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Schultz and Modified Schultz Polynomials for Edge – Identification Chain and Ring – for Square Graphs

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Abstract:

In a connected graph G, the distance function between each pair of two vertices from a set vertex $V(G)$ is the shortest distance between them and the vertex degree u denoted by $degu$ is the number of edges which are incident to the vertex u. The Schultz and modified Schultz polynomials of G are have defined as: $Sc(G;x) = \sum(degu + degv)\, x^{d(u,v)}$ *and* $Sc^*(G;x) = \sum (degu.degv)\, x^{d(u,v)}$, respectively, where the summations are taken over all unordered pairs of distinct vertices in $V(G)$ and $d(u,v)$ is the distance between u and v in $V(G)$. The general forms of Schultz and modified Schultz polynomials shall be found and indices of the edge – identification chain and ring – square graphs in the present work.

Keywords: Edge – Identification Chain and Ring , Schultz index , Modified Schultz index, Polynomials.

Introduction:

All the connected graphs were considered in this paper simple, finite and undirected. $V = V(G)$ and $E = E(G)$ denotes the set of vertices and the set of edges respectively of G, the number of vertices of G is called the order of G, that is $p = p(G) = |V(G)|$ and the number of edges of G is called the size of G, that is $q = q(G) = |E(G)|$. The vertex degree $deg_G u$ or $degu$ referes to the number of vertices that are incident to the vertex u. The distance between any two arbitrary vertices u and v of G is the length of the shortest path joining u to v, and it is denoted by $d_G(u,v)$ or $d(u,v)$. The diameter of a connected graph G is the maximum distance between any two vertices in $V(G)$ and denoted by $diamG$, [1,2], that is,

$$diamG = max_{u,v\in V(G)}\{d(u,v)\}.$$

There are many studies on polynomials and indices with respect the Schultz and modified Schultz,[3-6], and also, there are studies on applied of its,[7-11].

The Schultz index (*molecular topological index*) was introduced by Schultz in 1989,[12] ,and the modified Schultz index was defined by Klavžar and Gutman in 1997,[13].

The Schultz and modified Schultz indices are defined respectively as:

$$Sc(G) = \sum_{\{u,v\}\subseteq V(G)}(degu + degv)\ d(u,v).$$

$$Sc^*(G) = \sum_{\{u,v\}\subseteq V(G)}(degu \cdot degv)\ d(u,v).$$

The Schultz and Modified Schultz polynomials are important polynomials which can obtain the indices and study some properties of the coefficients. The Schultz polynomial is defined as**:**

$$Sc(G;x) = \sum_{\{u,v\}\subseteq V(G)}(degu + degv)\, x^{d(u,v)}\ .$$

In addition, the modified Schultz polynomial is defined as:

$$Sc^*(G;x) = \sum_{\{u,v\}\subseteq V(G)}(degv \cdot degu)\, x^{d(u,v)}\ .$$

The indices of Schultz and modified Schultz can be obtained by the derivative of Schultz and modified Schultz polynomials with respect to x at $x = 1$, that are:

$$Sc(G) = \frac{d}{dx}(Sc(G;x))|_{x=1} \text{ and}$$

$$Sc^*(G) = \frac{d}{dx}(Sc^*(G;x))|_{x=1}.$$

The average distance of a connected graph G of order $p(G)$ with respect Schultz and modified Schultz are have defined as:

$\overline{Sc}(G) = 2Sc(G)/p(G)\,(p(G)\,-1)$ and $\overline{Sc^*}(G) = 2Sc^*(G)/p(G)\,(p(G)\,-1)$.

That the graph G is said to be r – regular graph if all vertex of G has r degree.

If G is an r-regular graph, then

$$Sc(G;x) = \sum_{\{u,v\}\subseteq V(G)} 2r\,x^{d(u,v)} \quad \text{and}$$
$$Sc^*(G;x) = \sum_{\{u,v\}\subseteq V(G)} r^2\,x^{d(u,v)}.$$

Hence $Sc^*(G,x) = \frac{r}{2}Sc(G,x)$. …(1.1)

With simplified, can be obtained as the following $Sc^*(G) = \frac{r}{2}Sc(G)$ and $\overline{Sc^*}(G) = \frac{r}{2}\overline{Sc}(G)$.…(1.2)

The number of pairs of vertices of G that are distance k which denoted by $D(G,k)$. Let $D_k(r,h)$ be the set of all unordered pairs of vertices (u,v) of G with distance k such $degu = r$ and $degv = h$. From clearly that :

$$\sum_{k=1}^{diam(G)}|D_k(G)| = p(G)(p(G)-1)/2,$$

where $D(G,k) = |D_k(G)|$.

Finally the topological indices such as Schultz and modified Schultz indices determine some properties of chemical structures, see,[14-16].

In this work, the general forms can be identified of Schultz and modified Schultz polynomials and indices of the edge – identification chain and ring – square graphs.

Main Results:

Binary operations are construct the new graph from any two graphs such as: Cartesian graph product, strong graph product, tenser graph product, … etc.. In this paper, the special definitions are given of operation for edge identification chain and ring graphs.

Definition 1: Edge – Identification Chain (EIC) – Graphs:

Let $\{G_1, G_2, \dots, G_n\}$ be a set of pairwise disjoint graphs with non – adjacent edge $u_iv_i\,, x_iy_i \in E(G_i), i = 1,2,\dots,n\,, n \geq 2$, then the edge-identification chain graph $C_e(G_1, G_2, \dots, G_n) \equiv C_e(G_1, G_2, \dots, G_n{:}\, u_1v_1\,, x_1y_1; \dots; u_nv_n\,, x_ny_n)$ of $\{G_i\}_{i=1}^{n}$ with respect to the edges $\{u_iv_i\,, x_iy_i\}_{i=1}^{n}$ is the graph obtained from the graphs $G_1, G_2, \dots, G_n$ by identifying the edge x_iy_i with the edge $u_{i+1}v_{i+1}$ for all $i = 1,2,\dots,n-1$. (Fig. 1) in which:

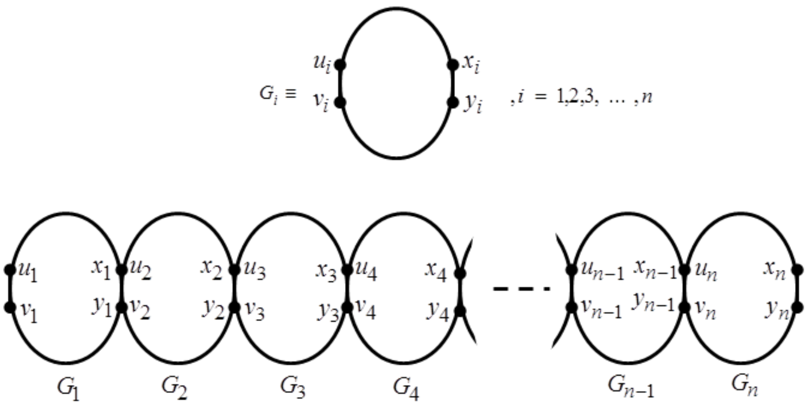


Figure 1. $C_e(G_1, G_2, \dots, G_n)$. Edge – Identification Chain (EIC) – Graphs.

The graph is noted as $C_e(G_1, G_2, \dots, G_n)$ has:

1. $p(C_e(G_1, G_2, \dots, G_n)) = \sum_{i=1}^{n} p(G_i) - 2(n-1)$.
2. $q(C_e(G_1, G_2, \dots, G_n)) = \sum_{i=1}^{n} q(G_i) - (n-1)$.
3. $n \leq diam\big(C_e(G_1, G_2, \dots, G_n)\big) \leq \sum_{i=1}^{n} diam(G_i) - (n-1)$.

The equality of lower bound is satisfied at complete graph but the upper bound is satisfied at path graph.

If $G_i \equiv H_p$, for all $1 \leq i \leq n$, where H_p is a connected graph of order p, the $C_e(H_p, H_p, \dots, H_p)$ by $C_e(H_p)_n$ is denoted.

Definition 2: Edge – Identification Ring (EIR) – Graph:

Let $\{G_1, G_2, \dots, G_n\}$ be a set of pairwise disjoint graphs with non – adjacent edge $u_iv_i\,, x_iy_i \in E(G_i), i = 1,2,\dots,n\,, n \geq 2$, then the edge-identification ring graph $R_e(G_1, G_2, \dots, G_n) \equiv R_e(G_1, G_2, \dots, G_n{:}\, u_1v_1\,, x_1y_1; \dots; u_nv_n\,, x_ny_n)$ of $\{G_i\}_{i=1}^{n}$ with respect to the edges $\{u_iv_i\,, x_iy_i\}_{i=1}^{n}$ is the graph obtained from the graphs $G_1, G_2, \dots, G_n$ by identifying the edge x_iy_i with the edge $u_{i+1}v_{i+1}$ for all $i = 1,2,\dots,n$, where $u_{n+1}v_{n+1} \equiv u_1v_1$. (Fig. 2).

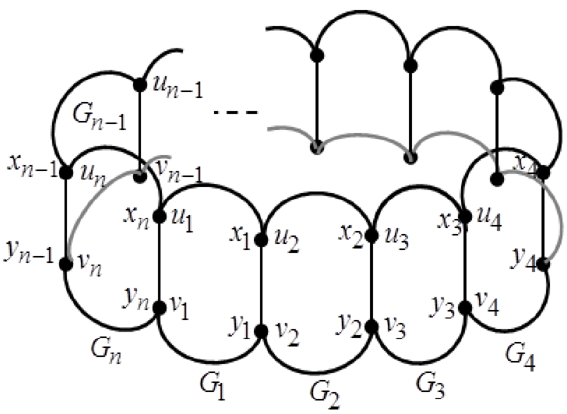


Figure 2. $R_e(G_1, G_2, \dots, G_n)$.
Edge – Identification Ring (EIR) – Graph.

In addition, the graph is denoted $R_e(G_1, G_2, \dots, G_n)$ has:

1. $p(R_e(G_1, G_2, \dots, G_n)) = \sum_{i=1}^{n} p(G_i) - 2n.$
2. $q(R_e(G_1, G_2, \dots, G_n)) = \sum_{i=1}^{n} q(G_i) - n.$
3. $\left\lceil \frac{n-1}{2} \right\rceil \le diam(R_e(G_1, G_2, \dots, G_n)) \le \left\lceil \frac{\sum_{i=1}^{n} diam(G_i) - n - 1}{2} \right\rceil.$

Also, the equality of lower bound is satisfied at complete graph but the upper bound is satisfied at path graph.

If $G_i \equiv H_p$, for all $1 \le i \le n$, where H_p is a connected graph of order p, the $R_e(H_p, H_p, \dots, H_p)$ by $R_e(H_p)_n$ is denoted.

Finally, there are more chains and rings consisting of special graphs about finding the polynomials and indices of types distance such as: ordinary distance, Detour distance, … etc., (17-19) . Therefore, the Schultz and modified Schultz will be continued to be found of them.

Schultz and modified Schultz of $C_e(C_4)_{p-1}$ and $R_e(C_4)_p$:

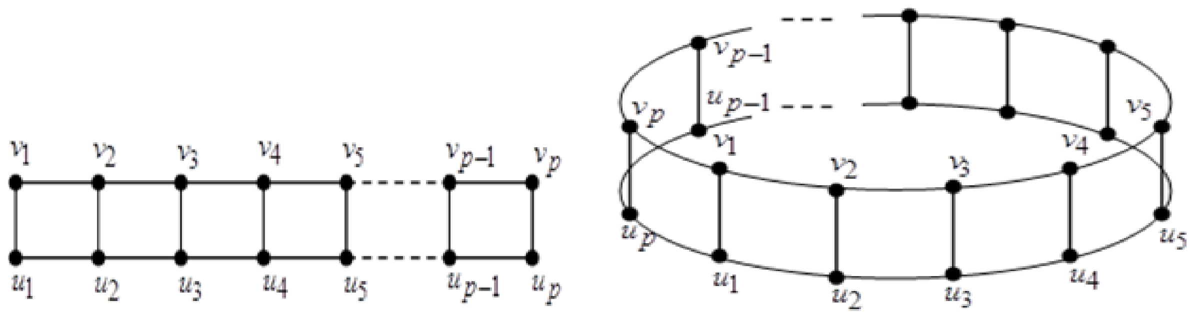


Figure 3.

