an isomorphism. Furthermore, if F is a hollow module with the property that every epimorphism from F to F is an isomorphism, thus End (F) is semilocal, in that case, non units would form an ideal. Thus it is enough to display that, for any endomorphism f: T/I→T/I, the above sequence becomes stable. If f ∈ End (T/I), then ∃ c ∈ F(I) such that f(t + I) = tc + I ∀ t + I ∈ T/I. Then for any n ≥ 1, Ker f^n = (I: c^n)/I and since the sequence (I: c) ⊆ (I: c^2) ⊆ .. stable, the proof is complete.

A T-module F is said to be an SLE module if its endomorphism ring End(F) is semilocal.

Theorem 12 Assume that F be an indecomposable T-module. if F is an SLE-module, then F has the finite exchange property.

Proof: Assume that S = End(F) is a semilocal ring and let f ∈ S. But S is semilocal, then either f or 1 − f is a unit in S. If f is unit, thus ${1_F}^2 = 1_F$ ∈ fS and ${0_F}^2 = 0_F$ ∈ (1−f)S. thus, by [12, cor.11.14(c)] F has the finite exchange property.

Proposition 13 Let F be a projective semihollow-lifting T-module. Then F = $B_1 \oplus B_2$, where B_1 is semilocal and B_2 is semihollow-lifting.

Proof: Take the projective semihollow-lifting module F. By prop.10, F has a semimaximal submodule. Then by prop. 9, F has a semilocal submodule which is a direct summand of F say B_1. Thus by Theorem11, End(B_1) is semilocal. Then by Theorem12, B_1 has the finite exchange property. Thus by prop. 8, $\frac{F}{B_1}$ is semihollow-lifting. But B_1 is a direct summand of F, thus F = $B_1 \oplus B_2$, for some $B_2 \subseteq$ F. By the (second isomorphism theorem), $\frac{F}{B_1} \cong B_2$. Then B_2 is semihollow-lifting.

Recall that a T-module F is called semilifting module or SD1-module if for every submodule V of F, there exists a direct summand Y of F such that Y $\subseteq_{sce}$ V in F [5].

Corollary 14 Let T be an indecomposable ring. Then the following are equivalent:
1. T is semilocal.
2. T is semilifting.
3. T is semihollow-lifting.

Proof: 1⇒2⇒3: Clear.

3⇒1Take T is semihollow-lifting. Since T is projective, thus by prop. 14, we have T = $B_1 \oplus B_2$, where B_1 is semilocal and B_2 is semihollow-lifting. Since R is indecomposable and $B_1 \neq 0$, then $B_2 = 0$ and hence T = B_1, Thus T is semilocal.

Conclusion:

This paper study the relation between semihollow-lifting modules and semiprojective covers. Also, give a condition under which a quotient of a semihollow-lifting module having a semiprojective cover.

1. Let K is a projective module. K is semihollow-lifting if and only if For every submodule A of K with $\frac{K}{A}$ is hollow, then $\frac{K}{A}$ has a semiprojective cover.
2. T is semihollow-lifting if and only if For every ideal J of T such that $\frac{T}{J}$ is hollow, then $\frac{T}{J}$ has a semiprojective cover.
3. If T is semihollow-lifting, then every semilocal T-module has semiprojective cover.
4. A module F is semihollow-lifting if and only if for every submodule V of F with $\frac{F}{V}$ is hollow has a strong semisupplement in F.
5. Let F_0 be a direct summand of an T-module F such that F_0 has the finite exchange property. If F is a semihollow-lifting module, then $\frac{F}{F_0}$ is semihollow-lifting.
6. Assume that F is a semihollow-lifting module that has a semimaximal submodule. Then F has a semilocal submodule which is a direct summand.
7. Let F be a projective semihollow-lifting T-module. Then F = $B_1 \oplus B_2$, where B_1 is semilocal and B_2 is semihollow-lifting.

Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in Diyala University.

Authors' contributions:

Mukdad Qaess Hussain, Anfal Hasan Dheyab and Rana Aziz Yousif contributed to the design and implementation of the research, to the analysis of the results and to the writing of the manuscript.

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مقاسات شبه الرفع المجوفة والاسقاطية

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الخلاصة:

لتكن T حلقة ذات عنصر محايد وليكن F مقاسا ايسر معرف على T. هذا البحث درس العلاقة بين المقاسات شبه الرفع المجوفة وغطاء المقاسات شبه الاسقاطية. بينت القضية 5 اذا كان T شبه رفع مجوف فان كل مقاس شبه محلي يمتلك غطاء شبه اسقاطي واعطى الشرط الذي يكون فيه المقاس الكسري للمقاسات شبه الرفع المجوفة يمتلك غطاء شبه اسقاطي. القضية 2 تبين انه اذا كان K مقاس اسقاطي فان K يكون شبه رفع مجوف اذا وفقط اذا كل مقاس جزئيA في K بحيث K/A مجوف فان K/A يمتلك غطاء شبه اسقاطي.

الكلمات المفتاحية: غطاء المقاسات الاسقاطية، المقاسات الاسقاطية، مقاسات شبه الرفع المجوفة.