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Solving the Hotdog Problem by Using the Joint Zero-order Finite Hankel - Elzaki Transform

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Abstract:

This paper is concerned with combining two different transforms to present a new joint transform FHET and its inverse transform IFHET. Also, the most important property of FHET was concluded and proved, which is called the finite Hankel – Elzaki transforms of the Bessel differential operator property, this property was discussed for two different boundary conditions, Dirichlet and Robin. Where the importance of this property is shown by solving axisymmetric partial differential equations and transitioning to an algebraic equation directly. Also, the joint Finite Hankel-Elzaki transform method was applied in solving a mathematical-physical problem, which is the Hotdog Problem. A steady state which does not depend on time was discussed for each obtained general solution, i.e. in the boiling and cooling states. To clarify the idea of temperature rise and fall over the time domain given in the problem, some figures were drawn manually using Microsoft PowerPoint. The obtained results confirm that the proposed transform technique is efficient, accurate, and fast in solving axisymmetric partial differential equations.

Keywords: Bessel differential operator, Boiling, Cooling, Elzaki transform, Finite Hankel transform, Hotdog, joint zero-order Finite Hankel - Elzaki transform, Partial Differential Equation.

Introduction:

The method of finite Hankel transform was first introduced by Sneddon in 1946 where the importance of this transform is shown by solving several initial-boundary problems that have an interest in solving axisymmetric physical mathematical problems [1]. In the early 1990s, a new integral transform, which is called the Sumudu transform, was introduced by Watugala that was used to find solutions to ordinary differential equations (ODEs).Elzaki TM modified the Sumudu transform and gave a new technique that was used in several applications such as solving initial and boundary problems [2,3] and also solving partial differential equations (PDEs) [4], where many phenomena that arise in the mathematical physics field can be described by partial differential equations (PDEs) [5].

The partial differential equations, appear in many applications of mathematics, physics, chemistry, and engineering, for this reason, the researcher presents a number of methods for solving them, such as Elzaki [6], Laplace and Hankel transform…etc. additionally, some researchers addressed these transforms and combined them [7] and the primary purpose of this research is to demonstrate an efficient new double transform. [8,9]

Double integral transforms, their properties, and theories are still new and under study [10], the Finite Hankel and Elzaki transform are of great importance in solving PDEs, where applying each signal transform to a PDE gives us an ODE, but combining these two transforms into one transform brings us directly to an algebraic equation instead of converting to ODE, when applying the joint transform to a partial differential equation, and this method is faster in solving than applying each transform individually.

This method is very effective in solving differential equations that are related to the engineering and physical sciences compared to other methods as they need only one step to get the exact solution, while the other methods need more steps to get the exact solution .[11,12]

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Consequently, the importance of the research lies in using a new method that was first used in solving axisymmetric PDEs with initial and boundary conditions, which is the joint finite Hankel - Elzaki transform method, and this research aims to develop methods for solving PDEs in faster ways that give an algebraic equation directly, and the joint zero-order finite Hankel – Elzaki transform has been studied in particular in this research, as this method transfers us directly to an algebraic equation.

The main aim of this research is to define the FHET method and apply it to solve the hot dog problem, which consists of a heat equation and two different cases: the first stage is the boiling stage with an initial condition and a boundary condition (Dirichlet) and the second stage is the cooling stage with an initial condition and a boundary condition (Robin), and then applying the FHET method to find the general solution to the heat equation at each stage, studying the steady state of the general solution to the heat equation also at each stage, and drawing an illustration of the time-domain for the rise and fall of temperature, and also study and draw the center point temperature.

Materials and Methods:

Basic Concepts:

Cooling: Cooling is a process in which heat is drawn or removed. It is the technique that specializes in lowering the temperature of a space or a substance to a temperature lower than the temperature of the surrounding atmosphere and maintaining it continuously at this lowered degree.

Boiling: Boiling is a mode of vaporization (the change of phase of a substance from a liquid to a gas), which is (i.e., boiling) the rapid transformation of a liquid into vapor when its temperature reaches its boiling point, the temperature at which the vapor pressure of the liquid becomes equal to the external pressure applied to its surface of this liquid

Hotdog: A hot dog (sausage) is a long thin casing containing minced meat with seasonings and spices. This covering is usually the animal intestine. However, sometimes you will see synthetic casings as well. Some types of sausage go through the cooking process when they are prepared. Sometimes the casing is removed after that.

Bessel Functions:[13]

Bessel functions are solutions to the differential equation

$$x^2y'' + xy' + (x^2 - v^2)y = 0 \qquad 1$$

where v is a parameter, $J_v(x)$ is the Bessel function of the first kind and order v and can be called a cylindrical function.

$J_v(x)$ is defined by:

$$J_v(x) = \sum_{m=0}^{\infty}(-1)^m \frac{\left(\frac{x}{2}\right)^{2m+v}}{m!\Gamma(m+v+1)} \quad ; v \geq 0 \qquad 2$$

Bessel Differential Operator: [13]

The Bessel differential operator

$$\Delta_v = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{v^2}{r^2} = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) - \frac{v^2}{r^2} \qquad 3$$

is derived from the Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \qquad 4$$

after the separation of variables in cylindrical coordinates (r,θ,z).

The Bessel differential operator may be a useful tool in solving problems with cylindrical symmetry.

Some Recurrence Relations of Bessel Functions: [13]

$$J'_v(x) = \frac{v}{x}J_v(x) - J_{v+1}(x) \qquad 5$$

$$\frac{d}{dx}\left(x^vJ_v(x)\right) = x^vJ_{v-1}(x) \qquad 6$$

where $v \geq 0$ and integer.

The Binomial Series: [13]

The binomial series is defined by:

$$(1+x)^{-\alpha} = \sum_{n=0}^{\infty}\binom{-\alpha}{n}x^n \quad ; |x| < 1 \qquad 7$$

where:

$$\binom{-\alpha}{n} = \frac{(-1)^n(\alpha)_n}{n!} \quad \Rightarrow (1+x)^{-\alpha} = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}(-x)^n \quad ; |x| < 1 \qquad 8$$

where $(\alpha)_n$ is called Pochhammer Symbol or Appel's Symbol and is defined by:

$$(\alpha)_n = \frac{1.2\ldots(\alpha-1)\alpha(\alpha+1)\ldots\ldots(\alpha+n-1)}{1.2\ldots(\alpha-1)} = \frac{(\alpha+n-1)!}{(\alpha-1)!} = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \qquad 9$$

Legendre Formula:[1]

Legendre Formula has the form:

$$\sqrt{\pi}\Gamma(2x) = 2^{2x-1}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) \qquad (10)$$

Finite Hankel Transform: [1]

Definition 1: The finite Hankel transform (FHT) of order v of a function $f(r)$ is denoted by $\tilde{f}_v(k_i) = H_v\{f(r)\}$ and is defined by

$$\tilde{f}_v(k_i) = H_v\{f(r)\} = \int_0^a rf(r)J_v(rk_i)\,dr \;;\; v > -\frac{1}{2} \qquad 11$$

where $rJ_v(rk_i)$ kernel finite Hankel transform ; $r \geq 0$ and $J_v(rk_i)$ is the Bessel function of the first kind and order v.

where the sufficient condition for the function $f(r)$ to exist is:

- $f(r)$ is a piecewise continuous function on the interval $[0,a]$.
- $\int_0^a \sqrt{r}|f(r)|dr < \infty$.

