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Three-Dimensional Nonlinear Integral Operator with the Modelling of Majorant Function

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Abstract:

In this paper, the process for finding an approximate solution of nonlinear three-dimensional (3D) Volterra type integral operator equation (N3D-VIOE) in R^3 is introduced. The modelling of the majorant function (MF) with the modified Newton method (MNM) is employed to convert N3D-VIOE to the linear 3D Volterra type integral operator equation (L3D-VIOE). The method of trapezoidal rule (TR) and collocation points are utilized to determine the approximate solution of L3D-VIOE by dealing with the linear form of the algebraic system. The existence of the approximate solution and its uniqueness are proved, and illustrative examples are provided to show the accuracy and efficiency of the model.

Keywords: Majorant function, Modified Newton method, Non-linear integral operator, Three-dimensional Nonlinear Integral Operator, Three-dimensional Trapezoidal rule method.
Mathematical Subject Classification (2020): 45G15, 47H99.

Introduction:

Over the past years, integral operators have been increasingly utilized in various fields of sciences. This trend can be stated by the inference of science models that represent real phenomena [1-6]. Diverse problems in biology and mechanics arise in an integrated one and a multidimensional equation [7-10]. These equations also appear in mass and heat transfer, fluid mechanics, molecular physics and in many other problems [11-16].

There are few approaches to deal with the 3D integral equation of both kinds Fredholm and Volterra or mixed integral operators. Mirzaee et.al. in [17] used a method that is based on the 3D block pulse function to establish an approximate solution for the 3D nonlinear mixed Fredholm-Volterra integral equations. In [18], Mirzaee and Hadadiyan applied the modified block pulse approximation to solve the 3D nonlinear mixed Fredholm-Volterra integral equations of the second kind. While in [19], Mirzaee and Hadadiyan found the solution of the 3D nonlinear mixed Fredholm-Volterra integral equations based on the 3D triangular functions. Maleknejad et.al. in [20] used the Bernstein polynomials of three variable with all their properties to establish the approximate solution of the 3D Fredholm-Volterra operators of the first and second kind.

Numerous papers dealt with MF to reach the accurate approximate and numerical solution of nonlinear problems. In [21, 22], it was proposed that the majorant modelling technique would be used to solve the system of 2×2 Volterra integral equation. Eshkuvatov et.al. in [23] itemised MF to solve one-dimensional nonlinear integral operator of Volterra type. Hameed et.al. applied MF to solve the nonlinear system of a two dimensional Volterra integral operator in [24] and multidimensional nonlinear integral equations in [25]. While Ezquerro and Hernandez-Veron in [26] applied the MF for concluding the approximate solution of some Hammerstein integral equations. The nonlinear singular integral equation in [27] has been solved via MF. In [28], Argyros and Hilout provided the conditions of MF in a Banach space to study the local convergence of MNM.

In this study, the N3D-VIOE of the second kind in R^3 is considered as

$$U(\xi_1,\zeta_2,\zeta_3)-\int_{a_1}^{\zeta_1}\int_{a_2}^{\zeta_2}\int_{a_3}^{\zeta_3}K(\zeta_1,\zeta_2,\zeta_3,x,y,z)G(x,y,z,U(x,y,z))dxdydz = f(\zeta_1,\zeta_2,\zeta_2), \quad (1)$$

where the variables $(\zeta_1,\zeta_2,\zeta_3)\in[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]$. The unknown function $U(\zeta_1,\zeta_2,\zeta_3)\in Q_1$, $f(\zeta_1,\zeta_2,\zeta_3)\in Q_1$ and $Q_1=C_{[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]}$. The kernel $K(\zeta_1,\zeta_2,\zeta_3,x,y,z)\in Q_1\times Q_2$ where $Q_2=C_{[c_1,d_1]\times[c_2,d_2]\times[c_3,d_3]}$ and all $a_i,b_i,c_i,d_i, i=\{1,2,3\}$ are real numbers. The nonlinear function $G(x,y,z,U(x,y,z))$ is a defined continuous function in its domain.

The outline of this paper is as follows: First, MNM is used to linearise the N3D-VIOE. Next, TR is utilized to detect the solution of L3D-VIOE. After that, the convergence analysis including the existence and uniqueness theorems of the approximate solution and the theorem of MF are proved. Then illustrative examples are provided to confirm the efficiency and the accuracy the method used. Finally, the main ideas of the approximate method are concluded.

Linearizing N3D-VIOE via MNM

Let us consider the operator equation of the form

$$\beta(U(\zeta_1,\zeta_2,\zeta_3))=0, \quad (2)$$

to Eq. (1) to get the form:

$$\beta(U(\zeta_1,\zeta_2,\zeta_3))=U(\zeta_1,\zeta_2,\zeta_3)-f(\zeta_1,\zeta_2,\zeta_3)-\int_{a_1}^{\zeta_1}\int_{a_2}^{\zeta_2}\int_{a_3}^{\zeta_3}K(\zeta_1,\zeta_2,\zeta_3,x,y,z)G(x,y,z,U(x,y,z))dxdydz.$$

