Journal of Al-Qadisiyah for Computer Science and Mathematics Vol. 15(2) 2023 , pp Math. 31–40

University Of AL-Qadisiyah

Available online at www.qu.edu.iq/journalcm

JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS

ISSN:2521-3504(online) ISSN:2074-0204(print)

SAS-Injective Modules

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ARTICLE INFO

Article history:
Received: 05 /03/2023
Rrevised form: 15 /04/2023
Accepted: 19 /04/2023
Available online: 30 /06/2023

Keywords:

Injective modules.
Small injective modules.
SAS-injective modules.
Semiartinian small submodule.

ABSTRACT

We introduce and investigate SAS-injective modules as a generalization of small injectivity. A right module M over a ring R is said to be SAS-N-injective (where N is a right R-module) if every right R-homomorphism from a semiartinian small right submodule of N into M extends to N. A module M is said to be SAS-injective, if M is SAS-R-injective. Some characterizations and properties of SAS-injective modules are given. Some results on small injectivity are extended to SAS-injectivity.

2020MSC: Primary: 16D50, 16D10, 16D80; Secondary: 16P40, 16E30, 16S90.

https://doi.org/ 110.29304/jqcm.2023.15.2.1247

1. Introduction

Throughout R is an associative ring with identity and all modules are unitary R-modules. If not otherwise specified, by a module (resp. homomorphism) we will mean a right R-module (resp. right R-homomorphism). For a submodule N of a module M, the notations $N \leq M$, $N \ll M$, $N \leq^{ess} M$, $N \leq^{max} M$, and $N \leq^{\oplus} M$ mean, respectively, that N is a submodule, a small submodule, an essential submodule, a maximal submodule, and a direct summand of M, respectively. If a is an element of right R-module M, then we use r(a) to denote the right annihilator of a in R. Also, we use the symbols J(M), soc(M) and Z(M) to denote the Jacobson radical, the socle and singular submodule of M_R, respectively. A module M is called semiartinian, if soc$(M/N) \neq 0$, for any proper submodule N of M. For a right R-module M_R, we denote by Sa(M) to the sum of all semiartinian submodules of M. We refer the reader to [1,3,4,6,12], for general background materials.

Injective modules have been studied extensively, and several generalizations for these modules are given by many authors (see, for example, [2,10,9,7,8]).

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A module M is called small-injective if every homomorphism from a small right ideal of R into M can be extended to a homomorphism from R_R into M [10].

In this article, a proper generalization of small-injectivity is introduced and investigated, namely SAS-injective modules. Let N be a right R-module. A right R-module M is said to be SAS-N-injective if every R-homomorphism from a semiartinian small right submodule of N into M extends to N. If M is SAS-R-injective, then we say that M is SAS-injective. Firstly, we give an example to show that SAS-injective modules need not be small-injective. Several properties of the class of SAS-injective modules are given. For example, we show that the class of SAS-N-injective modules is closed under isomorphic copies, direct products, finite direct sums and summands. Some characterizations of SAS-injective modules are given. We prove the equivalence of the following statements: (1) Every right R-module is SAS-injective; (2) Every simple right R-module is SAS-injective (3) Every semiartinian small submodule of any right R-module SAS-injective; (4) Every semiartinian small right ideal of R is SAS-injective; (5) Every semiartinian small right ideal of R is a summand of R; (6) $\mathrm{Sa}(R_R) \cap J(R) = 0$. Conditions under which quotient of SAS-injective right R-modules is SAS-injective are given. For instance, we prove that the equivalence of the following: (1) The class of SAS-injective right R-modules is closed under quotient; (2) For any right R-module M, the sum of any two SAS-injective submodules of M is SAS-injective; (3) All semiartinian small submodules of R_R are projective. Finally, we give conditions such that the class of SAS-injective right R-modules is closed under direct sums. For instance, we prove that the equivalence of the following conditions: (1) $\mathrm{Sa}(R_R) \cap J(R)$ is Noetherian ; (2) All direct sums of injective modules are SAS-injective; (3) The class of SAS-injective modules is closed under direct sums.

2. SAS-Injective Modules

As a generalization of small injective modules, we introduce the concept of SAS-injective modules.

Definition 2.1. A right R-module M is said to be SAS-N-injective (where N is a right R-module), if any right R-homomorphism $f: K \to M$ extends to N, where K is any semiartinian small submodule of N. If M is SAS-R-injective, then M is said to be SAS-injective.

Examples 2.2.