Definition 2: The inverse finite Hankel transform (IFHT) of order v is denoted by $H_v^{-1}\{\tilde{f}_v(k_i)\} = f(r)$ and is defined by

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$$H_v^{-1}\{\tilde{f}_v(k_i)\} = f(r) = \frac{2}{a^2}\sum_{i=1}^{\infty}\tilde{f}_v(k_i)\frac{J_v(rk_i)}{J^2{}_{v+1}(ak_i)} \quad 12$$

where the summation is taken over all positive roots of $J_v(ak_i) = 0$.
$J_v(ak_i) = 0$ is called condition Dirichlet.
Note 1: If $v = 0$ then $H_0\{f(r)\}$ is called zero-order finite Hankel transform.
Example 1: let $f(r) = r^v$ then by applying Eq. 11

$$\tilde{f}_v(k_i) = H_v\{r^v\} = \int_0^a r^{v+1}J_v(rk_i)\,dr$$
$$= \frac{1}{k_i}\int_0^a \frac{d}{dr}r^{v+1}J_{v+1}(rk_i)\,dr$$
$$= \frac{a^{v+1}}{k_i}\,J_{v+1}(ak_i)$$

using the relation 6.

Basic Operational Properties of Finite Hankel Transform:[1]

- If $\tilde{f}_v(k_i) = H_v\{f(r)\}$ Finite Hankel Transform of the $f(r)$ and If $\tilde{g}_v(k_i) = H_v\{g(r)\}$ Finite Hankel Transform of the $g(r)$ then:

$$H_v\{c_1 f(r) + c_2 g(r)\} = c_1 H_1\{f(r)\} + c_2 H_2\{g(r)\} = c_1\tilde{f}_v(k_i) + c_2\tilde{g}_v(k_i) \quad 13$$

- If $\tilde{f}_v(k_i) = H_v\{f(r)\}$ Finite Hankel Transform of the $f(r)$ then:

$$1 - H_v\{f'^{(r)}\} = \frac{k_i}{2v}\left((v-1)H_{v+1}\{f(r)\} - (v+1)H_{v-1}\{f(r)\}\right) \quad ; \quad v \geq 1 \quad 14$$

where k_i positive roots of $J_v(ak_i) = 0$.

$$2 - H_v\left\{\frac{1}{r}\frac{d}{dr}\left(rf'(r)\right) - \frac{v^2}{r^2}f(r)\right\} = ak_i f(a)J_{v+1}(ak_i) - k_i{}^2\tilde{F}_v(k_i) \quad 15$$

where k_i positive roots of $J_v(ak_i) = 0$.

$$3 - H_v\left\{\frac{1}{r}\frac{d}{dr}\left(rf'(r)\right) - \frac{v^2}{r^2}f(r)\right\} = a\left(f'(a) + hf(a)\right)J_v(ak_i) - k_i{}^2\tilde{F}_v(k_i) \quad 16$$

where k_i positive roots of $hJ_v(ak_i) + k_i J'_v(ak_i) = 0$.

Modified Finite Hankel Transform: [13]

Definition 3: The finite Hankel transform of order v of a function $f(r)$ is defined by

$$\tilde{f}_v(k_i) = H_v\{f(r)\} = \int_0^a rf(r)J_v(rk_i)\,dr \;\; ; \;\; v > -\frac{1}{2} \quad 17$$

If k_i positive roots of $hJ_v(ak_i) + k_i J'_v(ak_i) = 0$ then Eq.17 is called modified finite Hankel transform (MFHT).

Definition 4: The inverse modified finite Hankel transform of order v (IMFHT) is defined by

$$H_v^{-1}\{\tilde{f}(k_i)\} = f(r) = 2\sum_{i=1}^{\infty}\frac{k_i{}^2}{\left((h^2+k_i{}^2)a^2 - v^2\right)}\frac{J_v(rk_i)}{(J_v(ak_i))^2}\,\tilde{f}_v(k_i) \quad 18$$

where k_i positive roots of $hJ_v(ak_i) + k_i J'_v(ak_i) = 0$.
$hJ_v(ak_i) + k_i J'_v(ak_i) = 0$ is called condition Robin.

Elzaki Transform (Modified Sumudu Transform): [3,14]

Definition 5: The Elzaki transform (ET) of a function $f(t)$ is denoted by $T(p) = E\left(f(t)\right)$ and is defined by

$$T(p) = E\left(f(t)\right) = p\int_0^{\infty} e^{-\frac{t}{p}}f(t)\,dt \quad 19$$

where $f(t)$ belonging to the set A defined by

$$A = \left\{f(t); \exists M, k_1, k_2 > 0\; ; \; |f(t)| < Me^{\frac{|t|}{k_i}}, if\;\; t \in (-1)^j \times [0,\infty[\right\}$$

also the sufficient condition for the function $f(t)$ to exist is:

- $f(t)$ should be piecewise continuous on the interval $[0,\infty]$.
- $f(t)$ should be of exponential order.

Definition 6: If $f(t)$ is original function, and if $E\{f(t)\} = T(p)$ then $E^{-1}\{T(p)\} = f(t)$ is called the inverse function of the Elzaki transform (IET).

Basic Properties of the Elzaki Transform: [3,4]

$$E\left(\frac{df(t)}{dt}\right) = \frac{1}{p}T(p) - pf(0) \quad 20$$
$$\Rightarrow \;\; E\left(\frac{df(x,t)}{dt}\right) = \frac{1}{p}T(x,p) - pf(x,0) \quad 21$$
$$E\{t^n\} = n!\,p^{n+2} = \Gamma(n+1)p^{n+2} \quad 22$$

Elzaki Transform of Some Basic Functions: [3,14]

If $f(t) = 1$ then:

$$E(1) = p\int_0^{\infty} e^{-\frac{t}{p}}\,dt = -p^2\, e^{-\frac{t}{p}}\Big|_0^{\infty} = p^2 \quad 23$$

if $f(t) = e^{-at}$ then:

$$E(e^{-at}) = p\int_0^{\infty} e^{-t\left(\frac{1}{p}+a\right)}\,dt = \frac{p}{\frac{1}{p}+a}\,e^{-t\left(\frac{1}{p}+a\right)}\Big|_0^{\infty}$$
$$= \frac{p^2}{1+ap} \quad \Rightarrow$$
$$E^{-1}\left(E(e^{-at})\right) = E^{-1}\left(\frac{p^2}{1+ap}\right) \;\; \Rightarrow \;\; E^{-1}\left(\frac{p^2}{1+ap}\right) = e^{-at} \quad 24$$

if $f(t) = 1 - e^{-at}$ then:

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$$E(1-e^{-at}) = E(1) - E(e^{-at}) = p^2 - \frac{p^2}{1+ap} = \frac{p^2+ap^3-p^2}{1+ap} = \frac{ap^3}{1+ap} \Rightarrow$$

$$E^{-1}E(1-e^{-at}) = aE^{-1}\left(\frac{p^3}{1+ap}\right) \Rightarrow$$

$$E^{-1}\left(\frac{p^3}{1+ap}\right) = \frac{1}{a}(1-e^{-at}) \qquad 25$$

The Relationship between Hankel Transform and Elzaki Transform

The Hankel transform (HT) of a function $f(r)$ is denoted by $F_v(s) = H_v\{f(r)\}$ and is defined by

$$F_v(s) = H_v\{f(r)\} = \int_0^\infty rJ_v(rs)f(r)\,dr\,;v \geq -\frac{1}{2} \qquad 26$$

