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On fuzzy fractional differential equations using Riemann-Liouville derivative

حول معادلات تفاضلية ضبابية كسورية باستخدام مشتقة ريمان- ليوفيل

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Abstract

The main aim of this paper is to find the formulas of fuzzy fractional derivatives of order $0<\beta<2$ and the formulas of fuzzy Laplace transforms for fuzzy fractional derivatives of order $1<\beta<2$ using Riemann-Liouville derivative and H-differentiability.

المستخلص

الهدف الرئيسي من هذا البحث إيجاد صيغ المشتقات الكسورية الضبابية من الرتبة $0<\beta<2$, وصيغ تحويلات لابلاس الضبابية للمشتقات الكسورية الضبابية من الرتبة $1<\beta<2$ باستخدام مشتقة ريمان- ليوفيل وقابلية الاشتقاق-H.

1. Introduction

Fractional calculus theory is a mathematical analysis tool applied to the study of integrals and derivatives of arbitrary order which unifies and generalizes the notions of integer of-order differentiation of n -fold integration [1-3].

There are many works in subject of fractional calculus, recently, Salahshour et al. [4] deal with the solution of fuzzy fractional differential equations under Riemann-Liouville H-differentiability by fuzzy Laplace transforms. Ahmad et al. [5] deal with fuzzy power series which is a generalization to the classical power series. Allahviranloo et al. [6] give the explicit solutions of uncertain fractional differential equations under Riemann-Liouville and H-differentiability. This paper is organized as follows: Section 2 contains basic concepts. In section 3, we find the formulas of fuzzy Riemann-Liouville fractional derivatives of order $0<\beta<2$ and fuzzy Laplace transforms for fuzzy fractional Riemann-Liouville derivatives of order $1<\beta<2$. Also, an example is solved. Finally conclusions are drawn in section 4.

2.Basic Concepts

Definition 2.1 [1] The Gamma function $\Gamma(x)$ is defined by the integral

$$\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt,\ x>0.$$

Definition 2.2 [7] A fuzzy number u in parametric form is a pair $(\underline{u},\overline{u})$ of function $\underline{u}(r)$ and $\overline{u}(r), 0\le r\le 1$ which satisfy the following requirements :

1. $\underline{u}(r)$ is bounded non – decreasing left continuous function in $(0,1]$, and right continuous at 0,
2. $\overline{u}(r)$ is bounded non – increasing left continuous function in $(0,1]$, and right continuous at 0,
3. $\underline{u}(r)\le\overline{u}(r)$.

Definition 2.3 [7] Let $x, y\in E$. If there exists $z\in E$ such that $x=y+z$, then z is called H-difference of x and y , and it is denoted by $x\ominus y$.

We note that $x\ominus y\ne x+(-1)y$ and E be the set of all fuzzy numbers on R .

Definition 2.4 [8] Let $f(x)$ be continuous fuzzy-valued function. Suppose that $f(x)e^{-sx}$ is improper fuzzy Rimann-integrable on $[0,\infty)$, then $\int_0^\infty f(x)e^{-sx}dx$ is called fuzzy Laplace transforms and is denoted as:

$$L[f(x)]=\int_0^\infty f(x)e^{-sx}dx \quad (s>0 \text{ and integer}).$$

We have:

$$\int_0^\infty f(x)e^{-sx}dx=\left[\int_0^\infty \underline{f}(x;r)e^{-sx}dx,\int_0^\infty \bar{f}(x;r)e^{-sx}dx\right],$$

also by using the definition of classical Laplace transform:

$$\ell[\underline{f}(x;r)]=\int_0^\infty \underline{f}(x;r)e^{-sx}dx \text{ and } \ell[\bar{f}(x;r)]=\int_0^\infty \bar{f}(x;r)e^{-sx}dx,$$

then we follow:

$$L[f(x)]=\left[\ell[\underline{f}(x;r)],\ell[\bar{f}(x;r)]\right].$$

Definition 2.5 [5] A real function $f(x), x>0$ is said to be in the space $C_\mu, \mu\in R$ if there exists a real number $p>\mu$ such that $f(x)=x^p f_1(x)$, where $f_1(x)\in C[0,\infty)$, and it is said to be in the space C_μ^n if $f^{(n)}(x)\in C_\mu, n\in N$.

Definition 2.6 [5] The Riemann-Liouville fractional integral operator of order $\alpha\geq 0$ of function $f(x)\in C_\mu, \mu\geq -1$ is defined as :

$$J_s^\alpha f(x)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\int_s^x (x-t)^{\alpha-1}f(t)dt, & x>t>s\geq 0,\alpha>0,\\ f(x), & \alpha=0.\end{cases}$$

Definition 2.7 [5] The Riemann-Liouville fractional derivative of order $\alpha>0$ of $f\in C_{-1}^n, n\in N$ is defined as:

$$D_s^\alpha f(x)=\begin{cases}\dfrac{d^n}{dx^n}J^{n-\alpha}f(x), & n-1<\alpha<n,\\ \dfrac{d^n}{dx^n}f(x), & \alpha=n\end{cases}$$

Now, we denote $C^F[a,b]$ as the space of all continuous fuzzy-valued functions on $[a,b]$. Also we denote $L^F[a,b]$ as the space of all Lebesque integrable fuzzy-valued functions on the bounded interval $[a,b]\subset R$. [4]

Definition 2.8 [4] Let $f(x)\in C^F[a,b]\cap L^F[a,b], x_0$ in (a,b) and $\phi(x)=\dfrac{1}{\Gamma[1-\beta]}\int_a^x \dfrac{f(t)dt}{(x-t)^\beta}$.