(a) – Edge – Identification Chain of C_4, $C_e(C_4)_{p-1}$. **(b) – Edge – Identification Ring of C_4, $R_e(C_4)_p$.**

From Fig. 3 (a), can be noted that $\left(C_e(C_4)_{p-1}\right) = 2p, q(C_e(C_4)_{p-1}) = 3p - 2,$ $diam\left(C_e(C_4)_{p-1}\right) = p$ and $2 \le i, j \le p-1, i \ne j$, and there will be:

Table 1. The degree vertices for edge – identification chain of C_4, $C_e(C_4)_{p-1}$.

+ ×	$degu_1 = 2$ (+)	(×)	$degv_1 = 2$ (+)	(×)	$degu_i = 3$ (+)	(×)	$degv_i = 3$ (+)	(×)	$degu_p = 2$ (+)	(×)	$degv_p = 2$ (+)	(×)
$degu_1 = 2$			4	4	5	6	5	6	4	4	4	4
$degv_1 = 2$	4	4			5	6	5	6	4	4	4	4
$degu_j = 3$	5	6	5	6	6	9	6	9	5	6	5	6
$degv_j = 3$	5	6	5	6	6	9	6	9	5	6	5	6
$degu_p = 2$	4	4	4	4	5	6	5	6			4	4
$degv_p = 2$	4	4	4	4	5	6	5	6	4	4		

Theorem 1.1: For $p \ge 4$, then they will be given:

1. $Sc\left(C_e(C_4)_{p-1}; x\right) = 2(9p - 10)x + 4\sum_{k=2}^{p-1}(6p - 6k + 1)x^k + 8x^p.$

2. $Sc^*\left(C_e(C_4)_{p-1}; x\right) = (27p - 40)x + 6\sum_{k=2}^{p-2}(6p - 6k - 1)x^k + 32x^{p-1} + 8x^p.$

Proof: For all $p \ge 5$ and every two vertices $u, v \in V\left(C_e(C_4)_{p-1}\right)$, there is $d(u, v) = k, 1 \le k \le p$. There will be five partitions for proof:

P1. If $d(u, v) = 1$, then $|D_1| = 3p - 2 = q\left(C_e(C_4)_{p-1}\right)$, three subsets will be of it:

P1.1 $|D_1(2,2)| = |\{(u_i, v_i) : i = 1, p\}| = 2.$

P1.2 $|D_1(2,3)| = |\{(u_1, u_2), (u_p, u_{p-1}), (v_1, v_2), (v_p, v_{p-1})\}| = 4.$

P1.3 $|D_1(3,3)| = |\{(u_i, u_{i+1}), (v_i, v_{i+1}) : 2 \le i \le p - 2\} \cup \{(u_i, v_i) : 2 \le i \le p - 1\}| = 3p - 8.$

P2. If $d(u, v) = k, 2 \le k \le p - 3$, then $|D_k| = 4p - 4k + 2$, two subsets will be of it:

P2.1 $|D_k(2,3)| = |\{(u_1,u_{1+k}),(u_p,u_{p-k}),(v_1,v_{1+k}),(v_p,v_{p-k}),(u_1,v_k),(v_p,u_{p-k+1}),(v_1,u_k),(u_p,v_{p-k+1})\}| = 8.$

P2.2 $|D_k(3,3)| = |\{(u_i,u_{i+k}),(v_i,v_{i+k}): 2 \le i \le p-k-1\} \cup \{(u_i,v_{i+k-1}),(v_i,u_{i+k-1}): 2 \le i \le p-k\} = 4p-4k-6.$

P3. If $d(u,v) = p-2$, then $|D_{p-2}| = 10$, two subsets will be of it:

P3.1 $|D_{p-2}(2,3)| = |\{(u_1,u_{p-1}),(u_1,v_{p-2}),(v_1,v_{p-1}),(v_1,u_{p-2}),((u_p,u_2),(u_p,v_3),(v_p,v_2),(v_p,u_3)\}| = 8.$

P3.2 $|D_{p-2}(3,3)| = |\{(u_2,v_{p-1}),(v_2,u_{p-1})\}| = 2.$

P4. If $d(u,v) = p-1$, then $|D_{p-1}| = 6$, two subsets will be of it:

P4.1 $|D_{p-1}(2,2)| = |\{(u_1,u_p),(v_1,v_p)\}| = 2.$

P4.2 $|D_{p-1}(2,3)| = |\{(u_1,v_{p-1}),(v_p,u_2),(v_1,u_{p-1}),(u_p,v_2)\}| = 4.$

P5. If $d(u,v) = p$, then $|D_p| = 2$, one subset will be of it: $|D_p(2,2)| = |\{(u_1,v_p),(v_1,u_p)\}| = 2.$

From P1 – P5 and Table 1, there will be:

1. $Sc(C_e(C_4)_{p-1};x) = 2(9p-10)x + 4\sum_{k=2}^{p-1}(6p-6k+1)x^k + 8x^p.$

Now, modified Shultz polynomial can be found:

2. $Sc^*(C_e(C_4)_{p-1};x) = (27p-40)x + 6\sum_{k=2}^{p-2}(6p-6k-1)x^k + 32x^{p-1} + 8x^p.$

Simply, calculating the following:

$Sc(C_e(C_4)_3;x) = 52x + 52x^2 + 28x^3 + 8x^4.$

$Sc^*(C_e(C_4)_3;x) = 68x + 66x^2 + 32x^3 + 8x^4.$

With this, the proof is completed. ■

Remark:

1. $Sc(C_e(C_4)_2;x) = 34x + 28x^2 + 8x^3.$
2. $Sc^*(C_e(C_4)_2;x) = 41x + 32x^2 + 8x^3.$

Corollary 1.2: For $p \ge 3$, then they will be given:

1. $Sc(C_e(C_4)_{p-1}) = 2p(2p^2+p-2).$
2. $Sc^*(C_e(C_4)_{p-1}) = p(6p^2-3p-2).$ ■

Corollary 1.3: If n is a number of rings $C_4, n \ge 2$, then they will be given:

1. $Sc(C_e(C_4)_n) = 2(2n^3+7n^2+6n+1).$
2. $Sc^*(C_e(C_4)_n) = (6n^3+15n^2+10n+1).$ ■

Corollary 1.4: For $p \ge 3$, the they will be given:

1. $\overline{Sc}(C_e(C_4)_{p-1}) = 2(p+1) - \frac{2}{2p-1}.$
2. $\overline{Sc^*}(C_e(C_4)_{p-1}) = 3p - \frac{2}{2p-1}.$ ■

From Fig. 3(b), can be noted that $p(R_e(C_4)_p) = 2p, q(R_e(C_4)_p) = 3p$, $diam(R_e(C_4)_p) = \left\lceil\frac{p+1}{2}\right\rceil$ and for $1 \le i,j \le p, i \ne j$, there will be:

Table 2: – Edge – Identification Ring of C_4, $R_e(C_4)_p$.

+	×	$degu_i = 3$		$degv_i = 3$	
$degu_j = 3$		6	9	6	9
$degv_j = 3$		6	9	6	9

Theorem 2.1: For $p \ge 6$, then they will be given:

1. $$Sc(R_e(C_4)_p;x) = 18px + 24p\sum_{k=2}^{\left\lceil\frac{p+1}{2}\right\rceil-2} x^k + 6p\begin{cases}4x^{\frac{p-1}{2}} + 2x^{\frac{p+1}{2}}, & when\ p\ is\ an\ odd,\\ 3x^{\frac{p}{2}} + x^{\frac{p}{2}+1}, & when\ p\ is\ an\ even.\end{cases}$$

2. $$Sc^*(R_e(C_4)_p;x) = 27px + 36p\sum_{k=2}^{\left\lceil\frac{p+1}{2}\right\rceil-2} x^k + 9p\begin{cases}4x^{\frac{p-1}{2}} + 2x^{\frac{p+1}{2}}, & when\ p\ is\ an\ odd,\\ 3x^{\frac{p}{2}} + x^{\frac{p}{2}+1}, & when\ p\ is\ an\ even.\end{cases}$$

Proof: For every two vertices $u,v \in V(R_e(C_4)_p)$ there is $d(u,v) = k, 1 \le k \le \left\lceil\frac{p+1}{2}\right\rceil$.

There will be four partitions for proof:

P1. If $d(u,v) = 1$, then $|D_1| = 3p = q(R_e(C_4)_p)$, and one subsets will be of it:

$|D_1(3,3)| = |\{(u_i,u_{i+1}),(v_i,v_{i+1}),(u_i,v_i): 1 \le i \le p\}| = 3p$, where $u_{p+1} \equiv u_1$ and $v_{p+1} \equiv v_1$.

P2. If $d(u,v) = k$, for $2 \le k \le \left\lceil\frac{p+1}{2}\right\rceil - 2$, then $|D_k| = 4p$, and one subsets will be of it:

$|D_k(3,3)| = |\{(u_i,u_{i+k}),(v_i,v_{i+k}),(u_i,v_{i+k-1})(v_i,u_{i+k-1}): 1 \le i \le p\}| = 4p$, where $u_{p+a} \equiv u_a$ and $v_{p+a} \equiv v_a, a = 1,2,3,\dots k$.

P3. If $d(u,v) = \left\lceil\frac{p+1}{2}\right\rceil - 1$, then

$$\left|D_{\left\lceil\frac{p+1}{2}\right\rceil-1}\right| = \begin{cases}4p, & when\ p\ is\ an\ odd,\\ 3p, & when\ p\ is\ an\ even.\end{cases}$$

a. If p is an odd, one subset will be of it:

$$\left|D_{\left\lceil\frac{p+1}{2}\right\rceil-1}(3,3)\right| = |\{(u_{\frac{p-1}{2}},u_{p-1}),(v_{\frac{p-1}{2}},v_{p-1})\} \cup \{(u_i,u_{i+\frac{p-1}{2}}),(v_i,v_{i+\frac{p-1}{2}}),(u_i,v_{i+\frac{p-3}{2}}),(v_i,u_{i+\frac{p-3}{2}}): 1 \le i \le \tfrac{p-3}{2}\}| = 4p,$$ where

$u_{p+a} \equiv u_a$ and $v_{p+a} \equiv v_a, a = 1,2,3,\dots,\frac{p-3}{2}$.
$\left|D_{\lceil\frac{p+1}{2}\rceil-1}(3,3)\right| = |\{\left(u_i, u_{i+\frac{p-1}{2}}\right), \left(v_i, v_{i+\frac{p-1}{2}}\right), \left(u_i, v_{i+\frac{p-3}{2}}\right), \left(v_i, u_{i+\frac{p-3}{2}}\right) : 1 \le i \le p\} = 4p$, where $u_{p+a} \equiv u_a$ and $v_{p+a} \equiv v_a, a = 1,2,3,\dots,\frac{p-1}{2}$.

b. If p is an even, one subset will be of it:
$\left|D_{\lceil\frac{p+1}{2}\rceil-1}(3,3)\right| = \left|\left\{\left(u_i, u_{i+\frac{p}{2}}\right), \left(v_i, v_{i+\frac{p}{2}}\right) : 1 \le i \le \frac{p}{2}\right\} \cup \left\{\left(u_i, v_{i+\frac{p}{2}-1}\right), \left(v_i, u_{i+\frac{p}{2}-1}\right) : 1 \le i \le p\right\}\right| = 3p$, where $u_{p+a} \equiv u_a$ and $v_{p+a} \equiv v_a, a = 1,2,3,\dots,\frac{p}{2}-1$.

