Al-Qadisiyah Journal for Administrative and Economic Sciences ISSN$_{Online}$: 2312-9883
QJAE, Volume 25, Issue 3 (2023) ISSN$_{Print}$: 1816-9171

New Bayesian Variable Selection for Binary quantile regression

Taha Alshaybawee
Mariam Qais Fahem
Al-Qadisiyah University - College of Administration and Economics
Corresponding Author : Mariam Qais Fahem

Abstract : Quantile regression extends the mean regression model to conditional quantiles of the response variable (Koenker and Bassett, 1978; Koenker, 2005). As a result of its capacity to take advantage of all available data in the analysis, the Bayesian method of analysis has become extremely popular. We proposed a novel estimation method and selected variables. Using Bayesian methodology for binary quantile regression. On the basis of the recently proposed scale mixture of normal distribution mixing Rayleigh density representation for Laplace distribution prior density of the parameters vector, new hierarchical models have been developed. Simulation Scenarios and actual data are used to evaluate the efficacy of the new method in comparison to two existing methods.

Keyword: Quantile regression, Bayesian Inference, Posterior distribution, Binary data.

INTRODUCTION: The classical theory of linear models concentrates on the conditional mean function, which defines the relationship between the response variable y and the covariate vector x. However, the mean may not be of paramount importance to the researcher, or additional information regarding the entire conditional distribution of the response variable may be required. For instance, Whittaker et al. (2005) demonstrate that various quantiles of the response distribution necessitate different predictors for bank account usage. In addition, least squares methods, which center on the conditional mean function, presume the error has the same distribution regardless of the value of x. It is anticipated that the components of the vector of x will affect only the location of the conditional distribution of y, and not its scale or any other attribute of its distributional shape. In practice, however, it is frequently challenging to uphold these assumptions.

Quantile regression extends the mean regression model to conditional quantiles of the response variable (Koenker and Bassett, 1978; Koenker, 2005). Note that quantile regression includes median regression (or equivalently L1-regression), as the median is the most central quantile that divides the upper and lower halves of a sample. The technique provides a more nuanced view of the relationship between the dependent variable and the covariates, as it enables the user to investigate the relationship between a set of covariates and the various sections of the response variable's distribution.

A further advantage of the quantile regression method is that parameter estimates are not biased by a location-scale shift in the conditional distribution of the dependent variable. Not only have theoretical statisticians acknowledged these two distinct benefits, but they have also encouraged researchers from a variety of disciplines to employ quantile regression in their studies. Applications range from ecology (Brown and Peet, 2003), to cancer research (Li and Zhu, 2007), to economics (Buchinski, 1994; 1998), among others. See Yu et al. (2003) for a comprehensive overview of the various applications of quantile regression. In addition, quantile regression has been expanded to model dependent variables besides ratio/scale variables. Among these extensions are models for left-censored data (Powell, 1986; Yu and Stander, 2007), count data (Machado and Santos Silva, 2005), and proportions (Hewson and Yu, 2008).

Adopting quantile regression in the case of a binary response variable is not an evident choice. The dependent variable has only two possible values; consequently, regression cannot be used to model continuous quantiles. Nonetheless, numerous authors have acknowledged the prospective advantages of binary quantile regression. Manski (1975; 1985) defined the general semi-parametric binary quantile regression estimator. The focus of succeeding research has been exclusively on the median case for unknown reasons (Koenker and Hallock, 2001). Recent research by Kordas (2006) has examined the ramifications of estimating quantiles other than the median for binary regression models and demonstrated that, even in the dichotomous case, the approach provides a much richer understanding of how covariates influence the response variable.

Benoit and Van den Poel (2012) list some of the disadvantages of the frequentist approach, including the difficulty in optimizing the estimated parameters and the structure of the confidence interval. Benoit and Van den Poel (2012) adopted a new estimation procedure based on the Bayesian approach due to these issues. As a result of its capacity to take advantage of all available data in the analysis, the Bayesian method of analysis has become extremely popular. Benoit et al. (2013) proposed Bayesian lasso variable selection for quantile regression when the response variable is binary and the Gibbs sampler technique is used to estimate the model parameters. Alhamzawi (2015) proposed the normal prior distribution with mean zero and unknown variance for the parameter vector for each quantile when the response variable is binary and obtained a censored quantile model as a result. Hashem and colleagues (2015) introduced the Bayesian group lasso penalty for binary quantile regressions. In this paper, we employ the new scale mixture of Normal distribution that mixes with independent Rayleigh distribution representation of the Laplace density proposed by Flaih et al. in 2020 to develop a new hierarchical Bayesian model for estimation and variable selection of binary quantile regression.

2. New Bayesian Binary quantile Regression:

Quantile regression has become a very important tool since the seminal work of Koenker and Bassett (1978). Empirical studies took the features of quantile regression ability to research the impact of prediction variables on the dependent variable Hao et al. (2007). The model can be described as follows:

$$y_i = x_i'\beta_\tau + e_{i\tau}, \qquad i = 1,2,\dots,n$$

where x = ($x_1,\dots,x_n$) be an $(n \times m)$ design matrix of covariates variables, $e_{i\tau}$ is the error term at τ quantile, $(y_1,\dots, y_n)$ is the response variable and $^T\beta_\tau = (\beta_{\tau 0}, \beta_{\tau 1}, \dots, \beta_{\tau m})$ are the unknown quantile parameters. Binary quantile regression has been addressed by many researchers in the last few decades and they receive more attention in the literature see (Horowiz (1992), and Benoit and van den poel (2012)). In this study, we suppose that the response variable y_i $(i = 1,2,\dots,n)$ are observed variable and take the values $(y_i = 0 \; or \; 1)$. Whereas this variable is determined by the unobserved Latent variable y_i^*. So that we can rewrite the model as follows:

$$y_i^* = x_i'\beta_\tau + e_{i\tau}, \; i = 1,2,\dots,n$$
$$y_i = g(y_i^*) \quad where \; y_i = \{1 \quad if \quad y_i^* \geq 0 \quad 0 \quad otherwise$$