We say that $f(x)$ is fuzzy Riemann-Liouville H- differentiable about order $0<\beta<1$ at x_0, if there exists an element $\left({}^{RL}D_{a+}^\beta f\right)(x_0)\in E$, such that for $h>0$, sufficiently small

(i) $$\left({}^{RL}D_{a+}^\beta f\right)(x_0)=\lim_{h\to 0+}\frac{\phi(x_0+h)\ominus\phi(x_0)}{h}=\lim_{h\to 0+}\frac{\phi(x_0)\ominus\phi(x_0-h)}{h}$$

or

(ii) $$\left({}^{RL}D_{a+}^\beta f\right)(x_0)=\lim_{h\to 0+}\frac{\phi(x_0)\ominus\phi(x_0+h)}{-h}=\lim_{h\to 0+}\frac{\phi(x_0-h)\ominus\phi(x_0)}{-h}$$

(iii) $\left({}^{RL}D_{a+}^{\beta} f\right)(x_0) = \lim_{h\to 0+} \frac{\phi(x_0+h) \ominus \phi(x_0)}{h} = \lim_{h\to 0+} \frac{\phi(x_0-h) \ominus \phi(x_0)}{-h}$

or

(iv) $\left({}^{RL}D_{a+}^{\beta} f\right)(x_0) = \lim_{h\to 0+} \frac{\phi(x_0) \ominus \phi(x_0+h)}{-h} = \lim_{h\to 0+} \frac{\phi(x_0) \ominus \phi(x_0-h)}{h}$

For the sake of simplicity , we say that the fuzzy-valued function f is ${}^{RL}\left[(i)-\beta\right]$ -differentiable if it is differentiable as in the definition 2.8 case (i), and f is ${}^{RL}\left[(ii)-\beta\right]$-differentiable if it is differentiable as in the definition 2.8 case (ii) and so on for the other cases.

Theorem 2.9 [4] Let $f(x) \in C^F[a,b] \cap L^F[a,b]$, x_0 in (a,b) and $0<\beta<1$. Then:

(i) Let us consider f is ${}^{RL}\left[(i)-\beta\right]$ - differentiable fuzzy-valued function, then:

$\left({}^{RL}D_{a+}^{\beta} f\right)(x_0) = \left[({}^{RL}D_{a+}^{\beta} \underline{f})(x_0;r), ({}^{RL}D_{a+}^{\beta} \overline{f})(x_0;r)\right],\ 0\le r\le 1$

(ii) Let us consider f is ${}^{RL}\left[(ii)-\beta\right]$ - differentiable fuzzy -valued function, then:

$\left({}^{RL}D_{a+}^{\beta} f\right)(x_0) = \left[({}^{RL}D_{a+}^{\beta} \overline{f})(x_0;r), ({}^{RL}D_{a+}^{\beta} \underline{f})(x_0;r)\right], 0\le \mathrm{r} \le 1$

where

$$({}^{RL}D_{a+}^{\beta} \underline{f})(x_0;r) = \left[\frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_a^x \frac{\underline{f}(t;r)dt}{(x-t)^{\beta}}\right]_{x=x_0}$$

and

$$({}^{RL}D_{a+}^{\beta} \overline{f})(x_0;r) = \left[\frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_a^x \frac{\overline{f}(t;r)dt}{(x-t)^{\beta}}\right]_{x=x_0}$$

Theorem 2.10 [4] [Derivative theorem] suppose that $f(x) \in C^{\mathrm{F}}[0,\infty) \cap L^{\mathrm{F}}[0,\infty)$. Then :

$$L\left[\left({}^{RL}D_{a^+}^{\beta} f\right)(x)\right] = s^{\beta} L\left[f(x)\right] \ominus \left({}^{RL}D_{a^+}^{\beta-1} f\right)(0),$$

if f is ${}^{RL}\left[(i)-\beta\right]$ -differentiable, and

$$L\left[\left({}^{RL}D_{a^+}^{\beta} f\right)(x)\right] = -\left({}^{RL}D_{a^+}^{\beta-1} f\right)(0) \ominus \left(-s^{\beta} L\left[f(x)\right]\right),$$

if f is ${}^{RL}\left[(ii)-\beta\right]$-differentiable, provided the mentioned Hukuhara difference exist.

3. Riemann-Liouville H-differentiability

In this section, we introduce definition of fuzzy Riemann-Liouville derivatives of order $0<\beta<2$ under H-differentiability and find fuzzy Laplace transforms for fuzzy fractional derivatives of order $1<\beta<2$.

Definition 3.1 Let $f(x) \in C^F[0,b] \cap L^F[0,b]$, $\phi(x) = \frac{1}{\Gamma(\lceil\beta\rceil-\beta)} \int_0^x \frac{f(t)dt}{(x-t)^{1-\lceil\beta\rceil+\beta}}$ where $\phi_1(x_0)$ and $\phi_2(x_0)$ are the limits defined in a_1 and a_2 respectively. $f(x)$ is the Riemann-Liouville type fuzzy fractional differentiable function of order $0<\beta<2$, $\beta\neq 1$ at $x_0\in(0,b)$, if there exists an element $\left({}^{RL}D^{\beta}f\right)(x_0)\in C^F$ such that for all $0\leq r\leq 1$ and for $h>0$ sufficiently near zero either:

a₁. $\left({}^{RL}D^{\beta}f\right)(x_0)=\lim_{h\to 0^+}\frac{\phi(x_0+h)\ominus\phi(x_0)}{h}=\lim_{h\to 0^+}\frac{\phi(x_0)\ominus\phi(x_0-h)}{h}$ or

a₂. $\left({}^{RL}D^{\beta}f\right)(x_0)=\lim_{h\to 0^+}\frac{\phi(x_0)\ominus\phi(x_0+h)}{-h}=\lim_{h\to 0^+}\frac{\phi(x_0-h)\ominus\phi(x_0)}{-h}$

for $0<\beta<1$ and either:

b₁ . $\left({}^{RL}D^{\beta}f\right)(x_0)=\lim_{h\to 0^+}\frac{\phi_1(x_0+h)\ominus\phi_1(x_0)}{h}=\lim_{h\to 0^+}\frac{\phi_1(x_0)\ominus\phi_1(x_0-h)}{h}$ or