Then the relationship between Hankel Transform and Elzaki Transform is given in the following form

$$F_v(s) = H_v\left\{ae^{-\frac{r}{a}}f(r)\right\} = a\int_0^\infty rJ_v(rs)e^{-\frac{r}{a}}f(r)\,dr = E\{rJ_v(rs)f(r)\} = T(s) \qquad 27$$

Example 2:

$$H_v\left\{ar^ve^{-\frac{r}{a}}\right\} = a\int_0^\infty rJ_v(rs)r^ve^{-\frac{r}{a}}\,dr = E\{r^{v+1}J_v(rs)\} = a\int_0^\infty r^{v+1}J_v(rs)e^{-\frac{r}{a}}\,dr \qquad 28$$

Substitution Eq. 2 in Eq. 28 and using Eq. 22, then

$$H_v\left\{ar^ve^{-\frac{r}{a}}\right\} = a\int_0^\infty r^{v+1}e^{-\frac{r}{a}}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{rs}{2}\right)^{2m+v}}{m!\,\Gamma(m+v+1)}dr$$

$$= a\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{s}{2}\right)^{2m+v}}{m!\,\Gamma(m+v+1)}\int_0^\infty r^{2v+2m+1}e^{-\frac{r}{a}}\,dr$$

$$= \sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{s}{2}\right)^{2m+v}}{m!\,\Gamma(m+v+1)}E\{r^{2v+2m+1}\}$$

$$= \sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{s}{2}\right)^{2m+v}}{m!\,\Gamma(m+v+1)}\Gamma(2v+2m+2)a^{2v+2m+3}$$

$$= \sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{s}{2}\right)^{2m+v}}{m!\,\Gamma(m+v+1)}\Gamma\big(2(v+m+1)\big)a^{2v+2m+3}$$

From Eq.10, then

$$\sqrt{\pi}\Gamma\big(2(v+m+1)\big) = 2^{2(v+m+1)-1}\Gamma(v+m+1)\Gamma\left(v+m+1+\frac{1}{2}\right)$$

$$\Gamma\big(2(v+m+1)\big) = \frac{2^{2v+2m+1}}{\sqrt{\pi}}\Gamma(v+m+1)\Gamma\left(v+m+\frac{3}{2}\right) \Rightarrow$$

$$H_v\left\{ar^ve^{-\frac{r}{a}}\right\} = \sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{s}{2}\right)^{2m+v}}{m!\,\Gamma(m+v+1)}a^{2v+2m+3}\left(\frac{2^{2v+2m+1}}{\sqrt{\pi}}\Gamma(v+m+1)\Gamma\left(v+m+\frac{3}{2}\right)\right)$$

$$= \frac{1}{\sqrt{\pi}}s^v2^{v+1}a^{2v+3}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}(as)^{2m}\Gamma\left(v+m+\frac{3}{2}\right)$$

From Eq. 9, then

$$\left(v+\frac{3}{2}\right)_m = \frac{\Gamma\left(v+m+\frac{3}{2}\right)}{\Gamma\left(v+\frac{3}{2}\right)} \Rightarrow$$

$$= \frac{1}{\sqrt{\pi}}s^v2^{v+1}a^{2v+3}\Gamma\left(v+\frac{3}{2}\right)\sum_{m=0}^{\infty}\frac{(-a^2s^2)^m}{m!}\left(v+\frac{3}{2}\right)_m$$

$$= \frac{1}{\sqrt{\pi}}s^v2^{v+1}a^{2v+3}\Gamma\left(v+\frac{3}{2}\right)(1+a^2s^2)^{-\left(v+\frac{3}{2}\right)}$$

$$= \frac{1}{\sqrt{\pi}}s^v2^{v+1}a^{2v+3}\Gamma\left(v+\frac{3}{2}\right)(1+a^2s^2)^{-\left(v+\frac{3}{2}\right)}$$

$$= \frac{1}{\sqrt{\pi}}s^v2^{v+1}a^{2v+3}\Gamma\left(v+\frac{3}{2}\right)\frac{1}{(1+a^2s^2)^{\left(v+\frac{3}{2}\right)}}$$

Results and Discussion:

Definition 7: Let $f(r,t)$ is a defined, piecewise continuous, and integral function on $[0,\infty[\times[0,a]$, then the joint v –order finite Hankel-Elzaki transform (FHET) is denoted by $\hat{\bar{F}}(k_i,p) = FH_vE\{f(r,t)\}$ and is defined by

$$\hat{\bar{F}}(k_i,p) = FH_vE\{f(r,t)\} = p\int_0^\infty\int_0^a f(r,t)rJ_v(rk_i)e^{-\frac{t}{p}}dr\,dt\,;\ v > -\frac{1}{2} \qquad 29$$

note that these integrals exist. where $rJ_v(rk_i)e^{-\frac{t}{p}}$ kernel joint finite Hankel-Elzaki transform; $r \geq 0$ and $J_v(rk_i)$ is the Bessel function of the first kind and order v.

Note 2: If $v = 0$ then $FH_0E\{f(r,t)\}$ is called the joint zero-order finite Hankel-Elzaki transform.

Definition 8: The inverse joint v–order finite Hankel-Elzaki transform (IFHET) is denoted by $FH_vE^{-1}\left\{\hat{\bar{F}}(k_i,p)\right\}=f(r,t)$ and is defined by

$$f(r,t)=\frac{2}{a^2}\sum_{i=1}^{\infty}E^{-1}\left(\hat{\bar{F}}(k_i,p)\right)\frac{J_v(rk_i)}{\left(J_{v+1}(ak_i)\right)^2} \quad 30$$

where the summation is taken over all positive roots of $J_v(ak_i)=0$.

And the inverse joint v –order finite Hankel-Elzaki transform is defined by

$$f(r,t)=\sum_{i=1}^{\infty}E^{-1}\left(\hat{\bar{F}}(k_i,p)\right)\frac{2k_i^2}{\left((h^2+k_i^2)a^2-v^2\right)}\frac{J_v(rk_i)}{\left(J_v(ak_i)\right)^2} \quad 31$$

where k_i positive roots of $hJ_v(ak_i)+k_iJ'_v(ak_i)=0$.

Property of FHET: Let's prove one of the most important properties of FHET which is called the Bessel differential operator property.