We can estimated the parameter vector β_τ at each quantile by minimizing the objective function :

$$E\{\rho_\tau(\, y_i^* - g_\tau(x_i'\beta_\tau)),$$

where $\rho_\tau(.)$ is the check function and can be determine as $\rho_\tau(u) = u\{\tau - I(u > 0)\}$, I(.) is the indicator function. Bayesian Inference supplies an important method for accurate inference even within the case of a small sample. Actually, the Bayesian, quantile regression model is based on the assumption that the error term is distributed as a symmetric Laplace distribution. Minimizing the check function is equivalent to maximizing the density function of symmetric Laplace distribution. (konker and Machado (1999)) and we can write as follows :

$$f_\tau(\frac{y^*}{\tau,\theta) =}\frac{\tau(1-\tau)}{\theta} \; exp \; exp \left\{-\frac{1}{\theta}\,\rho_\tau(\, e_{i\tau})\right\}$$

where τ is the skewness of asymmetric Laplace distribution and θ is the scale parameter.
The likelihood function of $y_i^* = (y_1^*, y_2^*, \dots y_n^*)$ can be shown as:

$$f_\tau(\frac{y^*}{\tau,\theta,\beta_\tau) =}\frac{\tau^n(1-\tau)^n}{\theta^n} \; exp \; exp \left\{-\frac{1}{\theta}\sum_{i=1}^{n} \quad \rho_\tau(\, y_i^* - x_i'\beta_\tau)\right\}$$

Asymmetric Laplace distribution can be re-write as a mixture of scale normal distribution and an exponential Kozumi and Kobayashi (2009). Therefore, the error term can be write as $e_{i\tau} = \varphi_1 u_i + \sqrt{\varphi_2\theta u_i}v_i$, then the model :

$$y_i^* = x_i'\beta_\tau + \varphi_1 u_i + \sqrt{\varphi_2\theta u_i}v_i$$

Where $u_i \sim exp(\frac{\theta}{\tau(1-\tau))}$ is an exponential distribution, v_i is the standard normal, $\varphi_1 = 1 - 2\tau$ and $\varphi_2 = 2$ Alshaybawee et al. (2016), so that the likelihood function become as follows:

$$f(\beta,\theta,x,u) = (4\pi\theta u_i)^{\frac{-n}{2}} exp\left\{-\frac{1}{4\theta u_i}\sum_{i=1}^{n} \quad (y_i^* - x_i'\beta_\tau - \varphi_1 u_i)^2\right\}$$

Here we consider quantile regression with Lasso penalty, Tibshirani (1996) proposed that the Lasso regularized can be explained as posterior mode estimated when the parameters of the model are independent and set Laplace distribution as priors, to the parameter β_τ:

$$\pi(\theta) = \frac{\lambda}{2\theta} exp\left\{-\frac{\lambda|\beta_\tau|}{\theta}\right\}$$

this prior is similar to the l_1- penalty (park and Casella (2008); Hans (2009)). New hierarchical model construct in this study, the prior distribution double exponential density for the parameters represented as scale mixture of normal mixing Rayleigh distribution (Fliah et al. 2020)

$$\frac{1}{2\gamma} exp\left\{-\frac{|X|}{\gamma}\right\} = \int_0^\infty \quad \frac{1}{\sqrt{2\pi t^2}} e^{\frac{-X^2}{2t^2}} \frac{t}{\gamma} e^{\frac{-t^2}{2\gamma}} dt$$

Here, let $\gamma = \frac{\theta}{\lambda}$, then equation can be rewrite as

$$\frac{1}{2\gamma} exp\left\{-\frac{|\beta_\tau|}{\gamma}\right\} = \int_0^\infty \quad \frac{1}{\sqrt{2\pi t^2}} e^{\frac{-\beta_\tau{}^2}{2t^2}} \frac{t}{\gamma} e^{\frac{-t^2}{2\gamma}} dt$$

The prior distribution for $\boldsymbol{\gamma}$ set as Gamma distribution and θ is inverse gamma distribution.
Based on the priors distribution the hierarchical model for Bayesian binary quantile regression will be given as follows:

$$y_i^* = x_i'\beta_\tau + e_{i\tau},$$
$$y_i = \{1 \quad if \quad y_i^* \geq 0 \quad 0 \quad otherwise$$
$$y^*/y,\beta,u_i,\theta \sim N(x_i'\beta_i + \varphi_1 u_i, 2\theta u_i) u_i \sim \frac{\tau(1-\tau)}{\theta} exp\left\{-\frac{\tau(1-\tau)u_i\}}{\theta}\right\}$$
$$v_i \quad \sim \frac{1}{\sqrt{2\pi}} exp\left\{-\frac{v_i^2}{2}\right\} \beta_\tau, t_j^2/\gamma \quad \sim \prod_{j=1}^{p} \frac{1}{\left(2\pi t_j^2\right)^{1/2}} exp\left\{-\frac{\beta_\tau^2}{2t^2}\right\} \prod_{j=1}^{p} \frac{\sqrt{t_j^2}}{\gamma} exp\left\{-\frac{t_j^2}{2\gamma}\right\} \gamma$$
$$\sim (\gamma)^{a-1} exp\{-b\gamma\} \, \theta \quad \sim (\theta)^{-c-1} exp\{-d/\theta\}$$