(3) Then the initial iteration of MNM of the form

$$\beta'(U(\zeta_1,\zeta_2,\zeta_3))(U(\zeta_1,\zeta_2,\zeta_3)-U_0(\zeta_1,\zeta_2,\zeta_3))+\beta(U_0(\zeta_1,\zeta_2,\zeta_3))=0, \quad (4)$$

is used to find the approximate solution, where $U_0(\zeta_1,\zeta_2,\zeta_3)$ is the initial condition (IC) and it may be any continuous function. The Fréchet derivative of $U(\zeta_1,\zeta_2,\zeta_3)$ can be found at $U_0(t_1,t_2,t_3)$ as follows:

$$\begin{aligned} B'(U_0(\zeta_1,\zeta_2,\zeta_3))U &= \lim_{r\to 0}\frac{1}{r}[\beta(U_0(\zeta_1,\zeta_2,\zeta_3)+rU(\zeta_1,\zeta_2,\zeta_3))-\beta(U_0(\zeta_1,\zeta_2,\zeta_3))], \\ &= \lim_{r\to 0}\frac{1}{r}\left[\frac{d\beta(U_0)}{dU}rU+\frac{1}{2}\frac{d^2\beta}{dU^2}(U_0+\theta rU)r^2U^2\right], \theta\in(0,1) \quad (5) \\ &= \frac{d\beta(U_0)}{dU}U. \end{aligned}$$

Utilizing Eq. (4) and Eq. (5) to get

$$\left.\frac{d\beta}{dU}\right|_{U_0}(\Delta U(\zeta_1,\zeta_2,\zeta_3))=-\beta(U_0(\zeta_1,\zeta_2,\zeta_3)), \quad (6)$$

where $\Delta U(\zeta_1,\zeta_2,\zeta_3)=U_1(\zeta_1,\zeta_2,\zeta_3)-U_0(\zeta_1,\zeta_2,\zeta_3)$. To solve Eq. (6) for $\Delta U(\zeta_1,\zeta_2,\zeta_3)$ the derivative is required to compute

$$\begin{aligned} \left.\frac{d\beta}{dU}\right|_{U_0} &= \lim_{r\to 0}\frac{1}{r}[\beta(U_0(\zeta_1,\zeta_2,\zeta_3)+rU(\zeta_1,\zeta_2,\zeta_3))-\beta(U_0(\zeta_1,\zeta_2,\zeta_3))], \\ &= \lim_{r\to 0}\frac{1}{r}\left[\begin{array}{l} rU(\zeta_1,\zeta_2,\zeta_3) \\ -\int_{a_1}^{\zeta_1}\int_{a_2}^{\zeta_2}\int_{a_3}^{\zeta_3}K(\zeta_1,\zeta_2,\zeta_3,x,y,z)\left[\begin{array}{l}G(x,y,z,U_0(x,y,z)+rU(x,y,z)) \\ -G(x,y,z,U_0(x,y,z))\end{array}\right]dxdydz\end{array}\right], \\ &= U(\zeta_1,\zeta_2,\zeta_3)-\int_{a_1}^{\zeta_1}\int_{a_2}^{\zeta_2}\int_{a_3}^{\zeta_3}K(\zeta_1,\zeta_2,\zeta_3,x,y,z)G'_U(x,y,z,U_0(x,y,z))U(x,y,z)dxdydz, \end{aligned} \quad (7)$$

where $G'_U(x,y,z,U_0(x,y,z))$ is the partial derivative of $G(x,y,z,U(x,y,z))$ for $U(x,y,z)$. From Eqs. (6) and (7):

$$\Delta U(\zeta_1,\zeta_2,\zeta_3)-\int_{a_1}^{\zeta_1}\int_{a_2}^{\zeta_2}\int_{a_3}^{\zeta_3}K(\zeta_1,\zeta_2,\zeta_3,x,y,z)G'_U(x,y,z,U_0(x,y,z))\Delta U(x,y,z)dxdydz$$

$$=f(\zeta_1,\zeta_2,\zeta_3)+\int_{a_1}^{\zeta_1}\int_{a_2}^{\zeta_2}\int_{a_3}^{\zeta_3}K(\zeta_1,\zeta_2,\zeta_3x,y,z)G(x,y,z,U_0(x,y,z))dxdydz-U_0(\zeta_1,\zeta_2,\zeta_3),$$

(8)

or

$$\Delta U(\zeta_1,\zeta_2,\zeta_3)-\int_{a_1}^{\zeta_1}\int_{a_2}^{\zeta_2}\int_{a_3}^{\zeta_3}K_0(\zeta_1,\zeta_2,\zeta_3,x,y,z;U_0)\Delta U(x,y,z)dxdydz=F_0(\zeta_1,\zeta_2,\zeta_3),$$

(9)

where

$$K_0(\zeta_1,\zeta_2,\zeta_3,x,y,z;U_0)=K(\zeta_1,\zeta_2,\zeta_3,x,y,z)G'_U(x,y,z,U_0(x,y,z))$$

(10)

$$F_0(\zeta_1,\zeta_2,\zeta_3)=f(\zeta_1,\zeta_2,\zeta_3)+\int_{a_1}^{\zeta_1}\int_{a_2}^{\zeta_2}\int_{a_3}^{\zeta_3}K(\zeta_1,\zeta_2,\zeta_3x,y,z)G(x,y,z,U_0(x,y,z))dxdydz-U_0(\zeta_1,\zeta_2,\zeta_3).$$