Let $\hat{\bar{F}}(k_i,p)=FH_vE\{f(r,t)\}$ then:

$$FH_vE\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}f(r,t)\right)-\frac{v^2}{r^2}f(r,t)\right\}=p\int_0^a\int_0^\infty r\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}f(r,t)\right)-\frac{v^2}{r^2}f(r,t)\right)J_v(rk_i)e^{-\frac{t}{p}}drdt$$

$$=p\int_0^\infty\int_0^a\left(\frac{d}{dr}\left(r\frac{d}{dr}f(r,t)\right)\right)J_v(rk_i)e^{-\frac{t}{p}}drdt-p\int_0^\infty\int_0^a\left(\frac{v^2}{r}f(r,t)\right)J_v(rk_i)e^{-\frac{t}{p}}drdt \quad 32$$

Suppose:

$$I=\int_0^a\left(\frac{d}{dr}\left(r\frac{d}{dr}f(r,t)\right)\right)J_v(rk_i)dr$$

which is, invoking integration by parts

$$u=J_v(rk_i)\ \Rightarrow\ du=k_iJ'_v(rk_i)dr$$

$$dv=\frac{d}{dr}\left(r\frac{d}{dr}f(r,t)\right)dr\ \Rightarrow\ v=r\frac{d}{dr}f(r,t)$$

then

$$I=r\frac{d}{dr}f(r,t)J_v(rk_i)\Big|_0^a-k_i\underbrace{\int_0^a r\frac{d}{dr}f(r,t)J'_v(rk_i)\,dr}_{I_1}$$

invoking integration by parts again of I_1

$$u=rJ'_v(rk_i)\quad\Rightarrow\ du=J'_v(rk_i)+rk_iJ''_v(rk_i)$$

$$dv=\frac{d}{dr}f(r,t)dr\ \Rightarrow\ v=f(r,t)$$

then

$$I_1=-k_irf(r,t)J'_v(rk_i)\Big|_0^a+k_i\int_0^a\left(f(r,t)J'_v(rk_i)+rk_if(r,t)J''_v(rk_i)\right)dr\ \Rightarrow$$

$$I=r\frac{d}{dr}f(r,t)J_v(rk_i)\Big|_0^a-k_irf(r,t)J'_v(rk_i)\Big|_0^a+k_i\int_0^a\left(f(r,t)J'_v(rk_i)+rk_if(r,t)J''_v(rk_i)\right)dr$$

Substitution in Eq. 32 becomes

$$FH_vE\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}f(r,t)\right)-\frac{v^2}{r^2}f(r,t)\right\}=p\int_0^\infty\left(rf'(r,t)J_v(rk_i)|_0^a-k_irf(r,t)J'_v(rk_i)\Big|_0^a\right)e^{-\frac{t}{p}}dt$$

$$+p\int_0^\infty\left(\int_0^a\left(k_if(r,t)J'_v(rk_i)+rk_i^2f(r,t)J''_v(rk_i)-\frac{v^2}{r}f(r,t)J_v(rk_i)\right)dr\right)e^{-\frac{t}{p}}dt$$

$$=p\int_0^\infty\left(af'^{(a,t)J_v(ak_i)}-ak_if(a,t)J'_v(ak_i)\right)e^{-\frac{t}{p}}dt+p\int_0^\infty\left(\frac{1}{r}\int_0^a\left((rk_i)^2J''_v(rk_i)+(rk_i)J'_v(rk_i)-v^2J_v(rk_i)\right)f(r,t)\,dr\right)e^{-\frac{t}{p}}dt \quad 33$$

from Eq.1, then

$$(rk_i)^2J''_v(rk_i)+(rk_i)J'_v(rk_i)+((rk_i)^2-v^2)J_v(rk_i)=0\ \Rightarrow$$

$$(rk_i)^2J''_v(rk_i)+(rk_i)J'_v(rk_i)-v^2J_v(rk_i)=-(rk_i)^2J_v(rk_i)$$

Substitution in Eq. 33 becomes

$$=p\int_0^\infty\left(af'(a,t)J_v(ak_i)-ak_if(a,t)J'_v(ak_i)\right)e^{-\frac{t}{p}}dt-p\int_0^\infty\left(\frac{1}{r}\int_0^a(rk_i)^2J_v(rk_i)f(r,t)\,dr\right)e^{-\frac{t}{p}}dt$$

$$= p\int_0^\infty \left(af'(a,t)J_v(ak_i) - ak_i f(a,t)J'_v(ak_i)\right)e^{-\frac{t}{p}}dt - p\int_0^\infty \left(k_i^2\int_0^a rJ_v(rk_i)f(r,t)\,dr\right)e^{-\frac{t}{p}}dt$$

$$= p\int_0^\infty \left(af'(a,t)J_v(ak_i) - ak_i f(a,t)J'_v(ak_i)\right)e^{-\frac{t}{p}}dt - k_i^2\hat{\tilde{F}}(k_i,p) \quad 34$$

Let's discuss the following cases:

1- If k_i positive roots of $J_v(ak_i) = 0$ and suppose $f(a,t) = A\,; A = constant$ and apply relation 23, then become Eq. 34

$$FH_vE\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}f(r,t)\right) - \frac{v^2}{r^2}f(r,t)\right\} = -aAp^2k_iJ'_v(ak_i) - k_i^2\hat{\tilde{F}}(k_i,p)$$

from relation 5 then

$$J'_v(ak_i) = \frac{v}{ak_i}J_v(ak_i) - J_{v+1}(ak_i)$$

$$J_v(ak_i) = 0 \;\Rightarrow\; J'_v(ak_i) = -J_{v+1}(ak_i) \;\Rightarrow$$

$$FH_vE\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}f(r,t)\right) - \frac{v^2}{r^2}f(r,t)\right\} = aAp^2k_iJ_{v+1}(ak_i) - k_i^2\hat{\tilde{F}}(k_i,p) \quad 35$$

2- If k_i positive roots of $hJ_v(ak_i) + k_iJ'_v(ak_i) = 0$ then $k_iJ'_v(ak_i) = -hJ_v(ak_i)$ and suppose $f'(a,t) + hf(a,t) = B\;; B = constant$ and apply relation 23, then become Eq. 34

$$FH_vE\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}f(r,t)\right) - \frac{v^2}{r^2}f(r,t)\right\} = \int_0^\infty \left(a\left(f'(a,t) + hf(a,t)\right)J_v(ak_i) - k_i^2\tilde{F}(k_i,t)\right)e^{-\frac{t}{p}}dt$$

$$= ap^2BJ_v(ak_i) - k_i^2\hat{\tilde{F}}(k_i,p) \quad 36$$

Algorithm for Finding the General Solution (Solution Steps):

1- Applying $\hat{\tilde{F}}(k_i,p) = FH_vE\{f(r,t)\}$ to both sides of the partial differential equation given.

2- Applying the property

$$FH_vE\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}f(r,t)\right) - \frac{v^2}{r^2}f(r,t)\right\}$$

3- Substituting the appropriate boundary condition either Dirichlet or Robin.

4- Applying the property

$$E\left(\frac{df(x,t)}{dt}\right) = \frac{1}{p}T(x,p) - pf(x,0)$$

5- Substituting the appropriate given initial condition produces an algebraic equation.

6- Taking the inverse transform $FH_vE^{-1}\left\{\hat{\tilde{F}}(k_i,p)\right\} = f(r,t)$ fitting either Eq. 30 or Eq. 31

it produces the general solution

Application of FHET:

The joint zero-order finite Hankel-Elzaki transform (FHET) will be applied to the following problem:

Hotdog Problem

Let the heat equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u(r,t)}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial u(r,t)}{\partial t} \quad 37$$

where $0 \le r < r_1$ Fig. 1 and α heat diffusion constant

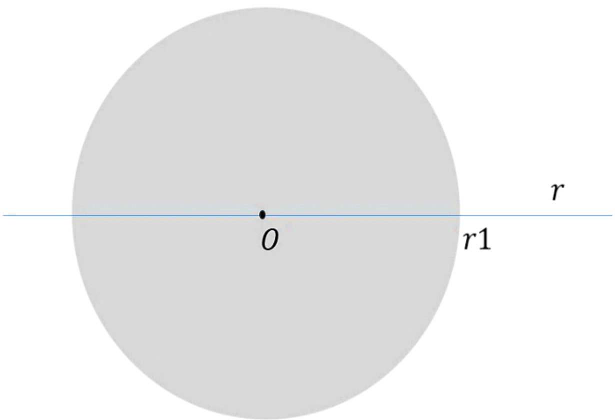

Figure 1. Represents the domain of study

Let's study the hotdog problem as follows: Let's have an ambient (the liquid), a body (the hotdog), and an external source for heating.