P4. If $d(u,v) = \left\lceil\frac{p+1}{2}\right\rceil$, then
$$\left|D_{\lceil\frac{p+1}{2}\rceil}\right| = \begin{cases} 2p, & when\ p\ is\ an\ odd, \\ p, & when\ p\ is\ an\ even. \end{cases}$$
a. If p is an odd, one subset will be of it:
$\left|D_{\lceil\frac{p+1}{2}\rceil}(3,3)\right| = \left|\left\{\left(u_i, v_{i+\frac{p-1}{2}}\right), \left(v_i, u_{i+\frac{p-1}{2}}\right) : 1 \le i \le p\right\}\right| = 2p,$ where $u_{p+a} \equiv u_a$ and $v_{p+a} \equiv v_a, a = 1,2,3,\dots,\frac{p-1}{2}$.

b. If p is an even one subset will be of it:
$\left|D_{\lceil\frac{p+1}{2}\rceil}(3,3)\right| = \left|\left\{\left(u_i, v_{i+\frac{p}{2}}\right), \left(v_i, u_{i+\frac{p}{2}}\right) : 1 \le i \le \frac{p}{2}\right\}\right| = p.$
$\left|D_{\lceil\frac{p+1}{2}\rceil}(3,3)\right| = \left|\left\{\left(u_i, v_{i+\frac{p}{2}}\right) : 1 \le i \le p\right\}\right| p$, where $v_{p+a} \equiv v_a, a = 1,2,3,\dots,\frac{p}{2}$.
From P1 and P4 and Table 2, there will be:

1. $Sc(R_e(C_4)_p; x) = 18px + 24p\sum_{k=2}^{\lceil\frac{p+1}{2}\rceil-2} x^k$
$$+6p\begin{cases} 4x^{\frac{p-1}{2}} + 2x^{\frac{p+1}{2}}, when\ p\ is\ an\ odd, \\ 3x^{\frac{p}{2}} + x^{\frac{p}{2}+1}, \ when\ p\ is\ an\ even. \end{cases}$$

Now, modified Schultz polynomial can be computed from the Schultz polynomial by Corollary (1.1). Since the graph $R_e(C_4)_p$ is 3-regular graph, then:

2. $Sc^*(R_e(C_4)_p; x) = \frac{3}{2}Sc(R_e(C_4)_p; x) = 27px$
$$+36p\sum_{k=2}^{\lceil\frac{p+1}{2}\rceil-2} x^k$$
$$+9p\begin{cases} 4x^{\frac{p-1}{2}} + 2x^{\frac{p+1}{2}}, \ when\ p\ is\ an\ odd, \\ 3x^{\frac{p}{2}} + x^{\frac{p}{2}+1}, \ when\ p\ is\ an\ even. \end{cases}$$

With this, the proof is completed. ■

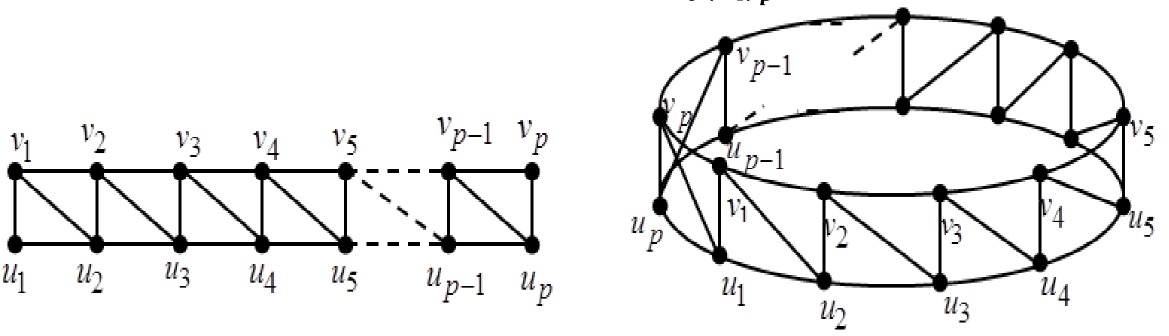
Remark:
1. $poly(R_e(C_4)_3; x) = \alpha\{9x + 6x^2\}$.
2. $poly(R_e(C_4)_4; x) = \alpha\{12x + 12x^2 + 4x^3\}$.
3. $poly(R_e(C_4)_5; x) = \alpha\{15x + 20x^2 + 10x^3\}$.

Where $\alpha = 6$ when the polynomial poly is Schultz and $\alpha = 9$ when the polynomial poly is modified Schultz.

Corollary 2.2: For $p \ge 3$, they will be given:
1. $Sc(R_e(C_4)_p) =$
$$3p\begin{cases} (p^2 + 2p - 1), & when\ p\ is\ an\ odd, \\ p(p+2), & when\ p\ is\ an\ even. \end{cases}$$
2. $Sc^*(R_e(C_4)_p) =$
$$\frac{9}{2}p\begin{cases} (p^2 + 2p - 1), & when\ p\ is\ an\ odd, \\ p(p+2), & when\ p\ is\ an\ even. \end{cases}$$
■

Corollary 2.3: For $p \ge 3$, they will be given:
1. $\overline{Sc}(R_e(C_e)_p) =$
$$\begin{cases} \frac{3\left(2p+5+\frac{1}{2p-1}\right)}{4}, when\ p\ is\ an\ odd, \\ \frac{3\left(2p+5+\frac{5}{2p-1}\right)}{4}, when\ p\ is\ an\ even. \end{cases}$$
2. $\overline{Sc^*}(R_e(C_e)_p) =$
$$\begin{cases} \frac{9\left(2p+5+\frac{1}{2p-1}\right)}{8}, when\ p\ is\ an\ odd, \\ \frac{9\left(2p+5+\frac{5}{2p-1}\right)}{8}, when\ p\ is\ an\ even. \end{cases}$$
■

Schultz and Modified Schultz of $C_e(\not{C}_4)_{p-1}$ and $R_e(\not{C}_4)_p$:



(a) – Edge – Identification Chain of $\not{C}_4$, $C_e(\not{C}_4)_{p-1}$. **(b) – Edge – Identification Ring of $\not{C}_4$, $R_e(\not{C}_4)_p$.**

Figure 4.