(11)

Eq. (9) is a linear equation for $\Delta U(\zeta_1,\zeta_2,\zeta_3)$ and by solving it to get $U_1(\zeta_1,\zeta_2,\zeta_3)$ as $U_1(\zeta_1,\zeta_2,\zeta_3)=\Delta U(\zeta_1,\zeta_2,\zeta_3)+U_0(\zeta_1,\zeta_2,\zeta_3)$. A sequence of approximate solution $U_n(\zeta_1,\zeta_2,\zeta_3)$ $(n=2,3,\cdots)$ can be established by continuing this process and can be evaluated from the following equation

$$\beta'(U_0(\zeta_1,\zeta_2,\zeta_3))\Delta U_n(\zeta_1,\zeta_2,\zeta_3)+\beta(U_{n-1}(\zeta_1,\zeta_2,\zeta_3))=0.$$

(12)

Eq. (12) is equivalent to the equation

$$\Delta U_n(\zeta_1,\zeta_2,\zeta_3)-\int_{a_1}^{\zeta_1}\int_{a_2}^{\zeta_2}\int_{a_3}^{\zeta_3}K_0(\zeta_1,\zeta_2,\zeta_3,x,y,z;U_0)\Delta U_n(x,y,z)dxdydz=F_{n-1}(\zeta_1,\zeta_2,\zeta_3),$$

(13)

where

$$\Delta U_n(\zeta_1,\zeta_2,\zeta_3)=U_n(\zeta_1,\zeta_2,\zeta_3)-U_{n-1}(\zeta_1,\zeta_2,\zeta_3),\ n=1,2,3,\cdots$$

(14)

and

$$F_{n-1}(\zeta_1,\zeta_2,\zeta_3)=f(\zeta_1,\zeta_2,\zeta_3)+\int_{a_1}^{\zeta_1}\int_{a_2}^{\zeta_2}\int_{a_3}^{\zeta_3}K(\zeta_1,\zeta_2,\zeta_3x,y,z)G(x,y,z,U_{n-1}(x,y,z))dxdydz-U_{n-1}(\zeta_1,\zeta_2,\zeta_3).$$

(15)

Solving Eq. (12) for $\Delta U_n(\zeta_1,\zeta_2,\zeta_3)$ gives a sequence of approximate solutions $U_n(\zeta_1,\zeta_2,\zeta_3)$, $(n=2,3,\cdots)$

Discretizing the Approximate Solution

A grid of points can be introduced as follows:

$$\Omega=\left\{\begin{matrix}(\zeta_{1i},\zeta_{2j},\zeta_{3k}):\zeta_{1i}=a_1+ih_1;\zeta_{2j}\\=a_2+jh_2;\zeta_{3k}=a_3+kh_3\end{matrix}\right\},$$

$$i=1,\cdots,m_1,\ j=1,\cdots,m_2,k=1,\cdots,m_3,$$

where $h_1=\frac{b_1-a_1}{m_1}$, $h_2=\frac{b2-a_2}{m_2}$, $h_3=\frac{b_3-a_3}{m_3}$ and m_i refers to the number of partitions in $[a_i,b_i],i=\{1,2,3\}$. Then, Eq. (13) becomes

$$\Delta U_n(\zeta_{1i},\zeta_{2j},\zeta_{3k})-\int_{a_1}^{\zeta_{1i}}\int_{a_2}^{\zeta_{2j}}\int_{a_3}^{\zeta_{3k}}K_0(\zeta_{1i},t\zeta_{2j},\zeta_{3k},x,y,z;U_0)\Delta U_n(x,y,z)dxdydz=F_{n-1}(\zeta_{1i},\zeta_{2j},\zeta_{3k}).$$

(16)

TR for triple integral on the arbitrary region $[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]$ can be estimated from TR for single integral as [29, pp.109]

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}g(x_1,x_2,x_3)dx_3dx_2dx_1\approx h_1h_2h_3\sum_{i=0}^{m_1}{}''\sum_{j=0}^{m_2}{}''\sum_{k=0}^{m_3}{}''\left(g(x_{1i},x_{2j},x_{3k})\right)$$

(17)

The notation $\sum''$ refers to the first and last terms are to be halved before summing.
Now, introducing a sub-grids Ω_{m_i}, Ω_{m_j} and Ω_{m_k} of ℓ_i, ℓ_j and ℓ_k points at each subinterval $[a_1,\zeta_{1i}]$, $[a_2,\zeta_{2j}]$ and $[a_3,\zeta_{3k}]$ which are included in $[a_1,b_1]$, $[a_2,b_2]$ and $[a_3,b_3]$ that appear in Eq. (16), such that

$$\begin{aligned}
\tau_{m_{1i}}^{z_1} &= h_{1i}z_1, h_{1i}=\frac{\zeta_{1i}-a_1}{\ell_1},\ i=1,2,\cdots,m_1,\ z_1=1,2,\cdots,\ell_1,\\
\tau_{m_{2j}}^{z_2} &= h_{2j}z_2, h_{2j}=\frac{\zeta_{2j}-a_2}{\ell_2},\ j=1,2,\cdots,m_2,\ z_2=1,2,\cdots,\ell_2,\\
\tau_{m_{3k}}^{z_3} &= h_{3k}z_3, h_{3k}=\frac{\zeta_{3k}-a_3}{\ell_3},\ k=1,2,\cdots,m_3,\ z_3=1,2,\cdots,\ell_3.
\end{aligned} \tag{18}$$