(1) All small-injective modules are SAS-injective, but the converse is not true in general, for example: let R be the localization ring of $\mathbb{Z}$ at the prime p, that is $R = \mathbb{Z}_{(p)} = \{\frac{m}{n}: \ p \text{ does not divide n}\}$. Then R is not small injective with $\mathrm{soc}(R_R) = 0$ (see [13, Example 4]). Since $\mathrm{soc}(R_R) = 0$, we have that $\mathrm{Sa}(R_R) = 0$ and hence the zero ideal is the only semiartinian small right ideal in R_R. Thus R_R is SAS-injective and hence SAS-injectivity is a proper generalization of small injectivity.

(2) Clearly, if $\mathrm{soc}(N_R) = 0$, then 0 is the only semiartinian small submodule of N and hence every module is SAS-N-injective. Particularly, all $\mathbb{Z}$-modules are SAS-injective.

Some properties of SAS-N-injective modules are given in the following theorem.

Theorem 2.3. Let M, N and K be right R-modules. Then the following statements hold:

(1) Let $\{M_i : i \in I\}$ be a class of modules. Then the direct product $\prod_{i\in I} M_i$ is SAS-N-injective if and only if all M_i are SAS-N-injective.
(2) If $K \subseteq N$ and M is SAS-N-injective, then M is SAS-K-injective.
(3) If M is SAS-K-injective and $M \cong N$, then N is SAS-K-injective.
(4) If M is SAS-K-injective and $K \cong N$, then M is SAS-N-injective.
(5) Any summand of an SAS-K-injective module is SAS-K-injective.

Proof. Obvious. □

Corollary 2.4. The next statements hold:

(1) A finite direct sum of SAS-N-injective modules is SAS-N-injective, for any module N. Moreover, a finite direct sum of SAS-injective modules is SAS-injective.
(2) A summand of an SAS-injective module is again SAS-injective.

Proof. (1) By applying Theorem 2.3 (1), when index I is taken to be a finite set.

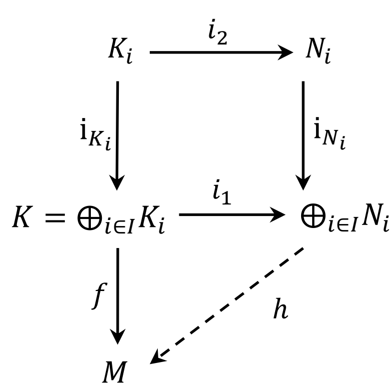
(2) This directly by using Theorem 2.3 (5). □

If for any submodule N of a right R-module M, there exists an ideal I of R such that $N = MI$, then M is called a multiplication module [11, p. 3839].

Proposition 2.5. Let $\{N_i : i \in I\}$ be a family of right R-modules and M be a right R-module. If $\oplus_{i\in I}N_i$ is a multiplication module, then M is SAS-$\oplus_{i\in I}N_i$-injective if and only if M is SAS-N_i-injective, for all $i \in I$.

Proof. (⇒) By Theorem 2.3 ((2), (4)).

(⇐) Let K be a semiartinian small submodule of $\oplus_{i\in I}N_i$. Since $\oplus_{i\in I}N_i$ is a multiplication module (by hypothesis), we have from [11, Theorem 2.2, p. 3844] that $K = \oplus_{i\in I}K_i$ with K_i is a submodule of N_i, for all i ∈ I. By [4, Lemma 5.1.3(c), p. 108], $K_i \ll N_i$. Since K is a semiartinian module, we have from [4, Exercises (7)(8), p. 238] that K_i is a semiartinian module and hence K_i is a semiartinian submodule of N_i. For $i \in I$, consider the following diagram:



where i_{K_i}, i_{N_i} are injection maps and i_1, i_2 are inclusion maps. The hypothesis implies that there exists a homomorphism $h_i: N_i \longrightarrow M$ such that $h_i \circ i_2 = f \circ i_{K_i}$. By [4, Theorem 4.1.6(2)], there exists exactly one homomorphism $h: \oplus_{i\in I}N_i \longrightarrow M$ satisfying $h_i = h \circ i_{N_i}$. Thus $f \circ i_{K_i} = h_i \circ i_2 = h \circ i_{N_i} \circ i_2 = h \circ i_1 \circ i_{K_i}$ for all $i \in I$. Let $(a_i)_{i\in I} \in \oplus_{i\in I}K_i$, thus $a_i \in K_i$, for all $i \in I$ and $f\,((a_i)_{i\in I}) = f\,(\sum_{i\in I} i_{K_i}((a_i)_{i\in I})) = (h \circ i_1)((a_i)_{i\in I})$ and hence $f = h \circ i_1$. □

If all right ideals of a ring R are ideals in R, then R is called right invariant [11, p.3839].

Corollary 2.6. Let R be a right invariant ring and let $1 = s_1 + s_2 \ldots + s_n$ in R, where the s_i are orthogonal idempotent, then a right R-module M is SAS-injective if and only if M is SAS-s_iR-injective for every $i = 1,2, \ldots \ldots, n$.