From Fig. 4 (a), can be noted that $p\left(C_e(\mathcal{C}_4)_{p-1}\right) = 2p$, $q(C_e(\mathcal{C}_4)_{p-1}) = 4p-3$ and $diam\left(C_e(C_4)_{p-1}\right) = p$. For all $2 \le i,j \le p-1$, $i \neq j$, there will be:

Table 3. Edge – Identification Chain of $\mathcal{C}_4$, $C_e(\mathcal{C}_4)_{p-1}$.

+ ×	$degu_1 = 2$		$degv_1 = 3$		$degu_i = 4$		$degv_i = 4$		$degu_p = 3$		$degv_p = 2$	
	+	×	+	×	+	×	+	×	+	×	+	×
$degu_1 = 2$			5	6	6	8	6	8	5	6	4	4
$degv_1 = 3$	5	6			7	12	7	12	6	9	5	6
$degu_j = 4$	6	8	7	12	8	16	8	16	7	12	6	8
$degv_j = 4$	6	8	7	12	8	16	8	16	7	12	6	8
$degu_p = 3$	5	6	6	9	7	12	7	12			5	6
$degv_p = 2$	4	4	6	8	6	8	5	6	5	6		

Theorem 3.1: For $p \ge 4$, then they will be given:

1. $Sc\left(C_e(\mathcal{C}_4)_{p-1};x\right) = 2(16p-19)x + 4\sum_{k=2}^{p-1}(8p-8k-1)x^k + 4x^p.$

2. $Sc^*\left(C_e(\mathcal{C}_4)_{p-1};x\right) = 4(16p-25)x + 32\sum_{k=2}^{p-2}(2p-2k-1)x^k + 37x^{p-1} + 4x^p.$

Proof: For all $p \ge 5$ and every two vertices $u,v \in V(C_e(\mathcal{C}_4)_{p-1})$, there is $d(u,v) = k$, $1 \le k \le p$. There will be five partitions for proof:

P1. If $d(u,v) = 1$, then $|D_1| = 4p-3 = q(C_e(\mathcal{C}_4)_{p-1})$ and four subsets will be of it:

P1.1 $|D_1(2,3)| = \left|\{(u_1,v_1),(v_p,u_p)\}\right| = 2.$

P1.2 $|D_1(2,4)| = \left|\{(u_1,u_2),(v_p,v_{p-1})\}\right| = 2.$

P1.3 $|D_1(3,4)| = |\{(v_1,v_2),(v_1,u_2),(u_p,u_{p-1}),(u_p,v_{p-1})\}| = 4.$

P1.4 $|D_1(4,4)| = |\{(u_i,u_{i+1}),(v_i,v_{i+1}),(u_i,v_i),(v_i,u_{i+1}): 2 \le i \le p-2\} \cup \{(v_{p-1},u_{p-1})\}| = 4p-11.$

P2. If $d(u,v) = k$, $2 \le k \le p-3$, then $|D_k| = 4p-4k+1$ and three subsets will be of it:

P2.1 $|D_k(2,4)| = |\{(u_1,u_{1+k}),(u_1,v_k),(v_p,v_{p-k}),(v_p,u_{p-k+1})\}| = 4.$

P2.2 $|D_k(3,4)| = |\{(v_1,v_{1+k}),(v_1,u_{1+k}),(u_p,v_{p-k}),(u_p,u_{p-k})\}| = 4.$

P2.3 $|D_k(4,4)| = |\{(u_i,u_{i+k}),(v_i,v_{i+k}),(v_i,u_{i+k})(u_i,v_{i+k-1}): 2 \le i \le p-k-1\} \cup \{(u_{p-k},v_{p-1})\}| = 4p-4k-7.$

P3. If $d(u,v) = p-2$, then $|D_{p-2}| = 9$, and three subsets will be of it:

P3.1 $|D_{p-2}(2,4)| = |\{(u_1,u_{p-1}),(u_1,v_{p-2}),(v_p,u_3),(v_p,v_2)\}| = 4.$

P3.2 $|D_{p-2}(3,4)| = |\{(v_1,v_{p-1}),(v_1,u_{p-1}),(u_p,v_2),(u_p,u_2)| = 4.$

P3.3 $|D_{p-2}(4,4)| = \left|\{(u_2,v_{p-1})\}\right| = 1.$

P4. If $d(u,v) = p-1$, then $|D_{p-1}| = 5$ and three subsets will be of it:

P4.1 $|D_{p-1}(2,3)| = \left|\{(u_1,u_p),(v_p,v_1)\}\right| = 2.$

P4.2 $|D_{p-1}(2,4)| = \left|\{(u_1,v_{p-1}),(v_p,u_2)\}\right| = 2.$

P4.3 $|D_{p-1}(3,3)| = \left|\{(v_1,u_p)\}\right| = 1.$

P5. If $d(u,v) = p$, then $|D_p| =1$ and one subsets will be of it: $|D_p(2,2)| = \left|\{(u_1,v_p)\}\right| = 1.$

From P1 – P5 and Table 3, there will be:

$$Sc\left(C_e(\mathcal{C}_4)_{p-1};x\right) = 2(16p-19)x + 4\sum_{k=2}^{p-1}(8p-8k-1)x^k + 4x^p.$$

And

$$Sc^*\left(C_e(\mathcal{C}_4)_{p-1};x\right) = 4(16p-25)x + 32\sum_{k=2}^{p-2}(2p-2k-1)x^k + 37x^{p-1} + 4x^p.$$

It is easy to calculate that:
$Sc(C_e(\mathcal{C}_4)_3;x) = 90x + 60x^2 + 28x^3 + 4x^4.$
$Sc^*(C_e(\mathcal{C}_4)_3;x) = 156x + 96x^2 + 37x^3 + 4x^4.$
With this, the proof is completed. ■