By applying these points on Eq. (16),

$$\begin{aligned}
&\Delta U_n\left(\tau_{m_{1i}}^{s_1},\tau_{m_{2j}}^{s_2},\tau_{m_{2k}}^{s_3}\right)-\\
&h_{1i}h_{2j}h_{3k}\sum_{z_1=1}^{i}{}''\sum_{z_2=1}^{j}{}''\sum_{z_3=1}^{k}{}''K_0\left(\tau_{m_{1i}}^{s_1},\tau_{m_{2j}}^{s_2},\tau_{m_{2k}}^{s_3},\tau_{m_{1i}}^{z_1},\tau_{m_{2j}}^{z_2},\tau_{m_{2k}}^{z_3};U_0\right)\Delta U_n\left(\tau_{m_{1i}}^{z_1},\tau_{m_{2j}}^{z_2},\tau_{m_{2j}}^{z_2}\right)\\
&\qquad = F_{n-1}\left(\tau_{m_{1i}}^{s_1},\tau_{m_{2j}}^{s_2},\tau_{m_{2k}}^{s_3}\right)\\
&\qquad i=1,2,\cdots,m_1,\ j=1,2,\cdots,m_2,k=1,2,\cdots,m_3\\
&\qquad s_1=1,2,\cdots,\ell_1,\ s_2=1,2,\cdots,\ell_2,s_3=1,2,\cdots,\ell_3
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
&F_{n-1}\left(\tau_{m_{1i}}^{s_1},\tau_{m_{2j}}^{s_2},\tau_{m_{2k}}^{s_3}\right)=\\
&h_{1i}h_{2j}h_{3k}\sum_{z_1=1}^{i}{}''\sum_{z_2=1}^{j}{}''\sum_{z_3=1}^{k}{}''\left[\begin{array}{l}K\left(\tau_{m_{1i}}^{s_1},\tau_{m_{2j}}^{s_2},\tau_{m_{2k}}^{s_3},\tau_{m_{1i}}^{z_1},\tau_{m_{2j}}^{z_2},\tau_{m_{2k}}^{z_3}\right)\\ G\left(\tau_{m_{1i}}^{z_1},\tau_{m_{2j}}^{z_2},\tau_{m_{2k}}^{z_3},U_{n-1}\left(\tau_{m_{1i}}^{z_1},\tau_{m_{2j}}^{z_2},\tau_{m_{2k}}^{z_3}\right)\right)\end{array}\right]\\
&\qquad -U_{n-1}\left(\tau_{m_{1i}}^{s_1},\tau_{m_{2j}}^{s_2},\tau_{m_{2k}}^{s_3}\right)
\end{aligned} \tag{20}$$

Eq. (19) is a linear algebraic system of $(m_1\times m_2\times m_3)\times(\ell_1\times\ell_2\times\ell_3)$ unknowns and with non-singularities, it has a unique solution in terms of $\Delta U_n\left(\tau_{m_{1i}}^{z_1},\tau_{m_{2j}}^{z_2},\tau_{m_{3k}}^{z_3}\right), n=2,3,\cdots,$ $z_i=1,2,\cdots,\ell_i, i=1,2,3$ and $i=1,2,\cdots,m_1,\ j=1,2,\cdots,m_2,k=1,2,\cdots,m_3$, then $U_n\left(\tau_{m_{1i}}^{z_1},\tau_{m_{2j}}^{z_2},\tau_{m_{3k}}^{z_3}\right)$ can be found as

$$\begin{aligned}
&U_n\left(\tau_{m_{1i}}^{z_1},\tau_{m_{2j}}^{z_2},\tau_{m_{3k}}^{z_3}\right)=\Delta U_n\left(\tau_{m_{1i}}^{z_1},\tau_{m_{2j}}^{z_2},\tau_{m_{3k}}^{z_3}\right)+\\
&U_{n-1}\left(\tau_{m_{1i}}^{z_1},\tau_{m_{2j}}^{z_2},\tau_{m_{3k}}^{z_3}\right)
\end{aligned} \tag{21}$$

The values of unknown functions $U_n(\zeta_1,\zeta_2,\zeta_3)$ can be evaluated via Newton forward interpolation formula, see [23], then

$$\begin{aligned}
&U_n(\zeta_1,\zeta_2,\zeta_3)=\Delta U_n(\zeta_1,\zeta_2,\zeta_3)+\\
&U_{n-1}(\zeta_1,\zeta_2,\zeta_3), n=2,3,\cdots
\end{aligned} \tag{22}$$

Convergence Analysis and MF

Since the functions $f(\zeta_1,\zeta_2,\zeta_3), U_0(\zeta_1,\zeta_2,\zeta_3),$ $K(\zeta_1,\zeta_2,\zeta_3,x,y,z),\ G(x,y,z,U(x,y,z)),$ $G'_U(x,y,z,U(x,y,z))$ and $G''_U(x,y,z,U(x,y,z))$ are continuous in their domain of definition, then they are bounded, see (30, pp 33) such that

$\max(f(\zeta_1,\zeta_2,\zeta_3))=M_f$,
$\max(U_0(\zeta_1,\zeta_2,\zeta_3))=M_{U_0}$,
$\max(K(\zeta_1,\zeta_2,\zeta_3,x,y,z))=M_K$,
$\max(G(x,y,z,U(x,y,z)))=M_G$,

$\max\left(G'_U(x,y,z,U(x,y,z))\right)=M_{G'}$,
$\max\left(G''_U(x,y,z,U(x,y,z))\right)=M_{G''}$.
The MF in [31] has been used, such that

$$\Psi(t)=\xi(t-t_0)^2-(1+\xi\eta)(t-t_0)+\eta, \quad (23)$$

where $\xi>0$ and $\eta\geq 0$ are real numbers. The following theorem states that the function in (23) is an MF of the operator as in Eq. (2).