Proof. By [1, Corollary 7.3, p. 96], we have $R = \oplus_{i=1}^{n} s_iR$. Since R is a right invariant ring, we get from [11, Proposition 3.1, p. 3855] that R is a multiplication module and hence Proposition 2.5 implies that M is SAS-injective if and only if M is SAS-s_iR-injective. □

The following proposition gives characterizations of SAS-injective modules.

Proposition 2.7. The next conditions are equivalent for a right R-module M:

(1) M is SAS-injective.

(2) The sequence $0 \longrightarrow \mathrm{Hom}_R(R/N, M) \xrightarrow{\pi^*} \mathrm{Hom}_R(R, M) \xrightarrow{i^*} \mathrm{Hom}_R(N, M) \longrightarrow 0$ is exact, for all submodule N of $Sa(R_R) \cap J(R)$, where i and π are the inclusion and canonical maps, respectively.

(3) $\mathrm{Ext}^1(R/N, M) = 0$, for all submodule $N \subseteq Sa(R_R) \cap J(R)$.

(4) For each semiartinian small right ideal N of R and for any R-homomorphism $f: N \longrightarrow M$, there exists an element $m \in M$ such that $f(r) = mr$ for all $r \in N$.

Proof. (**1**) ⇒ (**2**) Let N be a submodule of $\mathrm{Sa}(R_R) \cap J(R)$. It is clear that the sequence

$0 \longrightarrow \mathrm{Hom}_R(R/N, M) \xrightarrow{\pi^*} \mathrm{Hom}_R(R, M) \xrightarrow{i^*} \mathrm{Hom}_R(N, M)$ is exact. Let $g \in \mathrm{Hom}_R(N, M)$. Since M is SAS-injective, there exists a right R-homomorphism $f: R \longrightarrow M$ such that $fi = g$ and hence $i^*(f) = g$. Thus i^* is an R-epimorphism and hence the sequence $0 \longrightarrow \mathrm{Hom}_R(R/N, M) \xrightarrow{\pi^*} \mathrm{Hom}_R(R, M) \xrightarrow{i^*} \mathrm{Hom}_R(N, M) \longrightarrow 0$ is exact.

(**2**) ⇒ (**3**) By [5, Theorem 4.4(3), p.491], there is an exact sequence $0 \longrightarrow \mathrm{Hom}_R(R/N, M) \xrightarrow{\pi^*} \mathrm{Hom}_R(R, M) \xrightarrow{i^*} \mathrm{Hom}_R(N, M) \longrightarrow \mathrm{Ext}^1(R/N, M) \longrightarrow \mathrm{Ext}^1(R, M) \longrightarrow \mathrm{Ext}^1(N, M) \longrightarrow \cdots$. Since R_R is projective, it is follows from [5, Theorem 4.4(1), p.491] that $\mathrm{Ext}^1(R, M) = 0$ and hence the sequence $0 \longrightarrow \mathrm{Hom}_R(R/N, M) \xrightarrow{\pi^*} \mathrm{Hom}_R(R, M) \xrightarrow{i^*} \mathrm{Hom}_R(N, M) \longrightarrow \mathrm{Ext}^1(R/N, M) \longrightarrow 0$ is exact. By hypothesis, the sequence $0 \longrightarrow \mathrm{Hom}_R(R/N, M) \xrightarrow{\pi^*} \mathrm{Hom}_R(R, M) \xrightarrow{i^*} \mathrm{Hom}_R(N, M) \longrightarrow 0$ is exact and hence $\mathrm{Ext}^1(R/N, M) = 0$.

(3) ⇒ (4) Let $f: N \longrightarrow M$ be a R-homomorphism where N is a semiartinian small right ideal of R. Thus $N \leq \mathrm{Sa}(R_R) \cap J(R)$. As the proof of (2) ⇒ (3) we have that the sequence $0 \longrightarrow \mathrm{Hom}_R(R/N, M) \xrightarrow{\pi^*} \mathrm{Hom}_R(R, M) \xrightarrow{i^*} \mathrm{Hom}_R(N, M) \longrightarrow \mathrm{Ext}^1(R/N, M) \longrightarrow 0$ is exact. By hypothesis, $\mathrm{Ext}^1(R/N, M) = 0$ and hence the sequence $0 \longrightarrow \mathrm{Hom}_R(R/N, M) \xrightarrow{\pi^*} \mathrm{Hom}_R(R, M) \xrightarrow{i^*} \mathrm{Hom}_R(N, M) \longrightarrow 0$ is exact. Thus there is a right R-homomorphism $g \in \mathrm{Hom}_R(R, M)$ with $i^*(g) = f$, this means $gi = f$. Let $r \in N$, thus $f(r) = g(r) = g(1)r = mr$, where $m = g(1)$.