Remark:

1. $Sc(C_e(\mathcal{C}_4)_2;x) = 58x + 28x^2 + 4x^3.$

2. $Sc^*(C_e(\mathcal{C}_4)_2;x) = 92x + 37x^2 + 4x^3.$

Corollary 3.2: For $p \ge 3$, then they will be given:

1. $Sc\left(C_e(\mathcal{C}_4)_{p-1}\right) = \frac{2(8p^3-3p^2+p-3)}{3}.$

2. $Sc^*\left(C_e(\mathcal{C}_4)_{p-1}\right) = \frac{(32p^3-48p^2+43p-27)}{3}.$ ■

Corollary 3.3: If n is a number of rings $\mathcal{C}_4$, $n \ge 2$, then they will be given:

1. $Sc(C_e(\mathcal{C}_4)_n) = \frac{2(8n^3+21n^2+19n+3)}{3}.$

2. $Sc^*(C_e(\mathcal{C}_4)_n) = \frac{n(32n^2+48n+43)}{3}.$ ■

Corollary 3.4: For $p \ge 3$, then they will be given:

1. $\overline{Sc}\left(C_e(\mathcal{C}_4)_{p-1}\right) = \frac{8p+1}{3} + \frac{p-2}{p(2p-1)}.$

2. $\overline{Sc^*}(C_e(\not{C}_4)_{p-1}) = \frac{16(p-1)}{3} + \frac{9(p-1)}{p(2p-1)}$. ■

From Fig. 4 (b), can be noted that $p(R_e(\not{C}_4)_p) = 2p,\ q(R_e(\not{C}_4)_p) = 4p$ and $diam(R_e(\not{C}_4)_p) = \left\lceil\frac{p}{2}\right\rceil$. For $1 \le i,j \le p,\ i \ne j$, there will be:

Table 4. Edge – Identification Ring of $\not{C}_4$, $R_e(\not{C}_4)_p$.

+	×	$degu_i = 4$		$degv_i = 4$	
$degu_j = 4$		8	16	8	16
$degv_j = 4$		8	16	8	16

Theorem 4.1: For $p \ge 5$, then they will be given:

1. $Sc(R_e(\not{C}_4)_p;x) = 32p\sum_{k=1}^{\left\lceil\frac{p}{2}\right\rceil-1} x^k + 8p\begin{cases} x^{\frac{p+1}{2}}, & when\ p\ is\ an\ odd, \\ 3x^{\frac{p}{2}}, & when\ p\ is\ an\ even. \end{cases}$

2. $Sc^*(R_e(\not{C}_4)_p;x) = 64p\sum_{k=1}^{\left\lceil\frac{p}{2}\right\rceil-1} x^k + 16p\begin{cases} x^{\frac{p+1}{2}}, & when\ p\ is\ an\ odd, \\ 3x^{\frac{p}{2}}, & when\ p\ is\ an\ even. \end{cases}$

Proof: For every two vertices $u, v \in V(R_e(\not{C}_4)_p)$, there is $d(u,v) = k, 1 \le k \le \left\lceil\frac{p}{2}\right\rceil$.
There will be two partitions for proof:

P1. If $d(u,v) = k,\ 1 \le k \le \left\lceil\frac{p}{2}\right\rceil - 1$, then $|D_k| = 4p$ and one subsets will be of it:
$|D_k(4,4)| = |\{(u_i,u_{i+k}),(v_i,v_{i+k}),(u_i,v_{i+k-1}),(v_i,u_{i+k}): 1 \le i \le p\}| = 4p$, where $u_{p+a} \equiv u_a$ and $v_{p+a} \equiv v_a,\ a = 1,2,3,\dots,k$.

P2 . If $d(u,v) = \left\lceil\frac{p}{2}\right\rceil$, then
$\left|D_{\left\lceil\frac{p}{2}\right\rceil}\right| = \begin{cases} p\ , when\ p\ is\ an\ odd, \\ 3p\ , when\ is\ an\ even. \end{cases}$

a. If p is an odd, and one subset will be of it:
$\left|D_{\frac{p+1}{2}}(4,4)\right| = \left|\left\{\left(u_i, v_{i+\frac{p-1}{2}}\right): 1 \le i \le p\right\}\right| = p$,
where $v_{p+a} \equiv v_a,\ a = 1,2,3,\dots,\frac{p-1}{2}$.

b. If p is an even, and one subset will be of it:
$\left|D_{\frac{p}{2}}(4,4)\right| = \left|\left\{\left(u_i,u_{i+\frac{p}{2}}\right),\left(v_i,v_{i+\frac{p}{2}}\right): 1 \le i \le \frac{p}{2}\right\} \cup \left\{\left(u_i,v_{i+\frac{p}{2}-1}\right),\left(v_i,u_{i+\frac{p}{2}}\right): 1 \le i \le p\right\}\right| = 3p$,
where $u_{p+a} \equiv u_a$, and $v_{p+a} \equiv v_a,\ a = 1,2,3,\dots,\frac{p}{2}$.

From P1and P2 and Table 4, there will be:

1. $Sc(R_e(\not{C}_4)_p;x) = 32p\sum_{k=1}^{\left\lceil\frac{p}{2}\right\rceil-1} x^k + 8p\begin{cases} x^{\frac{p+1}{2}}, & when\ p\ is\ an\ odd, \\ 3x^{\frac{p}{2}}, & when\ p\ is\ an\ even. \end{cases}$

Now, modified Schultz polynomial can be computed from the Schultz polynomial by (1.1). Since the graph $R_e(\not{C}_4)_p$ is 4-regular graph, then:

2. $Sc^*(R_e(\not{C}_4)_p;x) = 2Sc(R_e(\not{C}_4)_p;x)$
$= 64p\sum_{k=1}^{\left\lceil\frac{p}{2}\right\rceil-1} x^k + 16p\begin{cases} x^{\frac{p+1}{2}}, & when\ p\ is\ an\ odd, \\ 3x^{\frac{p}{2}}, & when\ p\ is\ an\ even. \end{cases}$ ■

Remark:
1. $poly(R_e(\not{C}_4)_3,x) = \propto \{12x + 3x^2\}$.
2. $poly(R_e(\not{C}_4)_4,x) = \propto \{16x + 12x^2\}$.
Where $\propto = 8$ when *poly* is polynomial Schultz and $\propto = 16$ when *poly* is polynomial modified Schultz.