The first stage: the hotdog cooking stage (Boiling) Heating of the liquid begins with an initial temperature T_0 and this heat is derived from the external source, and the liquid heating continues until it reaches the saturated temperature T_{sat} (boiling) or it can be said that $T_{sat} = T_{max}$ which is the maximum temperature that the ambient (the liquid) reaches, after the liquid reaches T_{sat}, the body (hotdog) is put inside the boiling liquid with the persistence of the external source of heating Fig. 2, In the beginning, the hotdog's temperature is T_0, T_0 begins to change by gaining heat from the boiling liquid (ambient temperature) Fig. 3, the process of heat exchange between the liquid and the hotdog is carried out by Eq. 37 Which describes the heat transfer during time $0 \le t \le t_1$ Fig. 4, where the heat exchange process takes place from the hot medium to the cold body (from the top to the bottom) until a thermal equilibrium between the liquid and the hotdog is reached, that is until the temperature becomes equal between the ambient

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and the body which is T_{sat} (that is the hotdog has been cooked) at the moment $t = t_1$ Fig. 5 and this is the time that is needed to fully cook the hotdog so that it does not burn.

I Boiling

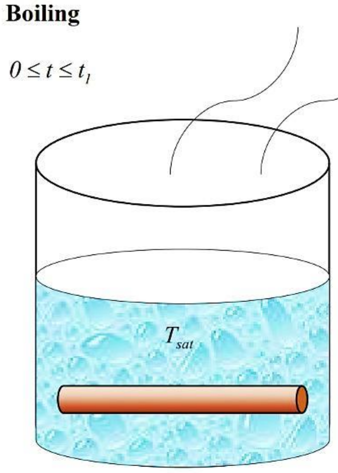

Figure 2. Represents the ambient in which a hotdog is cooked over time $0 \le t \le t_1$

An illustration of the time-domain for the rise and fall of temperature during time $0 \le t \le t_1$

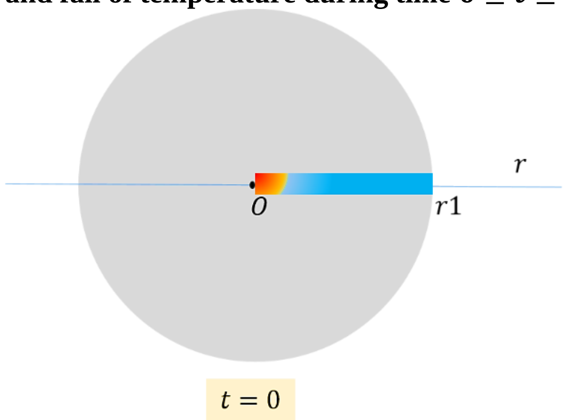

Figure 3. The start of heating at time $t = 0$

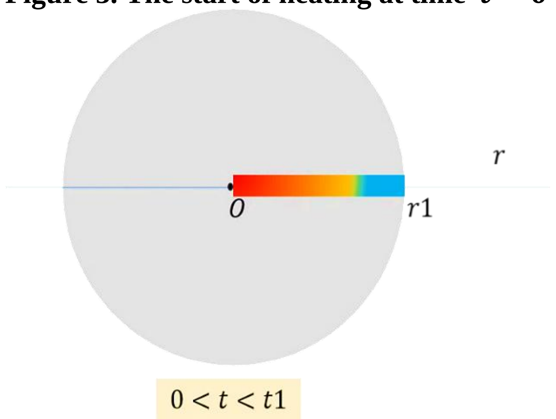

Figure 4. During time $0 < t < t_1$ the temperature increased

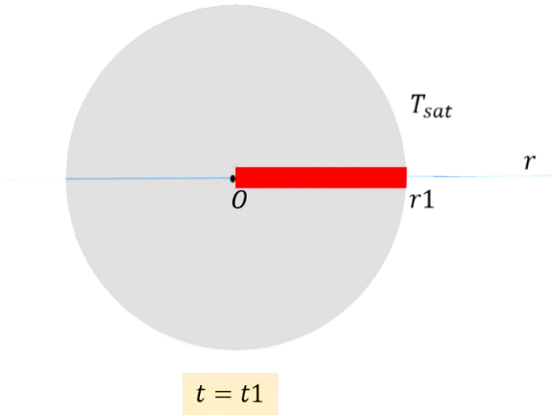

Figure 5. During the time $t = t_1$ the temperature reached saturation T_{sat} (boiling).

This stage has the following conditions:

- The first initial condition

$u^I(r, 0) = u_0(r) = T_0$
38

Where T_0 the initial temperature of the ambient and the body.

- The first boundary condition (Dirichlet)

$u(r_1, t) = u(r_1) = T_{sat}$
39

Where T_{sat} saturation temperature.

The second stage: the stage of getting the hot dog out of the boiling water (Cooling):
When the hotdog is gotten out from the first ambient (water) to another ambient (the room, for example) Fig. 6, the initial temperature T_{t_1} at this stage is the same as the final temperature at the boiling stage, that is, the cooling begins at time $t = t_1$ Fig. 7, the process of heat exchange between the ambient (the room) and the hotdog is carried out by Eq. 37 which describes the heat transfer during time $t > t_1$ (from the top "the body" to the bottom "the ambient") until a thermal equilibrium between ambient temperature T_a (the room for example) and the hotdog is reached, or it can be said that $T_a = T_{min}$ i.e. the lowest degree of cooling can get (i.e. the hotdog has been cooled) at the moment $t > t_1$ Fig. 8 is after the time t_1 the hotdog cooling process is carried out in Fig. 9.
It can be said that the time needed for a hotdog to cool down is $t_2 \ge t > t_1$ where t_2 is the lowest cooling time for the hotdog, That is, no matter how much time increases, the temperature will not drop below than T_{min}.