(4) ⇒ **(1)** It is clear. □

Proposition 2.8. For a module M, the next statements are equivalent:

(1) All modules are SAS-M-injective.

(2) All semiartinian small submodules of any module is SAS-M-injective .

(3) All semiartinian small submodules of M are SAS-M-injective.

(4) Every semiartinian small submodule of M is a summand of M.

(5) $Sa(M) \cap J(M) = 0$.

Proof. (1) ⇒ (**2**) ⇒ (**3**) and (**5**) ⇒ (**1**) are clear.

(3) ⇒ (4) Let W be a semiartinian small submodule of M. By hypothesis, W is SAS-M-injective and hence $g \circ i = I_W$ for some a homomorphism $g: M \longrightarrow W$, where i is the inclusion and I_W is the identity homomorphism. Hence i is split and this implies that W is a summand of M.

(4)⇒ (5) Let $x \in \mathrm{Sa}(M) \cap \mathrm{J}(M)$, thus $x \in \mathrm{Sa}(M)$ and $x \in \mathrm{J}(M)$. By [4, Exercises (7)(2), p.238], $\mathrm{Sa}(M)$ is a semiartinian module and hence from [4, Exercises (7)(8), p.238] we have xR is a semiartinian submodule of M. By [4, Corollary 9.1.3, p.214], xR is a small submodule of M. By hypothesis xR is a summand of M and hence $xR \oplus K = M$ for some submodule K of M. Since xR is a small submodule of M, we have that $K = M$ and hence $xR = 0$. So, $x = 0$ and hence $\mathrm{Sa}(M) \cap \mathrm{J}(M) = 0$. □

Corollary 2.9. For a ring R, the next conditions are equivalent:

(1) All right R-modules are SAS-injective.
(2) All semiartinian small submodules of any right R-module are SAS-injective.
(3) All semiartinian small right ideals of R are SAS-injective.
(4) All semiartinian small right ideals of R are summands of R.
(5) $Sa(R_R) \cap J(R_R) = 0$.

Proof. By applying Proposition 2.8 with $M = R$. □

Proposition 2.10. Let M be a right R-module. Then $\mathrm{Sa}(M) \cap \mathrm{J}(M)$ is a semisimple summand of M if and only if all modules are SAS-M-injective.

Proof. (⇒) Let $\mathrm{Sa}(M) \cap \mathrm{J}(M)$ be a semisimple summand of M and let N be a module. Let K be a semiartinian small submodule of M. Since $\mathrm{Sa}(M) \cap \mathrm{J}(M)$ is a semisimple summand of M, it follows that $M = (\mathrm{Sa}(M) \cap \mathrm{J}(M)) \oplus W$ for some submodule W of M. Since K is a submodule of $\mathrm{Sa}(M) \cap \mathrm{J}(M)$ and $\mathrm{Sa}(M) \cap \mathrm{J}(M)$ is semisimple, $\mathrm{Sa}(M) \cap \mathrm{J}(M) = K \oplus U$ for some submodule U of $\mathrm{Sa}(M) \cap \mathrm{J}(M)$. We obtain $M = K \oplus U \oplus W$ and hence all semiartinian small submodules of M are summand of M. By Proposition 2.8, all modules are SAS-M-injective.

(⇐) Suppose that every right R-module is SAS-M-injective. By Proposition 2.8, $\mathrm{Sa}(M) \cap \mathrm{J}(M) = 0$ and hence $\mathrm{Sa}(M) \cap \mathrm{J}(M)$ is a semisimple summand of M. □

Theorem 2.11. If all simple singular modules are SAS-injective, then $r(a) \leq^{\oplus} R_R$ and aR is projective, for every $a \in \mathrm{Sa}(R_R) \cap \mathrm{J}(R_R)$.