Theorem 1: Consider the operator $\beta(U)=0$ as in Eq. (2) is well known in
$\Omega_\beta=\{U\in C_{[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]}:\|U-U_0\|\leq R\}$
and its second derivative is continuous in a closed ball
$\Omega_{0\beta}=\{U\in C_{[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]}:\|U-U_0\|\leq r\}$
, where $T=t_0+r\leq t_0+R$. Let $\Gamma_0=[\beta'(U_0)]^{-1}$, and the following requirements are satisfied:

1) $\|\Gamma_0\beta(U_0)\|\leq\dfrac{\eta}{(1+\xi\eta)}$, and 2)

$\|\Gamma_0\beta''(U)\|<\dfrac{2\xi}{(1+\xi\eta)}$, where $\|U-U_0\|\leq t-t_0\leq r$

, ξ and η as in Eq.(23). Then the function $\Psi(t)$ defined in Eq. (23) majorizes the operator $\beta(U)$ defined in Eq. (2).

Proof: [31].

Now, the following theorem states the existence and uniqueness of the approximate solution.

Theorem 2: The integral operator as in Eq. (1) has a unique solution $U^*(\zeta_1,\zeta_2,\zeta_3)$ in the closed ball $\Omega_{0\beta}$ and the sequence approximation in Eq. (9) converges to the solution $U^*(\zeta_1,\zeta_2,\zeta_3)$ if the following conditions are satisfied:

1) The resolvent kernel $\Gamma(\zeta_1,\zeta_2,\zeta_3)$ of Eq. (23) has been existed, where $\|\Gamma\|\leq M_K M_{G'}e^{[M_K M_{G'}(b_1-a_1)(b_2-a_2)(b_3-a_3)]}$,
2) $\beta''(U)$ is bounded

The convergence rate is given by the following formula

$$\left|U^*-U_n\right|\leq\left(\frac{2}{1+\zeta\eta}\right)^m\left(\frac{1}{\zeta}\right), m=1,2,\cdots \quad (24)$$

Proof: It presented that Eq. (1) is formulated to be a linear integral operator for $\Delta U(\zeta_1,\zeta_2,\zeta_3)$ as in Eq. (9), so it has a unique solution for the term $\Delta U(\zeta_1,\zeta_2,\zeta_3)$ on condition that the kernel $K_0(\zeta_1,\zeta_2,\zeta_3,x,y,z;U_0)$ is a continuous function, then the existence of the operator Γ_0 is achieved. The resolvent kernel of Eq. (9) is established to show that Γ_0 is bounded. Now, let the integral operator P $C_{[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]}\to C_{[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]}$ is illustrated as follows

$$\mathrm{P}(\Delta U(\zeta_1,\zeta_2,\zeta_3))=\int_{a_1}^{\zeta_1}\int_{a_2}^{\zeta_2}\int_{a_3}^{\zeta_3}K_0(\zeta_1,\zeta_2,\zeta_3,x,y,z;U_0)\,\Delta U(x,y,z)\,dxdydz, \quad (25)$$

where $K_0(\zeta_1,\zeta_2,\zeta_3,x,y,z;U_0)$ is defined as in Eq. (10) . Relying on Eq.(25), Eq. (9) can be rewritten in the form

$$\Delta U(\zeta_1,\zeta_2,\zeta_3)-\mathrm{P}(\Delta U(\zeta_1,\zeta_2,\zeta_3))=F_0(\zeta_1,\zeta_2,\zeta_3). \quad (26)$$

Eq. (26) has a unique solution $\Delta U^*(\zeta_1,\zeta_2,\zeta_3)$ that can be expressed in terms of $F_0(\zeta_1,\zeta_2,\zeta_3)$ as

$$\Delta U^*(\zeta_1,\zeta_2,\zeta_3)=F_0(\zeta_1,\zeta_2,\zeta_3)+\mathrm{A}(F_0(\zeta_1,\zeta_2,\zeta_3)), \quad (27)$$

where the operator A is an integral operator which can be written in terms of powers of the operator P [32, pp.378]

$$\mathrm{A}(F_0(\zeta_1,\zeta_2,\zeta_3))=I+\mathrm{P}(F_0(\zeta_1,\zeta_2,\zeta_3))+P^2(F_0(\zeta_1,\zeta_2,\zeta_3))+\cdots+P^m(F_0(\zeta_1,\zeta_2,\zeta_3))+\cdots, \quad (28)$$

Such that the powers of P are also integrals operators.