b₂. $\left({}^{RL}D^{\beta}f\right)(x_0)=\lim_{h\to 0^+}\frac{\phi_1(x_0)\ominus\phi_1(x_0+h)}{-h}=\lim_{h\to 0^+}\frac{\phi_1(x_0-h)\ominus\phi_1(x_0)}{-h}$ or

b₃ . $\left({}^{RL}D^{\beta}f\right)(x_0)=\lim_{h\to 0^+}\frac{\phi_2(x_0+h)\ominus\phi_2(x_0)}{h}=\lim_{h\to 0^+}\frac{\phi_2(x_0)\ominus\phi_2(x_0-h)}{h}$ or

b₄ . $\left({}^{RL}D^{\beta}f\right)(x_0)=\lim_{h\to 0^+}\frac{\phi_2(x_0)\ominus\phi_2(x_0+h)}{-h}=\lim_{h\to 0^+}\frac{\phi_2(x_0-h)\ominus\phi_2(x_0)}{-h}$

for $1<\beta<2$.

If the fuzzy valued function $f(x)$ is differentiable as in definition 3.1 cases (a_1, b_1, b_3) it is the Riemann-Liouville type differentiable in the first form and denoted by $\left({}^{RL}D_1^{\beta}f\right)(x_0)$, $\left({}^{RL}D_{1,1}^{\beta}f\right)(x_0)$ and $\left({}^{RL}D_{2,1}^{\beta}f\right)(x_0)$ respectively. If the fuzzy valued function $f(x)$ is differentiable as in definition 3.1 cases (a_2, b_2, b_4) it is the Riemann-Liouville type differentiable in the second form and denoted by $\left({}^{RL}D_2^{\beta}f\right)(x_0)$, $\left({}^{RL}D_{1,2}^{\beta}f\right)(x_0)$ and $\left({}^{RL}D_{2,2}^{\beta}f\right)(x_0)$ respectively.

Theorem 3.2 Let $f(x)\in C^F[0,b]\cap L^F[0,b]$ be a fuzzy-valued function and $f(x)=\left[\underline{f}(x;r),\overline{f}(x;r)\right]$ for $r\in[0,1]$, $0<\beta<2$ and $x_0\in(0,b)$. Then:

a₁. If $f(x)$ is Riemann-Liouville type fuzzy fractional differentiable function in the first form, then for $0<\beta<1$

$$\left({}^{RL}D_1^{\beta}f\right)(x_0)=\left[\left({}^{RL}D^{\beta}\underline{f}\right)(x_0;r),\left({}^{RL}D^{\beta}\overline{f}\right)(x_0;r)\right]$$

a₂. If $f(x)$ is Riemann-Liouville type fuzzy fractional differentiable function in the second form, then for $0<\beta<1$

$$\left({}^{RL}D_2^{\beta}f\right)(x_0)=\left[\left({}^{RL}D^{\beta}\overline{f}\right)(x_0;r),\left({}^{RL}D^{\beta}\underline{f}\right)(x_0;r)\right]$$

b₁. If $\left({}^{RL}D_1^{\beta}f\right)(x)$ is Riemann-Liouville type fuzzy fractional differentiable function in the first form, then for $1<\beta<2$

$$\left({}^{RL}D_{1,1}^{\beta}f\right)(x_0)=\left[\left({}^{RL}D^{\beta}\underline{f}\right)(x_0;r),\left({}^{RL}D^{\beta}\overline{f}\right)(x_0;r)\right]$$

b₂. If $\left({}^{RL}D_1^{\beta}f\right)(x)$ is Riemann-Liouville type fuzzy fractional differentiable function in the second form, then for $1<\beta<2$

$$\left({}^{RL}D_{1,2}^{\beta}f\right)(x_0)=\left[\left({}^{RL}D^{\beta}\overline{f}\right)(x_0;r),\left({}^{RL}D^{\beta}\underline{f}\right)(x_0;r)\right]$$

b₃ . If $\left({}^{RL}D_2^{\beta}f\right)(x)$ is Riemann-Liouville type fuzzy fractional differentiable function in the first form, then for $1<\beta<2$

$$\left({}^{RL}D_{2,1}^{\beta} f\right)(x_0)=\left[\left({}^{RL}D^{\beta}\bar{f}\right)(x_0;r),\left({}^{RL}D^{\beta}\underline{f}\right)(x_0;r)\right]$$

b$_4$. If $\left({}^{RL}D_2^{\beta} f\right)(x)$ is Riemann-Liouville type fuzzy fractional differentiable function in the second form, then for $1<\beta<2$

$$\left({}^{RL}D_{2,2}^{\beta} f\right)(x_0)=\left[\left({}^{RL}D^{\beta}\underline{f}\right)(x_0;r),\left({}^{RL}D^{\beta}\bar{f}\right)(x_0;r)\right]$$

where

$$\left({}^{RL}D^{\beta}\underline{f}\right)(x_0;r)=\left[\frac{1}{\Gamma(\lceil\beta\rceil-\beta)}\left(\frac{d}{dx}\right)^{\lceil\beta\rceil}\int_0^x\frac{\underline{f}(t;r)}{(x-t)^{1-\lceil\beta\rceil+\beta}}dt\right]_{x=x_0}$$

$$\left({}^{RL}D^{\beta}\bar{f}\right)(x_0;r)=\left[\frac{1}{\Gamma(\lceil\beta\rceil-\beta)}\left(\frac{d}{dx}\right)^{\lceil\beta\rceil}\int_0^x\frac{\bar{f}(t;r)}{(x-t)^{1-\lceil\beta\rceil+\beta}}dt\right]_{x=x_0}$$