Posterior of Binary quantile regression:

1- Sampling y_i^* ,i= 1,2,…,n from truncated normal distribution

$$y^*|y,\beta_\tau,u_i,\theta = \{N(x_i'\beta_\tau + \varphi_1 u_i, 2\theta u_i) I(y_i^* \geqslant 0) \quad if \ y_i = 1 \ N(x_i'\beta_\tau + \varphi_1 u_i, 2\theta ui) I(y_i^* < 0) \quad if \ yi = 0$$

2- Sampling β_τ

$$\pi(\beta_\tau/x_i,u_i,t_i,\theta,\gamma) \propto \pi(y^*/y,\beta_\tau,\theta,\gamma,u_i) \times \pi(\beta_\tau/t_i)$$
$$\alpha \ exp \ exp \ \{-\frac{1}{4\theta u_i}\sum(y_i^* - x_i'\beta_\tau - \varphi_1 u_i)^2] \ *exp \ exp \left\{-\frac{\beta_\tau^2}{2t_i}\right\}$$

The conditional posterior dist. For β_τ is a normal distribution N(A, B^)

$$B\hat{} = \sum \quad \left(\frac{x_{ij}^2}{2\theta u_i} - \frac{1}{t_i^2}\right) A = B\hat{}\left\{\sum \quad \frac{x_i(y_i^* - \varphi_1 u_i)}{2\theta ui}\right\}$$

(3) Sampling u_i

$$\pi\big(u/y^*,x\beta_\tau,t_i,\theta,\gamma\big) \times \pi(y^*/y,\beta_\tau,\theta,\gamma,t_i) \times \pi(u_i/\theta)$$

$$\alpha \frac{1}{\sqrt{ui}} exp\left\{-\frac{1}{2}\frac{(y_i^* - x_i'\beta_\tau - \varphi_1 u_i)^2}{2\theta u_i}\right\} \cdot exp\left\{-\frac{\tau(\tau-1)u_i}{\theta}\right\}$$

The full conditional of u_i is $GIG\left(\frac{1}{2}\right) A_1, A_2)$

$$A_1 = \frac{(y_i^* - x'\beta_\tau)^2}{2\theta} \quad , \quad A_2 = \frac{\varphi_1}{2\theta} + \frac{2\tau(1-\tau)}{\theta}$$

(4) Sampling t_i

$$\pi(t_i/y^*,x,\theta,\beta_\tau,u_i,\gamma) \alpha \pi(\beta_\tau/t_i) \cdot \pi(t_i/\gamma) \ \alpha \frac{1}{\sqrt{t_j^2}} exp\left\{-\frac{\beta_\tau^2}{2t^2}\right\} t_j exp\left\{\frac{t^2}{2\gamma}\right\}$$

The full conditional for t_i is $GIG(1, \beta_{1j}, \beta_{2j})$

$$\beta_{1j} = \frac{\beta_\tau^2}{2}, \beta_{2j} = \frac{1}{\gamma}$$

(5) Sampling θ.

$$\pi(\theta/y^*,\beta_\tau,x,\gamma,u_i,t_i) \alpha \pi(y^*/y,\beta_\tau,\theta,\gamma,t_i) \cdot \pi(ui/\theta) \cdot \pi(\theta)$$
$$\propto (\theta)^{-n/2} exp\left\{\frac{1}{2}\sum \quad \frac{(y^* - x'\beta_\tau - \varphi_1 u_i)^2}{2\theta u_i}\right\} \cdot \frac{1}{\theta}\{exp\left\{-\frac{\tau(1-\tau)u_i\}}{\theta}\right\} * \theta^{-c-1} \cdot exp\left\{-\frac{d}{\theta}\right\}$$
$$\propto \theta^{-\frac{n}{2}-c-1-1} \cdot exp\left\{-\frac{1}{\theta}\left[\sum \quad \frac{(y_i^* - x'\beta_\tau - \varphi_1 u_i)^2}{4u_i} + \tau(\tau-1)u_i + d\right]\right\}$$

Then the conditional distribution of θ is
Inverse gamma

(6) sampling γ

$$\pi(\gamma/y^*,\beta_\tau,\theta,u_i,ti,x)\ \propto\ \pi(tj/\gamma)\cdot\pi(\gamma)$$
$$\propto\gamma^{-n}exp\left\{-\frac{1}{\gamma}\Sigma_{j=1}^{p}\frac{t_j^2}{2}\right\}\cdot\gamma^{-a-1}exp\left\{-\frac{b}{\gamma}\right\}\propto\gamma^{-n-a-1}\cdot exp\left\{-\frac{1}{\gamma}\left[\Sigma\frac{t_j^2}{2}+b\right]\right\}$$

then the distribution of t_i is Inverse gamma.

5. Simulation Study

In current study, we want to evaluated our proposed method via simulation approach and compared with other methods in the same filed. First method is Bayesian binary regression quantile denoted by (BBqr) that is proposed by (C. F. Manski,1975). The second method is Bayesian lasso binary quantile regression denoted by (BLBqr) that is proposed by (D. F. Benoit, Alhamzawi,,2011). R code will be used to implement each of these packages. We will discuss our proposed method Bayesian new lasso binary quantile regression denoted by (BNLBQR) where, our proposed method is focus on estimation and variable selection in binary quantile regression via using Bayesian approach. Two criterions are used in current study are: Median of mean absolute deviations, referred to as (MMAD) and computed by the following formula $MMAD=median(mean(|X^T\hat{\beta}-X^T\beta^R|))$ and Mean absolute error referred as (MAE) and computed by the following formula, $MAE=|X^T\hat{\beta}-X^T\beta^R|$.

where β^R is true parameters , $\hat{\beta}$ is estimation parameters. In this simulation, we will used three quantile levels , low quantile at $\theta=0.25$ and middle quantile $\theta=0.50$ and high quantile level $\theta=0.75$. In current simulating, we will use three different distribution for the error term $e_i\sim\chi_{(4)}$ chi-square distribution with 4 degrees of freedom, normal distribution with mean (0) and variance (7) $e_i\sim N(0,7)$ and Laplace distribution with location parameters (0) and scale parameters (1) $e_i\sim Lap(0,1)$. In the current study for each simulation example, The first 2000 iterations of our algorithm's 12000 total iterations were excluded as burn in.