II Cooling

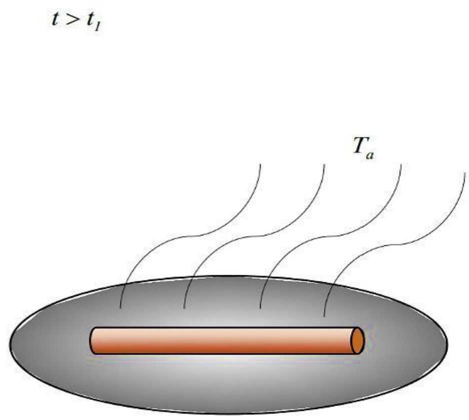

Figure 6. Represents the ambient in which a hotdog is cooled over time $t > t_1$

An illustration of the time-domain for the fall of temperature during time $t > t_1$:

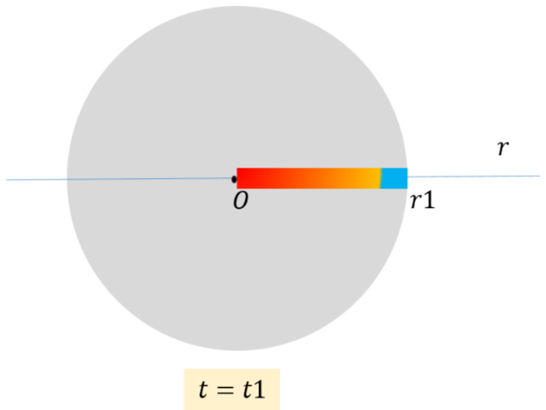

Figure 7. The start of cooling at time $t = t_1$

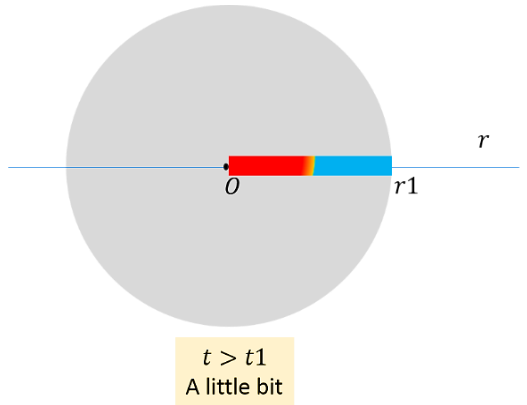

Figure 8. When t is slightly longer than t_1, it gets colder

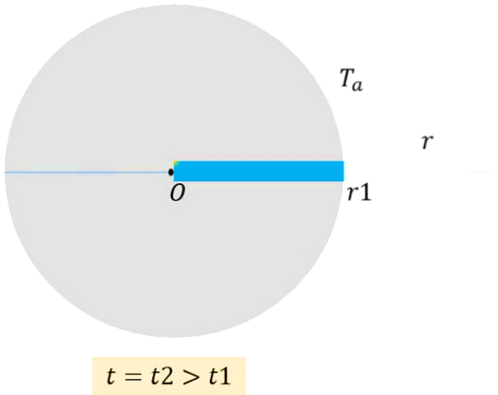

Figure 9. During the time $t = t_2 > t_1$, the coldness reached the point of equilibrium with the medium T_a

This stage has the following conditions:

- The second initial condition

$$u^{II}(r,0) = u^{II}{}_0(r) = u^{I}(r,t_1) = u_{t_1}(r) = T_{t_1} \quad 40$$

Where T_{t_1} initial temperature

- The second boundary condition (Robin)

$$\left.\frac{\partial u(r)}{\partial r} + hu(r)\right|_{r=r_1} = hT_a \quad 41$$

Where T_a ambient temperature.

<u>The solution:</u>

From Eq. 29 the joint zero-order finite Hankel-Elzaki transform for this problem will be defined as:

$$\hat{\tilde{u}}(k_i,p) = p\int_0^{\infty}\int_0^{r_1} u(r,t)rJ_0(k_ir)\,e^{-\frac{t}{p}}\,drdt \quad 42$$

1. Let's find the general solution for the first stage (boiling):

Applying Eq. 42 to Eq. 37, then:

$$\alpha p\int_0^{\infty} e^{-\frac{t}{p}}\int_0^{r_1} r\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right)J_0(k_ir)drdt = p\int_0^{\infty} e^{-\frac{t}{p}}\int_0^{r_1} r\left(\frac{\partial u}{\partial t}\right)J_0(k_ir)drdt$$

Applying property 35 to the first party with the boundary condition 39 where k_i positive roots of $J_0(r_1k_i) = 0$, then:

$$p^2\alpha r_1k_iT_{sat}J_1(r_1k_i) - \alpha k_i{}^2\hat{\tilde{u}}(k_i.p) = p\int_0^{\infty} e^{-\frac{t}{p}}\tilde{u}_t(k_i,t)dt$$

where $J'_0(r_1k_i) = -J_1(r_1k_i)$

Applying property 21 to the second party, then:

$$p^2\alpha r_1k_iT_{sat}J_1(r_1k_i) - \alpha k_i{}^2\hat{\tilde{u}}(k_i.p) = \frac{1}{p}\hat{\tilde{u}}(k_i.p) - p\tilde{u}(k_i,0) \quad 43$$

By applying the zero-order Hankel transform to the initial condition 38, then:

$$\tilde{u}(k_i.0) = u_0(k_i) = T_0 \int_0^{r_1} rJ_0(rk_i)\,dr = \frac{T_0}{k_i} r_1 J_1(r_1 k_i)$$

Substituting in Eq. 43 then:

$$p^2 \alpha r_1 k_i T_{sat} J_1(r_1 k_i) - \alpha {k_i}^2 \hat{\tilde{u}}(k_i.p) = \frac{1}{p}\hat{\tilde{u}}(k_i.p) - p\frac{T_0}{k_i} r_1 J_1(r_1 k_i) \Rightarrow$$

$$\left(\frac{1+\alpha {k_i}^2 p}{p}\right)\hat{\tilde{u}}(k_i.p) = p^2 \alpha r_1 k_i T_{sat} J_1(r_1 k_i) + p\frac{T_0}{k_i} r_1 J_1(r_1 k_i) \quad \Rightarrow$$

$$\hat{\tilde{u}}(k_i.p) = \frac{p^3}{1+\alpha {k_i}^2 p}\alpha r_1 k_i T_{sat} J_1(r_1 k_i) + \frac{p^2}{1+\alpha {k_i}^2 p}\frac{T_0}{k_i} r_1 J_1(r_1 k_i) \qquad 44$$

and this equation is an algebraic equation.

Applying Eq. 30 on Eq. 44 where k_i positive roots of $J_0(r_1 k_i) = 0$, then:

$$u(r,t) = \frac{2}{{r_1}^2}\sum_{i=1}^{\infty}\left(E^{-1}\left(\frac{p^3}{1+\alpha p {k_i}^2}\right)\alpha r_1 k_i T_{sat} J_1(r_1 k_i) + E^{-1}\left(\frac{p^2}{1+\alpha p {k_i}^2}\right)\frac{T_0}{k_i} r_1 J_1(r_1 k_i)\right)\frac{J_0(rk_i)}{\left(J_1(r_1 k_i)\right)^2} \qquad 45$$

Applying relations 24 and 25 on Eq. 45, then:

$$u(r,t) = \frac{2}{{r_1}^2}\sum_{i=1}^{\infty}\left(\frac{1}{\alpha {k_i}^2}\left(1 - e^{-\alpha {k_i}^2 t}\right)\alpha r_1 k_i T_{sat} J_1(r_1 k_i) + e^{-\alpha {k_i}^2 t}\frac{T_0}{k_i} r_1 J_1(r_1 k_i)\right)\frac{J_0(rk_i)}{\left(J_1(r_1 k_i)\right)^2}$$

$$u(r,t) = \frac{2}{r_1}\sum_{i=1}^{\infty}\left(\frac{T_0}{k_i}e^{-\alpha {k_i}^2 t} + \frac{T_{sat}}{k_i}\left(1 - e^{-\alpha {k_i}^2 t}\right)\right)\frac{J_0(rk_i)}{J_1(r_1 k_i)} \qquad 46$$

Which is the general solution for the first stage where $0 \le t \le t_1$.