Proof We shall prove b$_1$ as follows: Since $\left({}^{RL}D_1^{\beta} f\right)(x), 1<\beta<2$ is the Riemann-Liouville type fuzzy fractional differentiable function in the first form, then from b$_1$, of definition 3.1, we have:

$$\phi_1(x_0+h)\ominus\phi_1(x_0)=\left[\underline{\phi}_1(x_0+h;r)-\underline{\phi}_1(x_0;r),\bar{\phi}_1(x_0+h;r)-\bar{\phi}_1(x_0;r)\right],$$

$$\phi_1(x_0)\ominus\phi_1(x_0-h)=\left[\underline{\phi}_1(x_0;r)-\underline{\phi}_1(x_0-h;r),\bar{\phi}_1(x_0;r)-\bar{\phi}_1(x_0-h;r)\right].$$

Multiplying both sides by $\frac{1}{h}$, $h>0$, we get:

$$\frac{\phi_1(x_0+h)\ominus\phi_1(x_0)}{h}=\left[\frac{\underline{\phi}_1(x_0+h;r)-\underline{\phi}_1(x_0;r)}{h},\frac{\bar{\phi}_1(x_0+h;r)-\bar{\phi}_1(x_0;r)}{h}\right],$$

$$\frac{\phi_1(x_0)\ominus\phi_1(x_0-h)}{h}=\left[\frac{\underline{\phi}_1(x_0;r)-\underline{\phi}_1(x_0-h;r)}{h},\frac{\bar{\phi}_1(x_0;r)-\bar{\phi}_1(x_0-h;r)}{h}\right].$$

By taking $h\to 0^+$ on both sides of the above relation, we get:

$$\left({}^{RL}D^{\beta} f\right)(x_0)=\left[\frac{d}{dx}\underline{\phi}_1(x_0;r),\frac{d}{dx}\bar{\phi}_1(x_0;r)\right] \tag{3.1}$$

Now, since $\phi_1(x_0)$ is equal to the limits defined in a$_1$ of definition 3.1, then we have:

$$\phi(x_0+h)\ominus\phi(x_0)=\left[\underline{\phi}(x_0+h;r)-\underline{\phi}(x_0;r),\bar{\phi}(x_0+h;r)-\bar{\phi}(x_0;r)\right],$$

$$\phi(x_0)\ominus\phi(x_0-h)=\left[\underline{\phi}(x_0;r)-\underline{\phi}(x_0-h;r),\bar{\phi}(x_0;r)-\bar{\phi}(x_0-h;r)\right].$$

Multiplying both sides by $\frac{1}{h}$ $,h>0$, we obtain

$$\frac{\phi(x_0+h)\ominus\phi(x_0)}{h}=\left[\frac{\underline{\phi}(x_0+h;r)-\underline{\phi}(x_0;r)}{h},\frac{\bar{\phi}(x_0+h;r)-\bar{\phi}(x_0;r)}{h}\right],$$

$$\frac{\phi(x_0)\ominus\phi(x_0-h)}{h}=\left[\frac{\underline{\phi}(x_0;r)-\underline{\phi}(x_0-h;r)}{h},\frac{\bar{\phi}(x_0;r)-\bar{\phi}(x_0-h;r)}{h}\right].$$

By taking $h\to 0^+$ on both sides of the above relation ,we get:

$$\phi_1(x_0)=\left[\underline{\phi}'(x_0;r),\bar{\phi}'(x_0;r)\right],$$

Then

$$\underline{\phi}_1(x_0;r)=\underline{\phi}'(x_0;r)\ ,\ \bar{\phi}_1(x_0;r)=\bar{\phi}'(x_0;r) \tag{3.2}$$

Substituting (3.2) in (3.1) yields:

$$\left({}^{RL}D^{\beta} f\right)(x_0)=\left[\frac{d^2}{dx^2}\underline{\phi}\,(x_0;r),\frac{d^2}{dx^2}\overline{\phi}\,(x_0;r)\right]$$
$$=\left[\left({}^{RL}D^{\beta}\underline{f}\right)(x_0;r),\left({}^{RL}D^{\beta}\overline{f}\right)(x_0;r)\right].$$

The other proofs are similar .

Theorem 3.3 Suppose that $f\in C^F[0,\infty)\cap L^F[0,\infty)$ and, $1<\beta<2$ then :

1. If $\left({}^{RL}D_1^{\beta}f\right)(x)$ is ${}^{RL}[(i)-\beta]$-differentiable fuzzy–valued function, then

$$L\left[\left({}^{RL}D_{1,1}^{\beta} f\right)(x)\right]=s^{\beta}L\left[f\,(x)\right]\ominus s\left({}^{RL}D^{\beta-2} f\right)(0)\ominus\left({}^{RL}D^{\beta-1} f\right)(0)$$

2. If $\left({}^{RL}D_1^{\beta}f\right)(x)$ is ${}^{RL}[(ii)-\beta]$-differentiable fuzzy–valued function, then

$$L\left[\left({}^{RL}D_{1,2}^{\beta} f\right)(x)\right]=-s\left({}^{RL}D^{\beta-2} f\right)(0)\ominus\left(-s^{\beta}\right)L\left[f\,(x)\right]-\left({}^{RL}D^{\beta-1} f\right)(0)$$

3. If $\left({}^{RL}D_2^{\beta}f\right)(x)$ is ${}^{RL}[(i)-\beta]$-differentiable fuzzy–valued function, then

$$L\left[\left({}^{RL}D_{2,1}^{\beta} f\right)(x)\right]=-s\left({}^{RL}D^{\beta-2} f\right)(0)\ominus\left(-s^{\beta}\right)L\left[f\,(x)\right]\ominus\left({}^{RL}D^{\beta-1} f\right)(0)$$

4. If $\left({}^{RL}D_2^{\beta}f\right)(x)$ is ${}^{RL}[(ii)-\beta]$-differentiable fuzzy–valued function, then

$$L\left[\left({}^{RL}D_{2,2}^{\beta} f\right)(x)\right]=s^{\beta}L\left[f\,(x)\right]\ominus s\left({}^{RL}D^{\beta-2} f\right)(0)-\left({}^{RL}D^{\beta-1} f\right)(0)$$