$$\mathrm{P}^m(\Delta U(\zeta_1,\zeta_2,\zeta_3))=\int_{a_1}^{\zeta_1}\int_{a_2}^{\zeta_2}\int_{a_3}^{\zeta_3}K_0^m(\zeta_1,\zeta_2,\zeta_3,x,y,z;U_0)\,\Delta U(x,y,z)\,dxdydz, (m=1,2,\cdots), \quad (29)$$

where $K_0^m(\zeta_1,\zeta_2,\zeta_3,x,y,z;U_0)$ is the iterated kernel. By substituting Eq. (28) and Eq. (29) into Eq. (27), the solution of Eq. (26) has the form

$$\Delta U^*(\zeta_1,\zeta_2,\zeta_3)=F_0(\zeta_1,\zeta_2,\zeta_3)+\int_{a_1}^{\zeta_1}\int_{a_2}^{\zeta_2}\int_{a_3}^{\zeta_3}\Gamma_0(\zeta_1,\zeta_2,\zeta_3,x,y,z;U_0)F_0(x,y,z)\,dxdydz, \quad (30)$$

and the resolvent kernel in Eq. (30) has the form

$$\Gamma_0(\zeta_1,\zeta_2,\zeta_3,x,y,z;U_0)=\sum_{\rho=0}^{\infty}K_0^{\rho+1}(\zeta_1,\zeta_2,\zeta_3,x,y,z;U_0), \tag{31}$$

Converges uniformly for all $\zeta_1\in[a_1,b_1]$, $\zeta_2\in[a_2,b_2]$ and $\zeta_3\in[a_3,b_3]$. Since

$$\begin{aligned}\left|K_0(\zeta_1,\zeta_2,\zeta_3,x,y,z;U_0)\right| &= \left|K(\zeta_1,\zeta_2,\zeta_3,x,y,z)\right\|G'(x,y,z,U_0(x,y,z))\right| \\ &\le M_K M_{G'}.\end{aligned} \tag{32}$$

Consider $M_T=M_K M_{G'}$ and by mathematical induction:

$$\begin{aligned}\left|K_0^2(\zeta_1,\zeta_2,\zeta_3,x,y,z)\right| &\le \int_{a_1}^{\zeta_1}\int_{a_2}^{\zeta_2}\int_{a_3}^{\zeta}\left|K_0(\zeta_1,\zeta_2,\zeta_3,x,y,z;U_0)K_0(\zeta_1,\zeta_2,\zeta_3,x,y,z;U_0)\right|dxdydz,\\ &\le \frac{M_T^2(b_1-a_1)(b_2-a_2)(b_3-a_3)}{1!},\\ \left|K_0^3(\zeta_1,\zeta_2,\zeta_3,x,y,z)\right| &\le \int_{a_1}^{\zeta_1}\int_{a_2}^{\zeta_2}\int_{a_3}^{\zeta_3}\left|K_0(\zeta_1,\zeta_2,\zeta_3,x,y,z;U_0)K_0^2(\zeta_1,\zeta_2,\zeta_3,x,y,z;U_0)\right|dxdydz,\\ &\le \frac{M_T^3(b_1-a_1)^2(b_2-a_2)^2(b_3-a_3)^2}{2!},\\ &\vdots \qquad\qquad \vdots\\ \left|K_0^{\rho}(\zeta_1,\zeta_2,\zeta_3,x,y,z)\right| &\le \int_{a_1}^{\zeta_1}\int_{a_2}^{\zeta_2}\int_{a_3}^{\zeta_3}\left|K_0(\zeta_1,\zeta_2,\zeta_3,x,y,z;U_0)K_0^{\rho-1}(\zeta_1,\zeta_2,\zeta_3,x,y,z;U_0)\right|dxdydz,\\ &\le \frac{M_T^{\rho}(b_1-a_1)^{\rho-1}(b_2-a_2)^{\rho-1}(b_3-a_3)^{\rho-1}}{(\rho-1)!},\end{aligned}$$

then

$$\begin{aligned}\|\Gamma_0\|=\|A(F_0(\zeta_1,\zeta_2,\zeta_3))\| &\le \sum_{\rho=0}^{\infty}\left|K_0^{\rho+1}(\zeta_1,\zeta_2,\zeta_3,x,y,z;U_0)\right|\\ &\le \sum_{\rho=0}^{\infty}\frac{M_T^{\rho+1}(b_1-a_1)^{\rho}(b_2-a_2)^{\rho}(b_3-a_3)^{\rho}}{\rho!},\\ &= M_T\sum_{\rho=0}^{\infty}\frac{M_T^{\rho}(b_1-a_1)^{\rho}(b_2-a_2)^{\rho}(b_3-a_3)^{\rho}}{\rho!},\\ &= M_T e^{[M_K M_{G'}(b_1-a_1)(b_2-a_2)(b_3-a_3)]}.\end{aligned} \tag{33}$$