Proof. Let $a \in \mathrm{Sa}(R_R) \cap \mathrm{J}(R_R)$ and let $L = RaR + r(a)$. Thus there exists $N \leq R_R$ such that $L \oplus N \leq^{ess} R_R$. Assume that $L \oplus N \neq R_R$, then there exists $I \leq^{max} R_R$ with $L \oplus N \subseteq I$, and hence $I \leq^{ess} R_R$. By [6, Example 7.6 (3), p. 247], R/I is a singular right R-module. By [4, Corollary 3.1.14, p. 49], R/I is a simple module and hence the hypothesis implies that R/I is SAS-injective. Clearly, α is a well-defined R-homomorphism, where $\alpha : aR \longrightarrow R/I$ is defined by $\alpha(at) = t + I$, for any $t \in R$. It is obvious that aR is a semiartinian small right ideal of R. By SAS-injectivity of R/I, there is a right R-homomorphism $g : R \longrightarrow R/I$ with $\mathrm{g}(x) = f(x)$ for any $x \in aR$. Thus $1 + I = f(a) = g(a) = g(1)a = (c + I)a = ca + I$, for some $c \in R$ and hence $1 - ca \in I$. But $ca \in RaR \subseteq I$, so $1 \in I$, a contradiction. Hence $L \oplus N = R$ and so $RaR + (r(a) \oplus N) = R$ and this implies that $r(a) \oplus N = R$ (since $RaR \ll R_R$). We will prove that aR is projective. Since $r(a)$ is a summand of R_R, it follows that $r(a) = (1 - e)R$ for some an idempotent element e in R (by [12, 2.3(3), p.8]) with $R = eR \oplus (1 - e)R$. Define $\lambda : eR \longrightarrow aeR$ by $\lambda(er) = aer$, for all $r \in R$. It is clear that λ is an epimorphism. Let $x \in \ker(\lambda)$, thus $\lambda(x) = 0$ and so $x = er$ for some $r \in R$ and $aer = 0$. Hence $er \in r(a)$ and $er \in eR$, and this implies that $x \in eR \cap r(a)$ and so $\ker(\lambda) \subseteq eR \cap r(a)$. Let $y \in R \cap r(a)$, thus $y = er$ and $ay = 0$. So $aer = 0$ and hence $\lambda(y) = 0$. Thus $y \in \ker(\lambda)$ and so $eR \cap r(a) \subseteq \ker(\lambda)$. Thus $\ker(\lambda) = eR \cap r(a)$. Since $R = eR \oplus (1 - e)R$, we have $eR \cap (1 - e)R = 0$. Since $r(a) = (1 - e)R$, we have $eR \cap r(a) = 0$. Since $\ker(\lambda) = eR \cap r(a)$, we have $\ker(\lambda) = 0$. Thus $\lambda : eR \longrightarrow aeR$ is an isomorphism. Clearly $aR = aeR$, since $aeR \subseteq aR$ and if $x \in aR$, then $x = a \cdot r$ for some $r \in R$. So $x = ar = aer + a(1 - e)r$. Since $r(a) = (1 - e)R$, we have $a(1 - e)r = 0$ and so $x = aer \in aeR$. Thus $aR \subseteq aeR$ and hence $aR = aeR$. Since $R = eR \oplus (1 - e)R$, we have eR is projective. Since $eR \cong aeR$, we have aeR is projective. Since $aR = aeR$, we have that aR is projective. □

Corollary 2.12. If all simple singular right R-modules are SAS-injective, then $Z(R_R) \cap Sa(R_R) \cap \mathrm{J}(R_R) = 0$.

Proof. Assume that $Z(R_R) \cap \mathrm{Sa}(R_R) \cap \mathrm{J}(R_R) \neq 0$, then there exists $0 \neq a \in Z(R_R) \cap \mathrm{Sa}(R_R) \cap \mathrm{J}(R_R)$. Since $a \in Z(R_R)$, we have $r(a) \leq^{ess} R_R$. By Proposition 2.11, $r(a) \leq^{\oplus} R_R$ and so $r(a) \cap K = 0$ and $r(a) + K = R$ for some $K \leq R_R$. Since $r(a) \leq^{ess} R_R$, which implies that $K = 0$ and so $r(a) = R$ and hence $a = 0$ but this a contradiction. Thus $Z(R_R) \cap \mathrm{Sa}(R_R) \cap \mathrm{J}(R_R) = 0$. □

A ring R is named zero insertive, if for any $a, b \in R$ with $ab = 0$, then $aRb = 0$ [10].

Lemma 2.13. [10, Lemma 2.11] $RaR + r(a) \leq^{ess} R_R$, for any element a in a zero insertive ring R.

Proposition 2.14. If all simple singular right R-modules are SAS-injective and R is a zero insertive ring, then $\mathrm{Sa}(R_R) \cap \mathrm{J}(R_R) = 0$.

Proof. Assume that $\mathrm{Sa}(R_R) \cap \mathrm{J}(R_R) \neq 0$. Thus there is $0 \neq a \in \mathrm{Sa}(R_R) \cap \mathrm{J}(R_R)$, and hence $RaR \ll R_R$. If $RaR + r(a) \subsetneq R$, then $RaR + r(a) \subseteq K$ for some a maximal right ideal K of R. Using Lemma 2.13, we have $RaR + r(a)$ is an essential in R_R and hence K is an essential in R_R and so R/K is a simple singular right R-module (by [6, Example 7.6(3) p. 247]). By hypothesis, R/K is an SAS-injective module. Consider the mapping $f: aR \longrightarrow R/K$ defined by $f(ar) = r + K$ for all $r \in R$. Thus f is a well-defined right R-homomorphism. Since aR is a semiartinian small right ideal of R, it follows from SAS-injectivity of R/K, there is a right R-homomorphism $g: R \longrightarrow R/K$ with $g(x) = f(x)$ for any $x \in aR$. Thus $1 + K = f(a) = g(a) = g(1)a = (c + K)a = ca + K$, for some $c \in R$ and hence $1 - ca \in K$. Since $ca \in RaR \subseteq K$, we have $1 \in K$ and so $K = R$ and this is a contradiction. Therefore, $RaR + r(a) = R$. Since $RaR \ll R_R$ which implies that $r(a) = R$ and so $a = 0$ and this is a contradiction. Thus $\mathrm{Sa}(R_R) \cap \mathrm{J}(R_R) = 0$. □