Proof To prove **3,** let us consider $\left({}^{RL}D_2^{\beta}f\right)(x)$ is ${}^{RL}[(i)-\beta]$-differentiable, then by theorem 3.2 we get:

$$L\left[\left({}^{RL}D_{2,1}^{\beta} f\right)(x)\right]=L\left[\left(\underline{{}^{RL}D^{\beta} f}\right)(x;r),\left(\overline{{}^{RL}D^{\beta} f}\right)(x;r)\right]$$
$$=\left[\ell\left[\left({}^{RL}D^{\beta}\overline{f}\right)(x;r)\right],\ell\left[\left({}^{RL}D^{\beta}\underline{f}\right)(x;r)\right]\right] \tag{3.3}$$

We know that Laplace transform of Riemann-Liouville fractional derivative of order $1<\beta<2$ is:

$$\ell\left({}^{RL}D^{\beta}\underline{f}\right)(x;r)=s^{\beta}\ell\left[\underline{f}\,(x;r)\right]-\left({}^{RL}D^{\beta-1}\underline{f}\right)(0;r)-s\left({}^{RL}D^{\beta-2}\underline{f}\right)(0;r) \tag{3.4}$$
$$\ell\left({}^{RL}D^{\beta}\overline{f}\right)(x;r)=s^{\beta}\ell\left[\overline{f}\,(x;r)\right]-\left({}^{RL}D^{\beta-1}\overline{f}\right)(0;r)-s\left({}^{RL}D^{\beta-2}\overline{f}\right)(0;r) \tag{3.5}$$

Since $0<\beta-1<1$, then we have:

$$\left({}^{RL}D^{\beta-1}\underline{f}\right)(0;r)=\left(\overline{{}^{RL}D^{\beta-1}f}\right)(0;r),\ \left({}^{RL}D^{\beta-1}\overline{f}\right)(0;r)=\left(\underline{{}^{RL}D^{\beta-1}f}\right)(0;r). \tag{3.6}$$

Using the equations (3.4)-(3.6), then equation (3.3) becomes:

$$L\left[\left({}^{RL}D^{\beta} f\right)(x)\right]=\left[s^{\beta}\ell\left[\overline{f}\,(x;r)\right]-\left(\underline{{}^{RL}D^{\beta-1}f}\right)(0;r)-s\left({}^{RL}D^{\beta-2}\overline{f}\right)(0;r),s^{\beta}\ell\left[\underline{f}\,(x;r)\right]-\right.$$

$$\left.\left(\overline{{}^{RL}D^{\beta-1}f}\right)(0;r)-s\left({}^{RL}D^{\beta-2}\underline{f}\right)(0;r)\right]$$
$$=-s\left({}^{RL}D^{\beta-2}f\right)(0)\ominus\left(-s^{\beta}\right)L\left[f\,(x)\right]\ominus\left({}^{RL}D^{\beta-1}f\right)(0).$$

The other proofs are similar .

Example 3.4 Consider the following FFIVP:

$$\left({}^{RL}D^{\beta}y\right)(x)=\lambda y\,(x)\quad,\lambda\in(0,\infty),1<\beta<2 \tag{3.7}$$

$$\left({}^{RL}D^{\beta-1}y\right)(0)={}^{RL}y_0^{(\beta-1)}\in E\,,$$

$$\left({}^{RL}D^{\beta-2}y\right)(0)={}^{RL}y_0^{(\beta-2)}\in E.$$

We note that:

$$\left(\underline{{}^{RL}D^{\beta-1}y}\right)(0;r)={}^{RL}\underline{y}_0^{(\beta-1)}(r),\left(\overline{{}^{RL}D^{\beta-1}y}\right)(0;r)={}^{RL}\overline{y}_0^{(\beta-1)}(r),$$

$$\left({}^{RL}D^{\beta-2}\underline{y}\right)(0;r)={}^{RL}\underline{y}_0^{(\beta-2)}(r),\left({}^{RL}D^{\beta-2}\overline{y}\right)(0;r)={}^{RL}\overline{y}_0^{(\beta-2)}(r).$$

By taking fuzzy Laplace transform for both sides of equation (3.7) we get:

$$L\left[\left({}^{RL}D^{\beta}y\right)(x)\right]=\lambda L\left[y(x)\right]. \tag{3.8}$$

Now, we have $2^2=4$ cases as follows:

Case 1 Let us consider $\left({}^{RL}D_1^{\beta}y\right)(x)$ be ${}^{RL}\left[i-\beta\right]$-differentiable fuzzy-valued function, then equation (3.8) can be written as follows:

$$s^{\beta}L\left[y(x)\right]\ominus s\left({}^{RL}D^{\beta-2}y\right)(0)\ominus\left({}^{RL}D^{\beta-1}y\right)(0)=\lambda L\left[y(x)\right].$$

Then, we get :

$$s^{\beta}\ell\left[\underline{y}(x;r)\right]-s\left({}^{RL}D^{\beta-2}\underline{y}\right)(0;r)-\left(\underline{{}^{RL}D^{\beta-1}y}\right)(0;r)=\lambda\ell\left[\underline{y}(x;r)\right],$$

$$s^{\beta}\ell\left[\overline{y}(x;r)\right]-s\left({}^{RL}D^{\beta-2}\overline{y}\right)(0;r)-\left(\overline{{}^{RL}D^{\beta-1}y}\right)(0;r)=\lambda\ell\left[\overline{y}(x;r)\right].$$

Therefore we have :

$$\left(s^{\beta}-\lambda\right)\ell\left[\underline{y}(x;r)\right]={}^{RL}\underline{y}_0^{(\beta-2)}(r)s+{}^{RL}\underline{y}_0^{(\beta-1)}(r),$$
$$\left(s^{\beta}-\lambda\right)\ell\left[\overline{y}(x;r)\right]={}^{RL}\overline{y}_0^{(\beta-2)}(r)s+{}^{RL}\overline{y}_0^{(\beta-1)}(r).$$