5.1 First Simulation Scenario

In this simulation first scenario study, We demonstrate the efficiency of the our proposed method with sparse models The independent variables are simulated from the uniform distribution (0,1). The correlation between of two pairwise each independent variable equalizes to $0.5^{|l-g|}$. The true parameters $\beta=(1,0,2,0,0,2,0,1,0)$, therefore the true model taken the following formula :

$$y_i=\{1\quad if\quad y_i^*\geq 0\quad 0\quad otherwise\qquad\qquad i=1,\dots,n,$$
$$y_i{}^*=x_{i1}+2x_{i3}+2x_{i6}+x_{i8}+e_i\quad ,$$

y_i is observed depend variable for the unobserved $y_i{}^*$, $y_i{}^*$ is the latent dependent variable.

The results listed in Table 1, it presents a brief of the results for Median of mean absolute deviations (MMAD) and Mean absolute error (MAE) for the three methods under comparison at sparse model . It is clear from this table the MMAD and MAD that computed for our proposed method (BNLBQR) are much smaller than MMAD and MAD are computed by other two methods ($BBqr$) and ($BLBqr$). From these results, We judge that our proposed method has an outstanding performance in coefficient estimation and variable selection in our model study.

Table 1: Median of mean absolute deviations (MMAD) for first simulation example, the results are averaged over 100 independent simulations.

Methods	$e_i\sim\chi_{(4)}$	$e_i\sim N(0,7)$	$e_i\sim Lap(0,1)$.
$BBqr_{0.25}$	0.892 (0.618)	**0.925(**0.752**)**	1.120(0.948)
$BLBqr_{0.25}$	0.862 (0.607)	0. 871(0.692)	1.067(0.839)
$BNLBQR_{0.25}$	0.721(0.585)	0.716(0.572)	**0.973(**0.737**)**
$BBqr_{0.50}$	0.825(0.592)	**0.762(**0.572**)**	0.956(0.734)
$BLBqr_{0.50}$	0.752(0.548)	**0.749(**0.538**)**	0.946(0.692)
$BNLBQR_{0.50}$	0.729(0.528)	**0.692(**0.472**)**	0.847(0.631)
$BBqr_{0.75}$	0.753(0.572)	0.726(0.528)	0.869(0.651)
$BLBqr_{0.75}$	0.681(0.510)	0.708(0.528)	0.821(0.631)
$BNLBQR_{0.75}$	0.584(0.395)	0.573(0.417)	0.762(0.593)

Note: In the parentheses are MAE.

5.2 Second Simulation Scenario

In this simulation scenario study, We demonstrate the efficiency of the our proposed method with dense models The independent variables are simulated from the uniform distribution (0,1). The correlation between of two pairwise each independent variable equalizes to $0.5^{|l-g|}$. The true parameters $\beta = (0.85,0.85,0.85,0.85,0.85,0.85,0.85,0.85,0.85)$, therefore the true model taken the following formula :

$$y_i = \{1 \quad if \quad y_i^* \geq 0 \quad 0 \quad otherwise \qquad i = 1, \dots, n,$$

$$y_i{}^* = 0.85x_{i1} + 0.85x_{i2} + 0.85x_{i1}x_{i3} + 0.85x_{i4} + 0.85x_{i5} + 0.85x_{i6} + 0.85x_{i7} + 0.85x_{i8} + 0.85x_{i9} + e_i \quad ,$$

y_i is observed depend variable for the observation, $y_i{}^*$ is the latent dependent variable.

The results shown in Table 2 present summary of the results for Median of mean absolute deviations (MMAD) and Mean absolute error (MAE) for the three methods under comparison at dense model. It is clear the MMAD and MAD are reported in Table 2 that generated for our proposed method (BNLBQR) are smaller than the MMAD and MAD that computed by the other two methods ($BBqr$) and ($BLBqr$). Based on these results, We can conclude that the proposed method has a good performance in coefficient estimation and variable selection in our model study.

Table 2: Median of mean absolute deviations (MMAD) for second simulation example, the results are averaged over 100 independent simulations.

Methods	$e_i \sim \chi_{(4)}$	$e_i \sim N(0,7)$	$e_i \sim Lap(0,1)$.
$BBqr_{0.25}$	1.211 (0.954)	**1.185**(0.978)	1.006(0.879)
$BLBqr_{0.25}$	1.153 (0.923)	0. 962(0.783)	0.958(0.782)
$BNLBQR_{0.25}$	0.968(0.735)	0.828(0.633)	**0.748**(0.617)
$BBqr_{0.50}$	0.963(0.792)	**0.875**(0.718)	0.892(0.692)
$BLBqr_{0.50}$	0.894(0.747)	**0.817**(0.659)	0.775(0.573)
$BNLBQR_{0.50}$	0.682(0.492)	**0.717**(0.582)	0.648(0.492)
$BBqr_{0.75}$	0.828(0.682)	0.845(0.662)	0.761(0.538)
$BLBqr_{0.75}$	0.726(0.502)	0.782(0.519)	0.684(0.482)
$BNLBQR_{0.75}$	0.619(0.432)	0.525(0.385)	0.519(0.397)

Note: In the parentheses are MAE.