Corollary 2.15. If all simple singular right modules over a zero insertive ring R are SAS-injective, then all right R-modules are SAS-injective.

Proof. By Proposition 2.14 and Corollary 2.9. □

Theorem 2.16. Let R be ring. Then the following conditions are equivalent:

(1) $Sa(R_R) \cap J(R_R) = 0$.
(2) All right R-modules are SAS-injective.
(3) All simple right R-modules are SAS-injective.

Proof. Clearly, we have $(\mathbf{1}) \Rightarrow (\mathbf{2}) \Rightarrow (\mathbf{3})$.

$(\mathbf{3}) \Rightarrow (\mathbf{1})$. Assume that $\mathrm{Sa}(R_R) \cap \mathrm{J}(R_R) \neq 0$. Thus there is $0 \neq a \in \mathrm{Sa}(R_R) \cap \mathrm{J}(R_R)$, and hence $aR \ll R_R$. If $(\mathrm{Sa}(R_R) \cap \mathrm{J}(R_R)) + r(a) \subsetneq R$, then $(\mathrm{Sa}(R_R) \cap \mathrm{J}(R_R)) + r(a) \subseteq I$, for some maximal right ideal I of R. Thus R/I is a simple right R-module. By hypothesis, R/I is an SAS-injective module. We define $f: aR \rightarrow R/I$ by $f(ax) = x + I$ for any element x in R. Then clearly f is a well-defined right R-homomorphism. Since $aR \subseteq \mathrm{Sa}(R_R) \cap \mathrm{J}(R_R)$ it follows that aR is a semiartinian small right ideal of R. By SAS-injectivity of R/I, there is a right R-homomorphism $g: R \longrightarrow R/I$ with $g(x) = f(x)$ for any $x \in aR$. Thus $1 + I = f(a) = g(a) = g(1)a = (c + I)a = ca + I$, for some $c \in R$ and hence $1 - ca \in I$. Since $\mathrm{Sa}(R_R)$ and $\mathrm{J}(R_R)$ are predicals, we have that $\mathrm{Sa}(R_R)$ and $\mathrm{J}(R_R)$ are two-sided ideals. Thus $ca \in Sa(R_R) \cap J(R_R) \subseteq I$ and hence $1 \in I$, and so $I = R$ and this is a contradiction. Therefore, $(\mathrm{Sa}(R_R) \cap \mathrm{J}(R_R)) + r(a) = R$. Since $\mathrm{Sa}(R_R) \cap \mathrm{J}(R_R)$ is a small ideal in R_R, we have that $r(a) = R$ and so $a = 0$ and this is a contradiction. Thus $\mathrm{Sa}(R_R) \cap \mathrm{J}(R_R) = 0$. □

Remark 2.17. It is not necessary that all semiartinian small submodules of a projective module are projective, for example $< \bar{2} >$ is a semiartinian small submodule of the projective Z_4-module Z_4 but it is not projective, because it is not a summand of ${Z_4}^{(I)}$, for any index I.

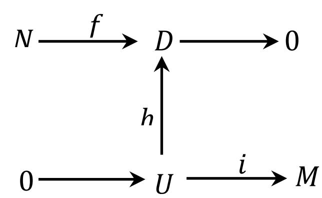
Theorem 2.18. The following conditions are equivalent for a projective module M:

(1) All epimorphic images of SAS-M-injective modules are SAS-M-injective.
(2) All epimorphic images of small-M-injective modules are SAS-M-injective.
(3) All epimorphic images of injective modules are SAS-M-injective.
(4) All sums of two SAS-M-injective submodules of any module are SAS-M-injective.
(5) All sums of two small-M-injective submodules of any module are SAS-M-injective.

(6) All sums of two injective submodules of any module are SAS-M-injective.
(7) All semiartinian small submodules of M are projective.

Proof. (1)⇒(2)⇒(3) and **(4)⇒(5)⇒(6)** are clear.