Consequently , applying inverse of Laplace transform on the both sides we have:

$$\underline{y}(x;r)={}^{RL}\underline{y}_0^{(\beta-2)}(r)\ell^{-1}\left[\frac{s}{s^{\beta}-\lambda}\right]+{}^{RL}\underline{y}_0^{(\beta-1)}(r)\ell^{-1}\left[\frac{1}{s^{\beta}-\lambda}\right],$$

$$\overline{y}(x;r)={}^{RL}\overline{y}_0^{(\beta-2)}(r)\ell^{-1}\left[\frac{s}{s^{\beta}-\lambda}\right]+{}^{RL}\overline{y}_0^{(\beta-1)}(r)\ell^{-1}\left[\frac{1}{s^{\beta}-\lambda}\right].$$

Finally, we determine the solution of FFIVP (3.7) as follows :

$$\underline{y}(x;r)={}^{RL}\underline{y}_0^{(\beta-2)}(r)x^{\beta-2}E_{\beta,\beta-1}\left(\lambda x^{\beta}\right)+{}^{RL}\underline{y}_0^{(\beta-1)}(r)x^{\beta-1}E_{\beta,\beta}\left(\lambda x^{\beta}\right),$$

$$\overline{y}(x;r)={}^{RL}\overline{y}_0^{(\beta-2)}(r)x^{\beta-2}E_{\beta,\beta-1}\left(\lambda x^{\beta}\right)+{}^{RL}\overline{y}_0^{(\beta-1)}(r)x^{\beta-1}E_{\beta,\beta}\left(\lambda x^{\beta}\right),$$

where $E_{\alpha,\beta}(z)$ denotes the Mittag-Leffler function .

Case 2 Let us consider $\left({}^{RL}D_1^{\beta}y\right)(x)$ be ${}^{RL}\left[ii-\beta\right]$-differentiable fuzzy-valued functions, then equation (3.8) can be written as follows:

$$-s\left({}^{RL}D^{\beta-2}\,y\right)(0)\ominus\left(-s^{\beta}\right)L\left[y(x)\right]-\left({}^{RL}D^{\beta-1}\,y\right)(0)=\lambda L\left[y(x)\right].$$

Then, we get :

$$-s\left({}^{RL}D^{\beta-2}\,\bar{y}\right)(0;r)+s^{\beta}\ell\left[\bar{y}(x;r)\right]-\left(\overline{{}^{RL}D^{\beta-1}\,y}\right)(0;r)=\lambda\ell\left[\underline{y}(x;r)\right],$$

$$-s\left({}^{RL}D^{\beta-2}\,\underline{y}\right)(0;r)+s^{\beta}\ell\left[\underline{y}(x;r)\right]-\left(\underline{{}^{RL}D^{\beta-1}\,y}\right)(0;r)=\lambda\ell\left[\bar{y}(x;r)\right].$$

Then we get the system:

$$s^{\beta}\ell\left[\bar{y}(x;r)\right]-\lambda\ell\left[\underline{y}(x;r)\right]={}^{RL}\bar{y}_0^{(\beta-2)}(r)s+{}^{RL}\bar{y}_0^{(\beta-1)}(r),$$
$$s^{\beta}\ell\left[\underline{y}(x;r)\right]-\lambda\ell\left[\bar{y}(x;r)\right]={}^{RL}\underline{y}_0^{(\beta-2)}(r)s+{}^{RL}\underline{y}_0^{(\beta-1)}(r).$$

The solution of the above system is:

$$\ell\left[\underline{y}(x;r)\right]=\frac{{}^{RL}\underline{y}_0^{(\beta-2)}(r)s^{\beta+1}+{}^{RL}\underline{y}_0^{(\beta-1)}(r)s^{\beta}+\lambda\;{}^{RL}\bar{y}_0^{(\beta-2)}(r)s+\lambda\;{}^{RL}\bar{y}_0^{(\beta-1)}(r)}{s^{2\beta}-\lambda^2},$$
$$\ell\left[\bar{y}(x;r)\right]=\frac{{}^{RL}\bar{y}_0^{(\beta-2)}(r)s^{\beta+1}+{}^{RL}\bar{y}_0^{(\beta-1)}(r)s^{\beta}+\lambda\;{}^{RL}\underline{y}_0^{(\beta-2)}(r)s+\lambda\;{}^{RL}\underline{y}_0^{(\beta-1)}(r)}{s^{2\beta}-\lambda^2}.$$

Consequently, applying inverse of Laplace transform on the both sides we have :

$$\underline{y}(x;r)={}^{RL}\underline{y}_0^{(\beta-2)}(r)\,\ell^{-1}\left[\frac{s^{\beta+1}}{s^{2\beta}-\lambda^2}\right]+{}^{RL}\underline{y}_0^{(\beta-1)}(r)\,\ell^{-1}\left[\frac{s^{\beta}}{s^{2\beta}-\lambda^2}\right]+\lambda\;{}^{RL}\bar{y}_0^{(\beta-2)}(r)\,\ell^{-1}\left[\frac{s}{s^{2\beta}-\lambda^2}\right]$$
$$+\lambda\;{}^{RL}\bar{y}_0^{(\beta-1)}(r)\,\ell^{-1}\left[\frac{1}{s^{2\beta}-\lambda^2}\right]$$

$$\bar{y}(x;r)={}^{RL}\bar{y}_0^{(\beta-2)}(r)\,\ell^{-1}\left[\frac{s^{\beta+1}}{s^{2\beta}-\lambda^2}\right]+{}^{RL}\bar{y}_0^{(\beta-1)}(r)\,\ell^{-1}\left[\frac{s^{\beta}}{s^{2\beta}-\lambda^2}\right]+\lambda\;{}^{RL}\underline{y}_0^{(\beta-2)}(r)\,\ell^{-1}\left[\frac{s}{s^{2\beta}-\lambda^2}\right]$$
$$+\lambda\;{}^{RL}\underline{y}_0^{(\beta-1)}(r)\,\ell^{-1}\left[\frac{1}{s^{2\beta}-\lambda^2}\right]$$