One may choose to look directly at the parameter estimations instead of the MMADs and MADs. Table 3 lists the parameters estimates in the first and second simulation. It can be concluded that our parameters estimates are much closer to the true parameters values than other methods under comparison.

Table 3. parameters estimates for the competing methods to first and second Simulation scenario with normal error distributions

First Simulation Scenario									
	$\theta = 0.25$								
	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9
Comparison Methods	**1**	**0**	**2**	**0**	**0**	**2**	**0**	**1**	**0**
$BBqr_{0.25}$	0.648	0.562	1.563	0.692	0.843	1.082	0.634	0.539	0.683
$BLBqr_{0.25}$	0.793	0.435	1.683	0.472	0.683	1.417	0.394	0.632	0.673
$BNLBQR_{0.25}$	**1.012**	**0.072**	**1.903**	**0.063**	**0.039**	**1.974**	**0.054**	**0.963**	**0.019**
$\theta = 0.50$									
$BBqr_{0.50}$	0.672	0.582	1.583	0.593	0.472	1.491	0.286	0.537	0.356
$BLBqr_{0.50}$	0.736	0.391	1.627	0.318	0.282	1.673	0.125	0.788	0.293
$BNLBQR_{0.50}$	**0.923**	**0.083**	**1.937**	**0.016**	**0.087**	**1.959**	**0.091**	**0.941**	**0.084**
$\theta = 0.75$									
$BBqr_{0.75}$	1.582	0.453	1.583	0.783	0.845	1.493	0.718	0.603	0.627
$BLBqr_{0.75}$	1.373	0.397	1.672	0.619	0.693	1.684	0.506	0.764	0.472
$BNLBQR_{0.75}$	**1.036**	**0.069**	**1.975**	**0.135**	**0.167**	**1.952**	**0.062**	**0.992**	**0.107**
second Simulation Scenario									

$\theta = 0.25$									
	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9
Comparison Methods	0.85	0.85	0.85	0.85	0.85	0.85	0.85	0.85	0.85
$BBqr_{0.75}$	0.153	0.946	0.573	0.683	0.293	0.567	0.854	0.183	0.967
$BLBqr_{0.75}$	0.473	0.649	0.721	0.736	0.483	0.613	1.471	0.276	0.502
$BNLBQR_{0.75}$	**0.746**	**0.926**	**0.810**	**0.836**	**0.719**	**0.685**	**0.862**	**0.933**	**0.821**
$\theta = 0.50$									
$BBqr_{0.75}$	0.492	0.583	0.572	0.810	1.037	**0.563**	**0.694**	**0.673**	0.603
$BLBqr_{0.75}$	0.693	0.782	0.683	0.921	0.985	**0.683**	**0.720**	**0.959**	0.793
$BNLBQR_{0.75}$	**0.923**	**0.808**	**0.789**	**0.893**	**0.890**	**0.837**	**0.839**	**0.846**	**0.906**
$\theta = 0.75$									
$BBqr_{0.75}$	0.283	0.434	0.683	0.367	0.467	0.268	0.384	0.583	0.684
$BLBqr_{0.75}$	0.598	0.663	0.743	0.563	0.563	0.478	0.452	0.793	0.679
$BNLBQR_{0.75}$	**0.704**	**0.832**	**0.904**	**0.903**	**0.826**	**0.683**	**0.693**	**0.869**	**0.907**

The MCMC chain for first simulation at quantile 0.25 as in Figures 1 show the rapidly MCMC our algorithm converges to the stationary . in first simulation at quantile 0.25 as in Figures 2 shows that the our posterior histograms show that the our conditional posterior distributions are closed from normal distributions.