(3)⇒(7) Let D and N be modules and U be a semiartinian small submodule of M. Consider the following diagram:



where f is epimorphism, h is a homomorphism, and i is the inclusion homomorphism. We can take N to be an injective R-module (by [3, Proposition 5.2.10, p. 148]). By hypothesis, D is SAS-M-injective and hence $\alpha i = h$ for some a homomorphism $\alpha: M \longrightarrow D$. By projectivity of M, we get that α can be lifted to an R-homomorphism $\tilde{\alpha}: M \longrightarrow N$ with $f\tilde{\alpha} = \alpha$. Let $\tilde{h}: U \longrightarrow N$ be the restriction of $\tilde{\alpha}$ over U. It is clear that $f\tilde{h} = h$ and hence U is projective.

(7)⇒(1) Let $h: A \longrightarrow B$ be an R-epimorphism, where A and B are right R-modules and A is an SAS-M-injective. Let K be a semiartinian small submodule of M, $f: K \longrightarrow B$ be an R-homomorphism and $i: K \rightarrow M$ the inclusion homomorphism. By (7), K is projective and hence $hg = f$ for some a homomorphism $g: K \longrightarrow A$. By SAS-M-injectivity of A, we get $\tilde{g}i = g$ for some a homomorphism $\tilde{g}: M \longrightarrow A$. Put $\alpha = h\tilde{g}: M \longrightarrow B$. Thus $\alpha i = h\tilde{g}i = hg = f$. Hence B is an SAS-M-injective right R-module.

(1)⇒(4) Let K_1 and K_2 be two SAS-M-injective submodules of a right R-module K. Then $K_1 + K_2$ is a homomorphic image of $K_1 \oplus K_2$. Since $K_1 \oplus K_2$ is SAS-M-injective (by Corollary 2.4.(1)), it follows from hypothesis that $K_1 + K_2$ is SAS-M-injective.

(6) ⇒ (3) Let E be an injective module and $N \leq E$. Let $Q = E \oplus E$, $H = \{(x, x) |\ x \in N\}$, $\bar{Q} = Q/H$, $K_1 = \{y + H \in \bar{Q} |\ y \in E \oplus 0\}$ and $K_2 = \{y + H \in \bar{Q} |\ y \in 0 \oplus E\}$. Then $\bar{Q} = K_1 + K_2$. Since $(E \oplus 0) \cap H = 0$ and $(0 \oplus E) \cap H = 0$, it follows that $E \cong K_i$, $i = 1, 2$. Clearly, $K_1 \cap K_2 \cong N$ under $y \longmapsto y + H$ for all $y \in N \oplus 0$. By hypothesis, $\bar{Q}$ is SAS-M-injective. Injectivity of K_1 implies that $\bar{Q} = K_1 \oplus A$ for some submodule A of $\bar{Q}$ and hence $A \cong (K_1 + K_2)/K_1 \cong K_2/(K_1 \cap K_2) \cong E/N$. By Theorem 2.3 ((3),(5)), E/N is SAS-M-injective. □

Corollary 2.19. The following statements are equivalent for a ring R:

(1) Every epimorphic image of an SAS-injective right R-module is SAS-injective.
(2) Every epimorphic image of a small injective right R-module is SAS-injective.
(3) Every epimorphic image of an injective right R-module is SAS-injective.
(4) Every sum of two SAS-injective submodules of any right R-module is SAS-injective.
(5) Every sum of two small injective submodules of any right R-module is SAS-injective.
(6) Every sum of two injective submodules of any right R-module is SAS-injective.
(7) Every semiartinian small submodule of R_R is projective.

Proof. By taking $M = R$ and applying Theorem 2.18. □

Let N be a right R-module. A right R-module M is called a rad-N-injective, if for any submodule K of $J(N)$, any right R-homomorphism $f: K \rightarrow M$ extends to N [14, p.412].

Theorem 2.20. If M is a finitely generated right R-module, then the following statements are equivalent:

(1) $Sa(M) \cap J(M)$ is a Noetherian R-module.

(2) Any direct sum of SAS-M-injective right R-modules is SAS-M-injective.
(3) Any direct sum of rad-M-injective right R-modules is SAS-M-injective.
(4) Any direct sum of small M-injective right R-modules is SAS-M-injective.
(5) Any direct sum of injective right R-modules is SAS-M-injective.
(6) $K^{(L)}$ is SAS-M-injective, for any injective right R-module K and for any index set L.
(7) $K^{(\mathbb{N})}$ is SAS-M-injective, for any injective right R-module K.

Proof. (2)⇒(3)⇒(4) ⇒ (5) ⇒(6) ⇒(7) are clear.