Finally, we determine the solution of FFIVP (3.7) as follows :

$$\underline{y}(x;r)={}^{RL}\underline{y}_0^{(\beta-2)}(r)x^{\beta-2}E_{2\beta,\beta-1}\left(\lambda^2x^{2\beta}\right)+{}^{RL}\underline{y}_0^{(\beta-1)}(r)x^{\beta-1}E_{2\beta,\beta}\left(\lambda^2x^{2\beta}\right)$$
$$+\lambda\;{}^{RL}\bar{y}_0^{(\beta-2)}(r)x^{2\beta-2}E_{2\beta,2\beta-1}\left(\lambda^2x^{2\beta}\right)+\lambda\;{}^{RL}\bar{y}_0^{(\beta-1)}(r)x^{2\beta-1}E_{2\beta,2\beta}\left(\lambda^2x^{2\beta}\right),$$

$$\bar{y}(x;r)={}^{RL}\bar{y}_0^{(\beta-2)}(r)x^{\beta-2}E_{2\beta,\beta-1}\left(\lambda^2x^{2\beta}\right)+{}^{RL}\bar{y}_0^{(\beta-1)}(r)x^{\beta-1}E_{2\beta,\beta}\left(\lambda^2x^{2\beta}\right)$$
$$+\lambda\;{}^{RL}\underline{y}_0^{(\beta-2)}(r)x^{2\beta-2}E_{2\beta,2\beta-1}\left(\lambda^2x^{2\beta}\right)+\lambda\;{}^{RL}\underline{y}_0^{(\beta-1)}(r)x^{2\beta-1}E_{2\beta,2\beta}\left(\lambda^2x^{2\beta}\right).$$

Case 3 Let us consider that $\left({}^{RL}D_2^{\beta}y\right)(x)$ be ${}^{RL}\left[i-\beta\right]$-differentiable fuzzy-valued function then equation (3.8) can be written as follows:

$$-s\left({}^{RL}D^{\beta-2}\,y\right)(0)\ominus\left(-s^{\beta}\right)L\left[y(x)\right]\ominus\left({}^{RL}D^{\beta-1}\,y\right)(0)=\lambda L\left[y(x)\right].$$

Then, we get :

$$-s\left({}^{RL}D^{\beta-2}\,\overline{y}\right)(0;r)+s^{\beta}\ell\left[\overline{y}(x;r)\right]-\left(\underline{{}^{RL}D^{\beta-1}\,y}\right)(0;r)=\lambda\ell\left[\underline{y}(x;r)\right],$$

$$-s\left({}^{RL}D^{\beta-2}\,\underline{y}\right)(0;r)+s^{\beta}\ell\left[\underline{y}(x;r)\right]-\left(\overline{{}^{RL}D^{\beta-1}\,y}\right)(0;r)=\lambda\ell\left[\overline{y}(x;r)\right].$$

Then we get the system:

$$s^{\beta}\ell\left[\overline{y}(x;r)\right]-\lambda\ell\left[\underline{y}(x;r)\right]={}^{RL}\overline{y}_0^{(\beta-2)}(r)s+{}^{RL}\underline{y}_0^{(\beta-1)}(r),$$
$$s^{\beta}\ell\left[\underline{y}(x;r)\right]-\lambda\ell\left[\overline{y}(x;r)\right]={}^{RL}\underline{y}_0^{(\beta-2)}(r)s+{}^{RL}\overline{y}_0^{(\beta-1)}(r).$$

The solution of the above system is

$$\ell\left[\underline{y}(x;r)\right]=\frac{{}^{RL}\underline{y}_0^{(\beta-2)}(r)s^{\beta+1}+{}^{RL}\overline{y}_0^{(\beta-1)}(r)s^{\beta}+\lambda\;{}^{RL}\overline{y}_0^{(\beta-2)}(r)s+\lambda\;{}^{RL}\underline{y}_0^{(\beta-1)}(r)}{s^{2\beta}-\lambda^2}$$
$$\ell\left[\overline{y}(x;r)\right]=\frac{{}^{RL}\overline{y}_0^{(\beta-2)}(r)s^{\beta+1}+{}^{RL}\underline{y}_0^{(\beta-1)}(r)s^{\beta}+\lambda\;{}^{RL}\underline{y}_0^{(\beta-2)}(r)s+\lambda\;{}^{RL}\overline{y}_0^{(\beta-1)}(r)}{s^{2\beta}-\lambda^2}$$

Consequently, applying inverse of Laplace transform on the both sides we have :

$$\underline{y}(x;r)={}^{RL}\underline{y}_0^{(\beta-2)}(r)\,\ell^{-1}\left[\frac{s^{\beta+1}}{s^{2\beta}-\lambda^2}\right]+{}^{RL}\overline{y}_0^{(\beta-1)}(r)\,\ell^{-1}\left[\frac{s^{\beta}}{s^{2\beta}-\lambda^2}\right]+\lambda\;{}^{RL}\overline{y}_0^{(\beta-2)}(r)\,\ell^{-1}\left[\frac{s}{s^{2\beta}-\lambda^2}\right]$$
$$+\lambda\;{}^{RL}\underline{y}_0^{(\beta-1)}(r)\,\ell^{-1}\left[\frac{1}{s^{2\beta}-\lambda^2}\right]$$
$$\overline{y}(x;r)={}^{RL}\overline{y}_0^{(\beta-2)}(r)\,\ell^{-1}\left[\frac{s^{\beta+1}}{s^{2\beta}-\lambda^2}\right]+{}^{RL}\underline{y}_0^{(\beta-1)}(r)\,\ell^{-1}\left[\frac{s^{\beta}}{s^{2\beta}-\lambda^2}\right]+\lambda\;{}^{RL}\underline{y}_0^{(\beta-2)}(r)\,\ell^{-1}\left[\frac{s}{s^{2\beta}-\lambda^2}\right]$$
$$+\lambda\;{}^{RL}\overline{y}_0^{(\beta-1)}(r)\,\ell^{-1}\left[\frac{1}{s^{2\beta}-\lambda^2}\right]$$