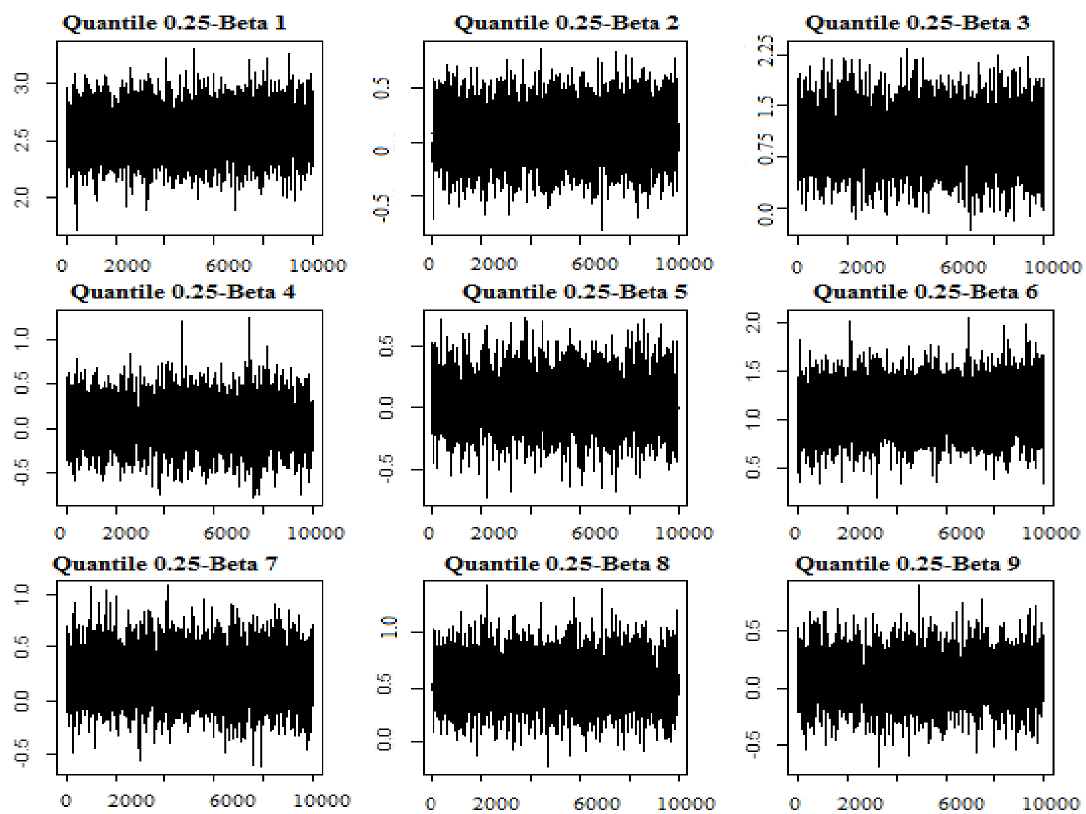

Figure 1. Trace plots of binary quantile regression parameters for first Simulation at quantile level $\theta = 0.25$.

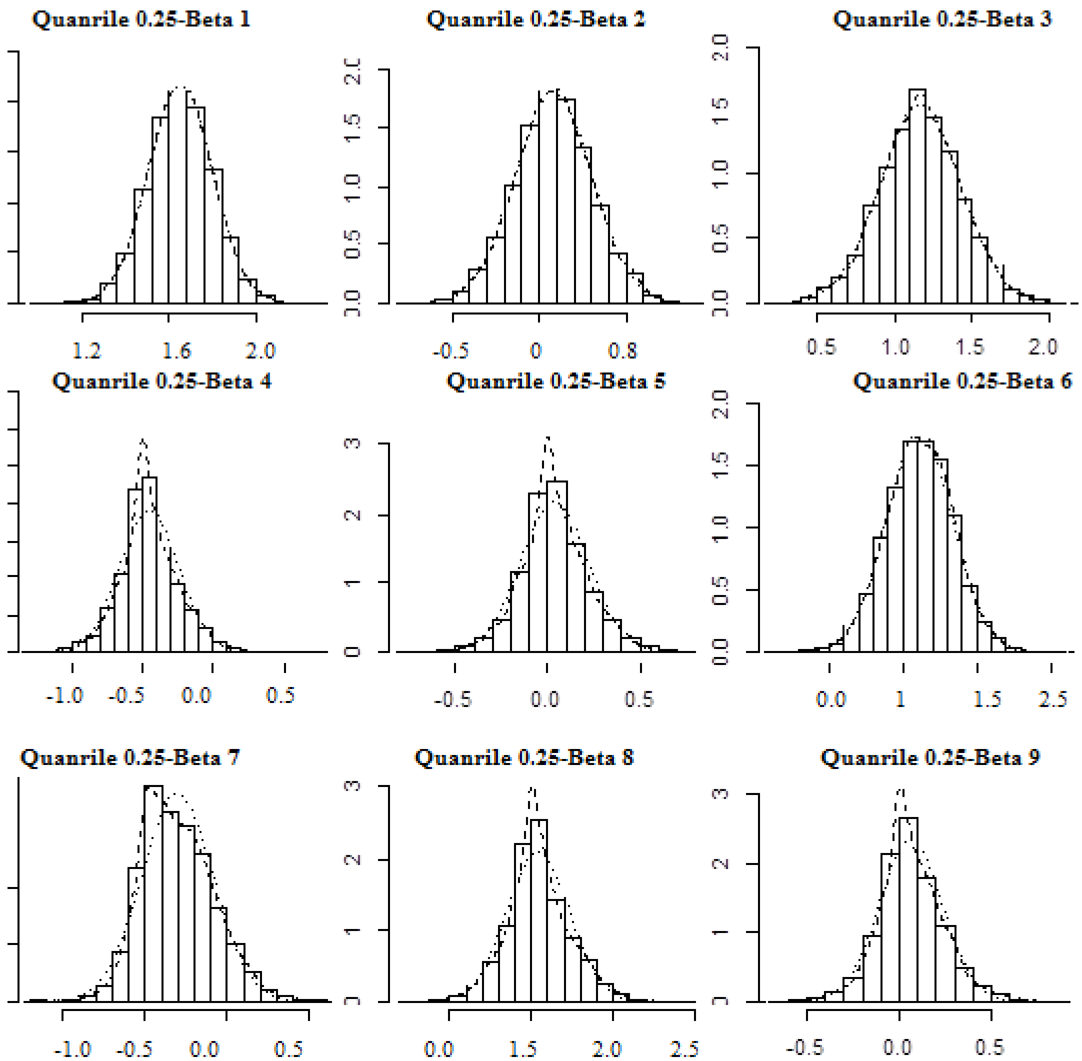

Figure 2. Histograms based on posterior distribution for our proposed method for Simulation 1 at quantile level $\theta = 0.25$.

6. Real data

In this section, we will use Pima Indians data to evaluation the performance of the proposed method ($BNLBQR$) compared to other two method ($BBqr$) and ($BNLBQR$). The data of Pima Indians are existing in R programs within caret package. These data have (532) observations, the important component in the analysis of Pima Indian data evaluates the relationship between diabetes and other cases using WHO criteria, (diabetes) as the dependent variable and 7 independent variables are : Number of pregnancies(x_1) referred by (npreg), Plasma glucose concentration in an oral glucose tolerance test (x_2) referred by (glu), diastolic blood pressure(x_3) referred by (bp), triceps skin fold thickness (x_4) referred by (skin), body mass index (x_5) referred by (bmi), diabetes pedigree function (x_6) referred by (ped) and age in years (x_7) referred by (age).