That implied the series in Eq. (33) is uniformly convergent for $\Gamma_0(\zeta_1,\zeta_2,\zeta_3,x,y,z;U_0)$. Next, to verify that $\beta''(U)$ is bounded for all $U(\zeta_1,\zeta_2,\zeta_3)\in\Omega_0$. Since the second derivative of the integral operator $\beta(U(\zeta_1,\zeta_2,\zeta_3))$ at $U_0(\zeta_1,\zeta_2,\zeta_3)$ is a bilinear operator, that implies $\beta''(U_0(\zeta_1,\zeta_2,\zeta_3))(U(\zeta_1,\zeta_2,\zeta_3))=\left.\frac{d^2\beta}{dU^2}\right|_{U_0}\bar{U}(\zeta_1,\zeta_2,\zeta_3)$ and the norm $\left\|\frac{d^2\beta}{dU^2}\right\|$ has the estimation:

$$\left\|\frac{d^2\beta}{dU^2}\right\| = \max\left|\int_{a_1}^{\zeta_1}\int_{a_2}^{\zeta_2}\int_{a_3}^{\zeta_3}\left[K(\zeta_1,\zeta_2,\zeta_3x,y,z)G''(x,y,z,U_{n-1}(x,y,z))\right]dxdydz\right|, \quad (34)$$
$$\leq M_K M_{G''}(b_1-a_1)(b_2-a_2)(b_3-a_3).$$

Thus, the second derivative is bounded therefore, the solution $U^*(\zeta_1,\zeta_2,\zeta_3)$ of Eq. (3) is unique see [32, pp. 532]. The rate of convergence is shown in (Husam Hameed et al. [31]). ■

Numerical Results and Discussions:

The goal of this section is to show the efficiency and ability of the method based on MNM and MF, the following example has been considered and MATLAB Rb 2013 has been used to get the results.

Example 1: Consider the following 3D-VIOE[18],

$$U(\zeta_1,\zeta_2,\zeta_3)=G(\zeta_1,\zeta_2,\zeta_3)-24\int_0^{\zeta_1}\int_0^{\zeta_2}\int_0^{\zeta_3}\zeta_1^2\zeta_2U(x,y,z)dxdydz,\ (\zeta_1,\zeta_2,\zeta_3)\in[0,1]\times[0,1]\times[0,1], \quad (35)$$

where $G(\zeta_1,\zeta_2,\zeta_3)=4\zeta_1^5\zeta_2^3\zeta_3+4\zeta_1^3\zeta_2^3\zeta_3^3+3\zeta_1^4\zeta_2^3\zeta_3^2+\zeta_1^2\zeta_2+\zeta_2\zeta_3^2+\zeta_1\zeta_2\zeta_3$ and the exact solution is $U(\zeta_1,\zeta_2,\zeta_3)=\zeta_1^2\zeta_2+\zeta_2\zeta_3^2+\zeta_1\zeta_2\zeta_3$.

The numerical result for Eq. (35) with $m_1=10$, $m_2=10$, $m_3=10$, $\ell_1=5,\ell_1=5,\ell_3=5$, $h_i=0.1,\{i=1,2,3\}$, and the initial guess $U_0(\zeta_1,\zeta_2,\zeta_3)=\zeta_1\frac{\zeta_2}{2}+\frac{\zeta_2}{2}\zeta_3+\frac{\zeta_1}{2}\zeta_2^2\frac{\zeta_3}{4}$.

The error results for this example is explained in Table 1 along with the comparison of the error computed the present method, the 3D block-pulse (3DBP) function method [33] and the modified block-pulse (MBP) functions method [18].

Table 1. Numerical result for Eq. (35) with $m_1=10, m_2=10, m_3=10, \ell_1=5,\ell_2=5, \ell_3=5$, $h_i=0.1,\{i=1,2,3\}$ and $U_0(\zeta_1,\zeta_2,\zeta_3)=\zeta_1\frac{\zeta_2}{2}+\frac{\zeta_2}{2}\zeta_3+\frac{\zeta_1}{2}\zeta_2^2\frac{\zeta_3}{4}$, the number of iterations n=10.

Nods $(\zeta_1,\zeta_2,\zeta_3)$ $\zeta_1=\zeta_2=\zeta_3=2^{-\ell}$	(3DBP) function method[33]	(MBP) function method[18]	MNM
	$\varepsilon_{U(\zeta_1,\zeta_2,\zeta_3)}$	$\varepsilon_{U(\zeta_1,\zeta_2,\zeta_3)}$	$\varepsilon_{U(\zeta_1,\zeta_2,\zeta_3)}$
1	0.1658187	0.0844986	0.0039541
2	0.0455175	0.0230859	6.0181763e-04
3	0.0144045	0.0073650	9.2851833e-05
4	0.0001628	0.0028076	1.3965325e-05
5	0.0008036	0.0004120	4.8432711e-06
6	0.0008837	0.0004921	2.2118647e-06

Table 1 indicates that ten iterations (n=10) are reasonably sufficient for the approximate solutions $U_n(\zeta_1,\zeta_2,\zeta_3)$ to be closed to the exact solution $U^*(\zeta_1,\zeta_2,\zeta_3)$ and to show that MNM is more accurate than the two other methods, (3DBP) function method and (MBP) function method.