The general solution is in two parts: the first part relates to the beginning of the heating phase when the temperature T_0 was a limit multiplied by the exponential function, which describes the change over time, and the second part expresses the final phase in which the temperature T_{sat} is reached.

2. Let's find the general solution for the second stage (cooling):

Applying Eq. 42 to Eq. 37, then:

$$\alpha p \int_0^{\infty} e^{-\frac{t}{p}} \int_0^{r_1} r\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right) J_0(k_i r)\,dr\,dt = p\int_0^{\infty} e^{-\frac{t}{p}} \int_0^{r_1} r\left(\frac{\partial u}{\partial t}\right) J_0(k_i r)\,dr\,dt$$

Applying property 36 to the first party with the boundary condition 41 where k_i positive roots of $hJ_0(r_1 k_i) + k_i {J'}_0(r_1 k_i) = 0$, then:

$$p^2 \alpha r_1 h T_a J_0(r_1 k_i) - \alpha {k_i}^2 \hat{\tilde{u}}(k_i.p) = p\int_0^{\infty} e^{-\frac{t}{p}} \tilde{u}_t(k_i,t)\,dt$$

Applying property 21 to the second party, then:

$$p^2 \alpha r_1 h T_a J_0(r_1 k_i) - \alpha {k_i}^2 \hat{\tilde{u}}(k_i.p) = \frac{1}{p}\hat{\tilde{u}}(k_i.p) - p\tilde{u}(k_i,t_1) \Rightarrow$$

$$\left(\frac{1+\alpha {k_i}^2 p}{p}\right)\hat{\tilde{u}}(k_i.p) = p^2 \alpha r_1 h T_a J_0(r_1 k_i) + p\tilde{u}(k_i,t_1) \qquad 47$$

Applying the zero-order Hankel transform to the initial condition 40, then:

$$\tilde{u}(k_i,t_1) = \frac{T_{t_1}}{k_i} r_1 J_1(r_1 k_i)$$

Substituting in Eq. 47, then:

$$\hat{\tilde{u}}(k_i,p) = \frac{p^3}{1+\alpha {k_i}^2 p}\alpha r_1 h T_a J_0(r_1 k_i) + \frac{p^2}{1+\alpha {k_i}^2 p}\frac{T_{t_1}}{k_i} r_1 J_1(r_1 k_i) \qquad 48$$

and this equation is an algebraic equation.

applying Eq. 31 on Eq. 48 where k_i positive roots of $hJ_0(r_1 k_i) + k_i {J'}_0(r_1 k_i) = 0$, then:

$$u(r,t) = \frac{2}{{r_1}^2}\sum_{i=1}^{\infty}\left(E^{-1}\left(\frac{p^3}{1+\alpha {k_i}^2 p}\right)\alpha r_1 h T_a J_0(r_1 k_i) + E^{-1}\left(\frac{p^2}{1+\alpha {k_i}^2 p}\right)\frac{T_{t_1}}{k_i} r_1 J_1(r_1 k_i)\right) \times \frac{{k_i}^2}{(h^2+{k_i}^2)}\frac{J_0(rk_i)}{\left(J_0(r_1 k_i)\right)^2} \qquad 49$$

Applying relations 24 and 25 on Eq. 49, then:

$$u(r,t) = \frac{2}{{r_1}^2}\sum_{i=1}^{\infty}\left(\frac{1}{\alpha {k_i}^2}\left(1 - e^{-\alpha {k_i}^2 t}\right)\alpha r_1 h T_a J_0(r_1 k_i) + e^{-\alpha {k_i}^2 t}\frac{T_{t_1}}{k_i} r_1 J_1(r_1 k_i)\right)\frac{{k_i}^2}{\left(h^2 + {k_i}^2\right)}\frac{J_0(rk_i)}{\left(J_0(r_1 k_i)\right)^2}$$

$$u(r,t) = \frac{2}{r_1}\sum_{i=1}^{\infty}\frac{1}{h^2+{k_i}^2}\left(T_{t_1} k_i J_1(r_1 k_i) e^{-\alpha {k_i}^2 t} + hT_a J_0(r_1 k_i)\left(1 - e^{-\alpha {k_i}^2 t}\right)\right)\frac{J_0(rk_i)}{\left(J_0(r_1 k_i)\right)^2} \qquad 50$$

Which is the general solution for the second stage where $> t_1$.

The general solution is in two parts: the first part relates to the beginning of the cooling stage when the temperature T_{t_1} was a limit multiplied by the exponential function that describes the change over time, and the second part expresses the final stage in which the temperature T_a is reached.

Steady-state Study:

- Steady-state study for the first stage of Eq. 46

$$u(r,t) = \frac{2}{r_1}\sum_{i=1}^{\infty}\left(\frac{T_0}{k_i}e^{-\alpha k_i^2 t} + \frac{T_{sat}}{k_i}\left(1 - e^{-\alpha k_i^2 t}\right)\right)\frac{J_0(rk_i)}{J_1(r_1k_i)}$$

The heat $u(r,t)$ can be thought of as the sum of two parts:

$$u(r,t) = steady\ state + transient\ state$$

where the transient state (variable with time t) and the steady state (constant with time t).

Steady-state: It is represented when $t \to \infty$ and therefore the exponential function decays (vanishes) to zero and the steady state is stayed, i.e. that does not depend on time t, from which the general solution becomes Eq. 46 when $t \to \infty$ is in the form:

$$u(r,t) = \frac{2T_{sat}}{r_1}\sum_{i=1}^{\infty}\frac{1}{k_i}\frac{J_0(rk_i)}{J_1(r_1k_i)} \quad 51$$

from Example 1, taking $v = 0$, then:

$$H_0\{1\} = \int_0^{r_1} rJ_0(rk_i)dr = \frac{1}{k_i}\int_0^{r_1}\frac{d}{dr}rJ_1(rk_i)\,dr = \frac{r_1}{k_i}J_1(r_1k_i)$$

and from Eq. 12 where $v = 0$, then:

$$1 = \frac{2}{r_1^2}\sum_{i=1}^{\infty}\frac{r_1}{k_i}J_1(r_1k_i)\frac{J_0(rk_i)}{\left(J_1(r_1k_i)\right)^2} = \frac{2}{r_1}\sum_{i=1}^{\infty}\frac{1}{k_i}\frac{J_0(rk_i)}{J_1(r_1k_i)}$$

where k_i positive roots of $J_1(r_1k_i) = 0$.

and by Substituting in Eq. 51, then:

$$u(r,t) = \frac{2T_{sat}}{r_1}\sum_{i=1}^{\infty}\frac{1}{k_i}\frac{J_0(rk_i)}{J_1(r_1k_i)} = T_{sat} \quad 52$$

Physically: it represents the saturated temperature.