The methods under consideration are evaluated based on two criterions are mean squared error (MSE) and standard division(SD) also use confidence interval at 95% reliability at $\theta \in (0.25, 0.50\ and\ 0.75)$.

Table 4 show the Mean squared errors (MSE) and standard division (SD)for the method under comparison for Pima Indians data.

Methods	$\theta = 0.25$	$\theta = 0.50$	$\theta = 0.75$
	MSE (SD)	MSE (SD)	MSE (SD)
$BBqr$	0.956 (0.721)	0.862 (0.584)	0.761 (0.525)
$BLBqr$	0.836 (0.672)	0.817 (0.584)	0.635 (0.437)
$BNLBQR$	**0.573 (0.375)**	**0.548 (0.402)**	**0.458 (0.294)**

The results shown in Table 4 are a summary of the results for Mean squared errors (MSE) and standard division (SD) for the three methods under comparison. It is clear from Table 4 the MSE and SD are computed for the proposed method (BNLBQR) are less than that the MSE and SD are computed for the other two methods ($BBqr$) and ($BLBqr$). From these results, We conclude that our proposed method has a good performance in coefficient estimation and variable selection even with real data.

Table 5 Coefficients estimates for the proposed method via three quantile level for the Pima Indians data

Name Variables	Variables	$\theta = 0.25$	$\theta = 0.50$	$\theta = 0.75$
npreg	x_1	0.835	1.026	0.041
glu	x_2	0.138	0.642	0.593
bp	x_3	0.006	0.008	0.000
skin	x_4	-1.023	-0.473	-0.452
bmi	x_5	0.009	0.011	0.634
ped	x_6	0.364	0.526	0.173
Age	x_7	0.000	0.000	0.000

the results are listed in table 5 show coefficients estimate in direct way for our proposed method via three quantile $\theta \in (0.25, 0.50\ and\ 0.75)$ as shown in the table above. In quantile level $\theta = 0.25$, the variables (age) ineffective on dependent variables , but the rest independent variables have positive and negative effects on dependent variables. Also in quantile level $\theta = 0.50$, the variable (age) is ineffective on dependent variables, but the rest independent variables have positive and negative effects on dependent variables. in quantile level $\theta = 0.75$, the variables (bp) and (age) are ineffective on age, but the rest independent variables have positive and negative effects on dependent variables. From the results listed in table -5 ,we see our proposed method has good performance for coefficient estimate and variable selection in binary quantile regression.

The following figure show the confidence interval of coefficients estimates of our proposed method (BNLBQR) through three quantile levels ($\theta \in (0.25, 0.50\ and\ 0.75)$)

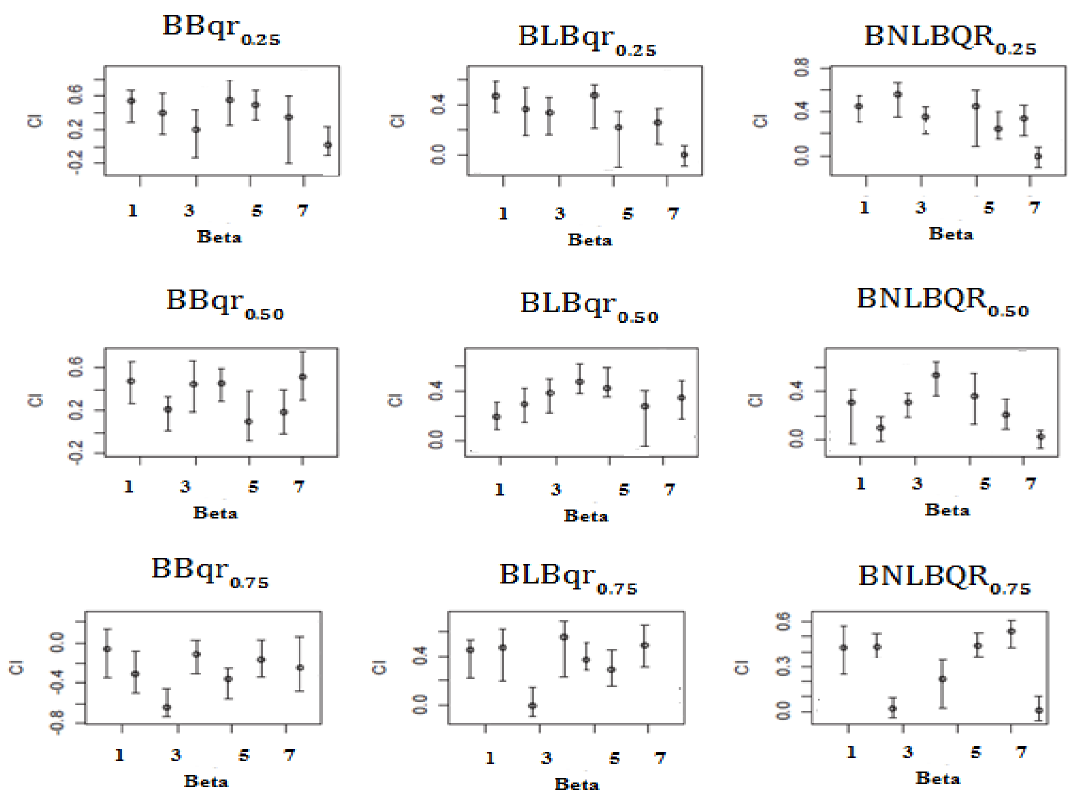

Figure 3. Confidence intervals of the parameter estimates for $\theta \in (0.25, 0.50\ and\ 0.75\}$ by the three methods for Pima Indians data .