Corollary 4.2: For all $p \ge 3$, then they will be given:
1. $Sc(R_e(\not{C}_4)_p) = 4p^2(p+1)$.
2. $Sc^*(R_e(\not{C}_4)_p) = 8p^2(p+1)$. ■

Corollary 4.3: For all $p \ge 3$, they will be given:
1. $\overline{Sc}(R_e(\not{C}_4)_p) = 2p + 3 + \frac{3}{2p-1}$.
2. $\overline{Sc^*}(R_e(\not{C}_4)_p) = 2\left(2p + 3 + \frac{3}{2p-1}\right)$. ■

Some Properties of the Coefficients of Schultz and Modified Schultz Polynomials:

A finite sequence $(a_1, a_2, \dots, a_h)$ of h positive integers is coefficients of polynomial $P(x) = \sum_{i=1}^{h} a_i x^i$. Then (Table 5):

1. The polynomial $P(x)$ is called j-unimodal if, for some index j, $a_1 \le a_2 \le \cdots \le a_j \ge a_{j+1} \ge \cdots \ge a_h$ and it is strictly j-unimodal if the inequality holds without equalities.
2. The polynomial $P(x)$ is called monotonically increasing (or monotonically decreasing) if, $a_i \le a_{i+1}$ or $a_i \ge a_{i+1}$, respectively, for all $1 \le i \le h$ and it is strictly-increasing or strictly-decreasing respectively if the inequalities holds without equalities.
3. The polynomial $P(x)$ is called palindromic if $a_i = a_{h-i+1}$, for all $1 \le i \le h$ and is called semi-palindromic if $a_j = a_{h-j+1}$, $1 + i \le j \le h - i$ and for all $1 \le i \le h - 2$.
4. The polynomial $P(x)$ is called troubled if $a_i \ne a_{i+1}$, for all $1 \le i \le h$.

5. The polynomial $P(x)$ is called equality if $a_i = a_{i+1}$, for all $1 \leq i \leq h$ and is called semi-equality if $a_i = a_{i+1}$ for some values of i.

Table 5. Some Properties of the Coefficients of Schultz and Modified Schultz Polynomials

Polynomials of Types graphs	Property 1	Property 2	Property 3	Property 4	property 5
$Sc(C_e(C_4)_{p-1};x)$	Satisfy at $j=2$	Not Satisfy	Not Satisfy	Satisfy	Not Satisfy
$Sc^*(C_e(C_4)_{p-1};x)$	Satisfy at $j=2$	Not Satisfy	Not Satisfy	Satisfy	Not Satisfy
$Sc(R_e(C_4)_p;x)$	Satisfy at $j=2$	Not Satisfy	Satisfy semi at $j=2$	Not Satisfy	Satisfy semi at $j=2$ to $j=\lceil p/2\rceil-1$
$Sc^*(R_e(C_4)_p;x)$	Satisfy at $j=2$	Not Satisfy	Satisfy semi at $j=2$	Not Satisfy	semi at $j=2$ Satisfy to $j=\lceil p/2\rceil-1$
$Sc(C_e(\not{C}_4)_{p-1};x)$	Satisfy at $j=1$	satisfy	Not satisfy	Satisfy	Not Satisfy
$Sc^*(C_e(\not{C}_4)_{p-1};x)$	Satisfy at $j=1$	satisfy	Not satisfy	Satisfy	Not Satisfy
$Sc(R_e(\not{C}_4)_p;x)$	Satisfy at $j=\lceil p/2\rceil-1$	satisfy	Satisfy semi at $j=2$	Not satisfy	Satisfy semi from $i=1$to $\lceil p/2\rceil-1$
$Sc^*(R_e(\not{C}_4)_p;x)$	Satisfy at $j=\lceil p/2\rceil-1$	satisfy	Satisfy semi at $j=2$	Not satisfy	Satisfy semi from $j=1$ to $\lceil p/2\rceil-1$

Conclusion:

From this paper, the general formulas can be obtained for square graphs for the modified Schultz and Schultz polynomials and some of their properties have been discussed.

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Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
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Authors' contributions statement:

Mahmood M. and Ahmed M. are find the new polynomials which are called Schultz and modified Schultz using a new technique to finding general formulas, indices, average and then studied some properties of coefficients polynomials of Schultz and modified Schultz polynomials.

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متعددات حدود شوالتز وشوالتز المعدلة لتطابق حافة لسلسلة وحلقة للبيانات المربعة

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الخلاصة:

في البيان المتصل G، دالة المسافة بين أي رأسين من رؤوس البيان $V(G)$ هي أقصر مسافة بينهما، كما تعرف درجة الرأس u والتي يرمز لها بـ $degu$ بأنها عدد الحافات الواقعة عليه. متعددة حدود شوالتز وشوالتز المعدلة تعرف كالاتي:
$Sc(G;x) = \sum(degu + degv)\, x^{d(u,v)}$ $and\ Sc*(G;x) = \sum (degu.degv)\, x^{d(u,v)}$,
على التوالي، حيث أن المجموع يؤخذ لكل الازواج غير المرتبة من الرؤوس المختلفة في $V(G)$ وأن $d(u,v)$ هي المسافة بين الرأسين u و v في $V(G)$. في هذا البحث استطعنا الحصول على صيغ عامة لكل من متعددة حدود شوالتز وشوالتز المعدلة ودليليهما ومعدلهما لتطابق الحافة لسلسلة وحلقة للبيانات المربعة.

الكلمات المفتاحية: تطابق الحافة لسلسلة وحلقة للبيانات المربعة، متعددات الحدود، الأدلة.