Finally, we determine the solution of FFIVP (3.7) as follows :

$$\underline{y}(x;r)={}^{RL}\underline{y}_0^{(\beta-2)}(r)x^{\beta-2}E_{2\beta,\beta-1}\left(\lambda^2x^{2\beta}\right)+{}^{RL}\overline{y}_0^{(\beta-1)}(r)x^{\beta-1}E_{2\beta,\beta}\left(\lambda^2x^{2\beta}\right)$$
$$+\lambda\;{}^{RL}\overline{y}_0^{(\beta-2)}(r)x^{2\beta-2}E_{2\beta,2\beta-1}\left(\lambda^2x^{2\beta}\right)+\lambda\;{}^{RL}\underline{y}_0^{(\beta-1)}(r)x^{2\beta-1}E_{2\beta,2\beta}\left(\lambda^2x^{2\beta}\right),$$

$$\overline{y}(x;r)={}^{RL}\overline{y}_0^{(\beta-2)}(r)x^{\beta-2}E_{2\beta,\beta-1}\left(\lambda^2x^{2\beta}\right)+{}^{RL}\underline{y}_0^{(\beta-1)}(r)x^{\beta-1}E_{2\beta,\beta}\left(\lambda^2x^{2\beta}\right)$$
$$+\lambda\;{}^{RL}\underline{y}_0^{(\beta-2)}(r)x^{2\beta-2}E_{2\beta,2\beta-1}\left(\lambda^2x^{2\beta}\right)+\lambda\;{}^{RL}\overline{y}_0^{(\beta-1)}(r)x^{2\beta-1}E_{2\beta,2\beta}\left(\lambda^2x^{2\beta}\right).$$

Case 4 Let us consider that $\left({}^{RL}D_2^{\beta}y\right)(x)$ be ${}^{RL}[ii-\beta]$-differentiable fuzzy-valued function then equation (3.8) can be written as follows:

$$s^{\beta}L\left[y(x)\right]\ominus s\left({}^{RL}D^{\beta-2}\,y\right)(0)-\left({}^{RL}D^{\beta-1}\,y\right)(0)=\lambda L\left[y(x)\right].$$

Then, we get :

$$s^{\beta}\ell\left[\underline{y}(x;r)\right]-s\left({}^{RL}D^{\beta-2}\,\underline{y}\right)(0;r)-\left(\overline{{}^{RL}D^{\beta-1}\,y}\right)(0;r)=\lambda\ell\left[\underline{y}(x;r)\right],$$

$$s^{\beta}\ell\left[\overline{y}(x;r)\right]-s\left({}^{RL}D^{\beta-2}\,\overline{y}\right)(0;r)-\left(\underline{{}^{RL}D^{\beta-1}\,y}\right)(0;r)=\lambda\ell\left[\overline{y}(x;r)\right].$$

Therefore we have :

$$\left(s^{\beta}-\lambda\right)\ell\left[\underline{y}(x;r)\right]={}^{RL}\underline{y}_0^{(\beta-2)}(r)s+{}^{RL}\overline{y}_0^{(\beta-1)}(r),$$
$$\left(s^{\beta}-\lambda\right)\ell\left[\overline{y}(x;r)\right]={}^{RL}\overline{y}_0^{(\beta-2)}(r)s+{}^{RL}\underline{y}_0^{(\beta-1)}(r).$$

Consequently, applying inverse of Laplace transform on the both sides we have :

$$\underline{y}(x;r)={}^{RL}\underline{y}_0^{(\beta-2)}(r)\,\ell^{-1}\left[\frac{s}{s^{\beta}-\lambda}\right]+{}^{RL}\overline{y}_0^{(\beta-1)}(r)\,\ell^{-1}\left[\frac{1}{s^{\beta}-\lambda}\right],$$

$$\overline{y}(x;r)={}^{RL}\overline{y}_0^{(\beta-2)}(r)\,\ell^{-1}\left[\frac{s}{s^{\beta}-\lambda}\right]+{}^{RL}\underline{y}_0^{(\beta-1)}(r)\,\ell^{-1}\left[\frac{1}{s^{\beta}-\lambda}\right].$$

Finally, we determine the solution of FFIVP (3.7) as follows :

$$\underline{y}(x;r)={}^{RL}\underline{y}_0^{(\beta-2)}(r)x^{\beta-2}E_{\beta,\beta-1}\left(\lambda x^{\beta}\right)+{}^{RL}\overline{y}_0^{(\beta-1)}(r)x^{\beta-1}E_{\beta,\beta}\left(\lambda x^{\beta}\right),$$

$$\overline{y}(x;r)={}^{RL}\overline{y}_0^{(\beta-2)}(r)x^{\beta-2}E_{\beta,\beta-1}\left(\lambda x^{\beta}\right)+{}^{RL}\underline{y}_0^{(\beta-1)}(r)x^{\beta-1}E_{\beta,\beta}\left(\lambda x^{\beta}\right).$$

4. Conclusions

In this paper the Riemann-Liouville fractional derivatives of order $0<\beta<2$ for fuzzy-valued function f are studied. Also the fuzzy Laplace transforms for the Riemann-Liouville fractional derivatives of order $1<\beta<2$ under H-differentiability are discussed. Also an fuzzy fractional initial value problem of order $1<\beta<2$ is solved.

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