7. Conclusion:

In this study, we introduced a novel approach for estimation, along with certain factors to consider. Utilizing Bayesian methodology for binary quantile regression. A new hierarchical model has been created based on a scale mixture of normal distribution mixing Rayleigh density representation. This representation was only recently presented for the Laplace distribution before density of the parameters vector.

Simulation Scenarios and actual data are taken into consideration in order to test the efficacy of the newly developed approach, which is based on a scale mixture of normal distribution mixing Rayleigh density, and to evaluate the performance of this method in comparison to the other two methods that are currently in use. The results of the simulation and the actual data that are provided in the tables that are located above have proved that the newly suggested approach is superior to the other ways that are currently in competition.

References :-

1. Alhamzawi, R. (2015). Model selection in quantile regression models. Journal of Applied Statistics 42 (2), 445-458.
2. Alshaybawee, T., H. Midi, and R. Alhamzawi (2016). Bayesian elastic net single index quantile regression. Journal of Applied Statistics, 1-19.
3. Benoit DF, Van den Poel D (2012) Binary Quantile Regression: A Bayesian Approach Based on the Asymmetric Laplace Distribution. J App Econom 27:1174-1188.
4. Benoit, D. F., Alhamzawi, R., and Yu, K. (2013). Bayesian lasso binary quantile regression. Computational Statistics, 28(6):2861-2873
5. Benoit, D., Al-Hamzawi, R., Yu, K., and Van den Poel, D. (2011). “bayesQR: Bayesian quantile regression.”
6. Brown RL, Peet RK. 2003. Diversity and invisibility of southern Appalachian plant communities. Ecology 84(1): 32-39.
7. Buchinski M. 1994. Changes in the U.S. wage structure 1963-1987: Application of quantile regression. Econometrica 62(2): 405-458.
8. Buchinski M. 1998. The dynamics of changes in the female wage distribution in the USA: a quantile regression approach. Journal of Applied Econometrics 13(1): 1-30.
9. Flaih et al. (2020) . Sparsity via new Bayesian Lasso . Periodicals of Engineering and Natural Sciences Vol. 8, No. 1, March 2020, pp.345-359.
10. Hans, C. (2009). “Bayesian lasso regression.” Biometrika, 96(4): 835–845.
11. Hao L., Naiman D. Q. (2007). Quantile regression. Thousand Oaks, CA: Sage.
12. Hashem, H., Vinciotti, V., Alhamzawi, R., and Yu, K. (2015). “Quantile regression with group lasso for classification.” Advances in Data Analysis and Classification, 1–16.
13. Hewson P, Yu K. 2008. Quantile regression for binary performance indicators. Applied Stochastic Models in Business and Industry 24(5): 401-418.
14. Horowitz JL. 1992. A smoothed maximum score estimator for the binary response model. Econometrica 60(3): 505-531
15. Koenker R, Hallock F. 2001. Quantile Regression. Journal of Economic Perspectives 15(4): 143-156.
16. Koenker R, Machado J (1999) Goodness of fit and related inference processes for quantile regression. J Am Stat Assoc 94:1296-1310
17. KOENKER, R., AND G. BASSETT (1978): "Regression Quantiles," Econometrica, 46, 33-50.
18. Koenker.(2005). Quantile Regression. New York: Cambridge University Press. 533
19. Kordas G (2006) Smoothed binary regression quantiles. J Appl Econom 21:387-407
20. Kordas G. 2006. Smoothed binary regression quantiles. Journal of Applied Econometrics 21(3): 387-407.
21. Kozumi, H. and Kobayashi, G. (2009) Gibbs Sampling Methods for Bayesian Quantile Regression. Journal of Statistical Computation and Simulation, 81, 1565-1578.
22. Li YJ, Zhu J. 2007. Analysis of array CGH data for cancer studies using fused quantile regression. Bioinformatics 22(18): 2470-2476.
23. Machado J, Santos Silva J. 2005. Quantiles for counts. Journal of the American Statistical Association 100(Dec): 1226-1237
24. Manski CF (1975) Maximum score estimation of the stochastic utility model of choice. J Econom 3:205-228
25. Manski CF (1985) Semiparametric analysis of discrete response: asymptotic properties of the maximum score estimator. J Econom 27:313-333
26. Manski CF. 1975. Maximum score estimation of the stochastic utility model of choice. Journal of Econometrics 3(3): 205-228.

27. Manski CF. 1985. Semiparametric analysis of discrete response: asymptotic properties of the maximum score estimator. Journal of Econometrics 27(3): 313-333.
28. Park T. Casella G (2008) The Bayesian lasso. J Am Stat Assoc 103:681-686
29. Park, T. and Casella, G. (2008) The Bayesian Lasso. Journal of the American Statistical Association, 103, 681-686.
30. Powell J. 1986. Censored regression quantiles. Journal of Econometrics 32(1): 134-155.
31. Tibshirani, R. (1996). "Regression Shrinkage and Selection via the Lasso." Journal of the Royal Statistical Society, Ser. B, 58: 267-288 534 .
32. Whittaker J, Whitehead C, Somers M. 2005. The neglog transformation and quantile regression for the analysis of a large credit scoring database. Journal of the Royal Statistical Society Series C - Applied Statistics 54(5): 863-878
33. Yu K, Lu Z, Stander J. 2003. Quantile regression: applications and current research areas. Journal of the Royal Statistical Society Series D – The Statistician 52(3): 331-350.
34. Yu K, Stander J. 2007. Bayesian analysis of a tobit quantile regression model. Journal of Econometrics 137(1): 260-276.