- Steady-state study for the second stage of Eq. 50

$$u(r,t) = \frac{2}{r_1}\sum_{i=1}^{\infty}\frac{1}{h^2 + k_i^2}\left(T_{t_1}k_iJ_1(r_1k_i)e^{-\alpha k_i^2 t} + hT_aJ_0(r_1k_i)\left(1 - e^{-\alpha k_i^2 t}\right)\right)\frac{J_0(rk_i)}{\left(J_0(r_1k_i)\right)^2}$$

The heat $u(r,t)$ can be thought of as the sum of two parts:

$$u(r,t) = steady\ state + transient\ state$$

Steady-state: It is represented when $t \to \infty$ and therefore the exponential function decays (vanishes) to zero and the steady state is stayed, i.e. that does not depend on time t, from which the general solution becomes Eq. 50 when $t \to \infty$ is in the form:

$$u(r,t) = \frac{2hT_a}{r_1}\sum_{i=1}^{\infty}\frac{1}{h^2 + k_i^2}J_0(r_1k_i)\frac{J_0(rk_i)}{\left(J_0(r_1k_i)\right)^2}$$

$$u(r,t) = \frac{2hT_a}{r_1}\sum_{i=1}^{\infty}\frac{1}{h^2+k_i^2}\frac{J_0(rk_i)}{J_0(r_1k_i)} \quad 53$$

from Example1, taking $v = 0$, then:

$$H_0\{1\} = \frac{r_1}{k_i}J_1(r_1k_i)$$

and from Eq. 18 where $v = 0$, then:

$$1 = \frac{2}{r_1^2}\sum_{i=1}^{\infty}\left(\frac{r_1}{k_i}J_1(r_1k_i)\right)\frac{k_i^2}{h^2 + k_i^2}\frac{J_0(rk_i)}{\left(J_0(r_1k_i)\right)^2}$$

$$1 = \frac{2}{r_1}\sum_{i=1}^{\infty}\frac{1}{h^2+k_i^2}k_iJ_1(r_1k_i)\frac{J_0(rk_i)}{\left(J_0(r_1k_i)\right)^2} \quad 54$$

where k_i positive roots of $hJ_0(r_1k_i) + k_iJ'_0(r_1k_i) = 0$.

As $J'_0(r_1k_i) = -J_1(r_1k_i)$ and by substitution in the boundary condition, then:

$$hJ_0(r_1k_i) = k_iJ_1(r_1k_i)$$

Substitution in Eq. 54

$$1 = \frac{2}{r_1}\sum_{i=1}^{\infty}\frac{h}{h^2 + k_i^2}J_0(r_1k_i)\frac{J_0(rk_i)}{\left(J_0(r_1k_i)\right)^2}$$

$$1 = \frac{2h}{r_1}\sum_{i=1}^{\infty}\frac{1}{h^2 + k_i^2}\frac{J_0(rk_i)}{J_0(r_1k_i)}$$

$$\Rightarrow \sum_{i=1}^{\infty}\frac{1}{h^2 + k_i^2}\frac{J_0(rk_i)}{J_0(r_1k_i)} = \frac{r_1}{2h}$$

and by Substituting in Eq. 53, then:

$$u(r,t) = \frac{2hT_a}{r_1}\left(\frac{r_1}{2h}\right) = T_a \quad 55$$

Physically: it represents the ambient temperature.

Discussion: the Ambient in Which a Hotdog is Cooked over Time $0 \le t \le t_1$

The heating of the liquid begins in time t = 0, drawing heat from an external source. over time, for example, in time t = 3, the temperature of the liquid had increased, also over more time, for example, in time t = 7, the temperature of the liquid had increased more and more, at $t_1 = 10$ the water reaches its boiling (saturation point). Then a hotdog is put in boiling water with the persistence of the external source of heating for ten minutes so that a thermal equilibrium has been achieved between the water and the hotdog and the hotdog has been cooked.

Center Point Temperature Graph:

The center point takes two values T_{sat} and T_a
T_{sat} represents the end of the first phase (the boiling phase).

T_a represents the beginning of the second phase (the cooling phase).

In the first phase of the heating process, the initial temperature was T_0, it began to rise gradually until it reached the saturation temperature (peak) T_{sat}, after a certain time, and let it be t_1, as a result of heat exchange (Fig. 10.)

Then in the second phase during the cooling process, the central point began to decline gradually until it reached T_a after a certain time, and let it be t_2, also as a result of heat exchange.

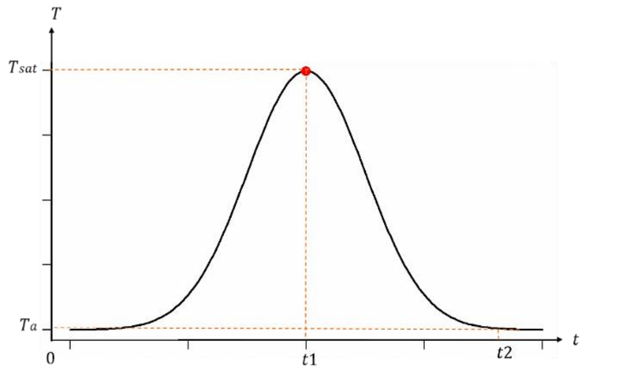

Figure 10. Represents the temperature of the central point takes two values, T_{sat} and T_a

Conclusion:

In this paper, two different single transforms were combined to present a new joint transform FHET and its inverse transform IFHET. Moreover, the most important property of FHET was concluded and proved, which is called the finite Hankel – Elzaki transforms of the Bessel differential operator property, this property was discussed for two different boundary conditions, Dirichlet and Robin, where the importance of this property is shown by solving axisymmetric partial differential equations. The results of the FHET study indicated that FHET is a powerful technique to deal with axisymmetric partial differential equations. In addition, it is an efficient, accurate, and fast technique for solving axisymmetric partial differential equations by transferring them directly to algebraic equations.

Authors' Declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Albaath.

Authors' Contributions Statement:

R. K.R. conceived of the presented idea using a new method combining two different single transforms to present a new joint transform FHET for solving axisymmetric partial differential equations, and wrote the manuscript. M.M.A. supervised this work.

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حل مسألة هوت دوغ بطريقة تحويل هانكل المنتهي من الرتبة الصفرية – الزاكي المشتركة

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الخلاصة:
يختص هذا البحث بدمج تحويلين مختلفين معاً لتقديم تحويل مشترك جديد FHET وتحويله العكسي IFHET، كما أنَّه تمَّ إيجاد أهم خصائص FHET وإثباتها، والتي تسمى خاصية تحويل هانكل المنتهي- الزاكي لمؤثر بيسل التفاضلي، تمت مناقشة هذه الخاصة لأجل شرطين حديين مختلفين هما ديرخليه وروبين. حيث تظهر أهمية هذه الخاصة من خلال حل المعادلات التفاضلية الجزئية ذات التماثل المحوري والانتقال إلى معادلة جبرية بشكل مباشر. أيضاً تمَّ تطبيق طريقة تحويل هانكل المنتهي-الزاكي المشتركة في حل مسألة رياضية فيزيائية وهي مسألة هوت دوغ (النقانق). تمَّ مناقشة الحالة المستقرة التي لا تعتمد على الزمن لكل حل عام حصلنا عليه أي في الحالتين الغليان والتبريد. تمّ رسم الأشكال من الشكل 4 إلى الشكل 9 رسماً يدوياً على برنامج بوربوينت وذلك لتوضيح فكرة ارتفاع وانخفاض الحرارة على المجال الزمني المعطى في المسألة. تؤكد النتائج التي حصلنا عليها أن تقنية التحويل المقترحة فعالة ودقيقة وسريعة في حل المعادلات التفاضلية الجزئية المتماثلة المحور.

الكلمات مفتاحية: مؤثر بيسل التفاضلي، غليان، تبريد، تحويل الزاكي، تحويل هانكل المنتهي، هوت دوغ (نقانق)، تحويل هانكل المنتهي-الزاكي المشترك، معادلة تفاضلية جزئية.