Example 2: Consider the following N3D-VIOE [34],

$$U(\zeta_1,\zeta_2,\zeta_3)=\zeta_1\zeta_2\zeta_3-\frac{(\zeta_1\zeta_2\zeta_3)^3}{27}+\int_0^{\zeta_1}\int_0^{\zeta_2}\int_0^{\zeta_3}U^2(x,y,z)dxdydz,$$
$$(\zeta_1,\zeta_2,\zeta_3)\in[0,1]\times[0,1]\times[0,1] \quad (36)$$

The exact solution for this integral operator is $U(\zeta_1,\zeta_2,\zeta_3)=\zeta_1\zeta_2\zeta_3$

Numerical results for Eq. (36) with $m_1=10$, $m_2=10$, $m_3=10$, $\ell_1=5,\ell_1=5$, $\ell_3=5$, $h_i=0.1,\{i=1,2,3\}$ and initial guess

$U_0(\zeta_1,\zeta_2,\zeta_3)=\frac{\zeta_1}{2}\zeta_2^2\frac{\zeta_3}{4}$ and

$U_0(\zeta_1,\zeta_2,\zeta_3)=\sqrt{\zeta_1\zeta_2\zeta_3}$ are given in Table 2-3

Table 2. Numerical result for Eq. (36) with $m_1=10, m_2=10,\ m_3=10,\ \ell_1=5, \ell_2=5,$ $\ell_3=5$, $h_i=0.1,\ \{i=1,2,3\}$ **and** $U_0(\zeta_1,\zeta_2,\zeta_3)=\frac{\zeta_1}{2}\zeta_2^2\frac{\zeta_3}{4}$

n	$\varepsilon_{U(\zeta_1,\zeta_2,\zeta_3)}$
1	0.0215944320000
2	0.016030131430
3	0.002777682726
5	1.21026369 e-04
10	3.20188615 e-06
20	2.44142574 e-10

Table 3. Numerical result for Eq. (36) with $m_1=10, m_2=10,\ m_3=10,\ \ell_1=5, \ell_2=5,$ $\ell_3=5, h_i=0.1,\ \{i=1,2,3\}$ **and** $U_0(\zeta_1,\zeta_2,\zeta_3)=\sqrt{\zeta_1\zeta_2\zeta_3}$

n	$\varepsilon_{U(\zeta_1,\zeta_2,\zeta_3)}$
1	0. 5281727300000
2	0.0287294638216
3	0.0138255823805
5	1.210263695 e-03
10	3.201886157 e-04
20	2.441425741 e-07
30	6.2694121930e-10

Table 2 shows that few iterations are required for the approximate solutions $U_n(\zeta_1,\zeta_2,\zeta_3)$ to be very closed to the exact solution $U^*(\zeta_1,\zeta_2,\zeta_3)$, while Table 3 indicates that when IC is chosen to be far from the exact solution, more iterations are required to reach the perfect approximate solution.

Notations used here are: m_1, m_2, and m_3 are the number of the main partitions on $[a_1,b_1]=[0,1]$, $[a_2,b_2]=[0,1]$, and $[a_3,b_3]=[0,1]$ respectively, ℓ_1, ℓ_2, and ℓ_3 are the subpartitions on $[a_1,\zeta_{1i}]=[0,\zeta_{1i}]$, $[a_2,\zeta_{2j}]=[0,\zeta_{2j}]$, and $[a_3,\zeta_{3k}]=[0,\zeta_{3k}]$ respectively, $i=1,2,\cdots,m_1$, $j=1,2,\cdots,m_2$, $k=1,2,\cdots,m_3$, n is the number of iterations and

$$\varepsilon_{U(\zeta_1,\zeta_2,\zeta_3)}=\max_{(\zeta_1,\zeta_2,\zeta_3)\in[0,1]\times[0,1]\times[0,1]}\left|U_n(\zeta_1,\zeta_2,\zeta_3)-U^*(\zeta_1,\zeta_2,\zeta_3)\right|$$

Conclusion:

In this note, the 3D nonlinear integral operator of Volterra type is solved using the modelling of MF and MNM. The existence and uniqueness theorem of the approximate solution is proved depending on the general theorems of MNM. Tables 1 and 2 show that the convergence of the method is rapid and its approximation is accurate. In general, it can be concluded that MNM and MF modelling are a powerful tool for solving many types nonlinear 3D integral operator equations of Volterra.

Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in The Middle Technical University.

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مفهوم (Majorant function) لايجاد حل للمؤثر التكاملي ذي ثلاث ابعاد

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الخلاصة:
تقدم هذه الورقة البحثية طريقة لايجاد الحل التقريبي لمؤثر فولتيرا التكاملي الثلاثي الأبعاد غير الخطي في R^3. حيث يتم استخدام مفهوم (Majorant function) وباستخدام طريقة نيوتن المعدلة لتحويل مؤثر فولتيرا التكاملي الثلاثي الأبعاد غير الخطي إلى متتالية لمؤثر فولتيرا التكاملي الثلاثي الأبعاد الخطي ومن يتم استخدام طريقة (Gaussian-Legendre) التربيعية لايجاد الحل التقريبي لمؤثر فولتيرا التكاملي الثلاثي الأبعاد الخطي من خلال التعامل مع نظام جبري خطي.تم مناقشة وجود ووحدانية الحل للطريقة المستخدمة مع اعطاء أمثلة توضيحية لإظهار دقة وكفاءة الطريقة.

الكلمات المفتاحية: المؤثر غير الخطي، المؤثر غير الخطي ثلاثي الابعاد، دالة Majorant، طريقة شبه المنحرف ثلاثية الابعاد، طريقة نيوتن المعدلة.