(1)⇒(2) Let $E = \oplus_{i\in I} M_i$ be a direct sum of SAS-M-injective right R-modules. Let K be a semiartinian small submodule of M and $f: K \longrightarrow E$ be a homomorphism. Thus $K \subseteq \text{Sa}(M) \cap \text{J}(M)$. Since $\text{Sa}(M) \cap \text{J}(M)$ is a Noetherian (by hypothesis), K is finitely generated and hence $f(K) \subseteq \oplus_{j\in J} M_j$, for some finite subset J of I. Since a finite direct sum of SAS-M-injective modules is SAS-M-injective (by Corollary 2.4(1)), we have $\oplus_{j\in J} M_j$ is SAS-M-injective. Define $\alpha: K \longrightarrow \oplus_{j\in J} M_j$ by $\alpha(x) = f(x)$, for every $x \in K$. It is clear that α is a right R-homomorphism. By SAS-M-injectivity of $\oplus_{j\in J} M_j$, there exists a right R-homomorphism $g: M \longrightarrow \oplus_{j\in J} M_j$ such that $g(a) = \alpha(a)$, for all $a \in K$. Define $h: M \longrightarrow E = \oplus_{i\in I} M_i$ by $h(x) = (ig)(x)$ for every $x \in M$, where $i: \oplus_{j\in J} M_j \to \oplus_{i\in I} M_i$ is the inclusion. Thus, for all $a \in K$, we have that $h(a) = ig(a) = g(a) = \alpha(a) = f(a)$ and hence E is SAS-M-injective.

(7) ⇒(1) Let $K_1 \subseteq K_2 \ldots$ be a chain of submodules of $\text{Sa}(M) \cap \text{J}(M)$. For each $i \geq 1$, let $E_i = E(M/K_i)$ and $E = \oplus_{i=1}^{\infty} E_i$. For every $i \geq 1$, we put $M_i = \prod_{j=1}^{\infty} E_j = E_i \oplus \left(\prod_{\substack{j=1 \\ i\neq j}}^{\infty} E_j\right)$, then M_i is injective. By hypothesis, $\oplus_{i=1}^{\infty} M_i = (\oplus_{i=1}^{\infty} E_i) \oplus \left(\oplus_{i=1}^{\infty} \prod_{\substack{j=1 \\ i\neq j}}^{\infty} E_j\right)$ is SAS-M-injective. By using Theorem 2.3(5) we obtain that E is SAS-M-injective. Define $f: H = \cup_{i=1}^{\infty} K_i \longrightarrow E$ by $f(x) = (x + K_i)_i$. Obviously, f is a well-defined right R-homomorphism. Since M is finitely generated, $\text{Sa}(M) \cap \text{J}(M)$ is a semiartinian small submodule of M, and so $\cup_{i=1}^{\infty} K_i$ is a semiartinian small submodule of M. By SAS-M-injectivity of E, there exists a right R-homomorphism $g: M \longrightarrow E = \oplus_{i=1}^{\infty} E_i$ such that $gi = f$, where $i: H \longrightarrow M$ is the inclusion homomorphism. Since M is finitely generated, $g(M) \subseteq \oplus_{i=1}^{n} E(M/K_i)$ for some n and hence $f(H) \subseteq \oplus_{i=1}^{n} E(M/K_i)$. Let $\pi_i: \oplus_{j=1}^{\infty} E(M/K_j) \longrightarrow E(M/K_i)$ be the projection homomorphism. Thus $\pi_i f(x) = \pi_i((x + K_j)_{j\geq 1}) = x + K_i$ for all $x \in H$ and $i \geq 1$ and hence $\pi_i f(H) = H/K_i$ for all $i \geq 1$. Since $f(H) \subseteq \oplus_{i=1}^{n} E(M/K_i)$, we have that $H/K_i = \pi_i f(H) = 0$ for all $i \geq n+1$. So $H = K_i$ for all $i \geq n+1$ and hence the chain $K_1 \subseteq K_2 \subseteq \cdots$ terminates at K_{n+1}. Thus $\text{Sa}(M) \cap \text{J}(M)$ is a Noetherian R-module. □

Corollary 2.21. If N is a finitely generated right R-module, then the following statements are equivalent:

(1) $Sa(N) \cap J(N)$ is a Noetherian R-module.
(2) $M^{(L)}$ is SAS-N-injective, for each SAS-N-injective right R-module M and for any index set L.
(3) $M^{(L)}$ is SAS-N-injective, for each rad-N-injective right R-module M and for any index set L.
(4) $M^{(L)}$ is SAS-N-injective, for each small N-injective right R-module M and for any index set L.
(5) $M^{(\mathbb{N})}$ is SAS-N-injective, for each SAS-N-injective right R-module M.
(6) $M^{(\mathbb{N})}$ is SAS-N-injective, for each rad-N-injective right R-module M.
(7) $M^{(\mathbb{N})}$ is SAS-N-injective, for each small N-injective right R-module M.

Proof. By Theorem 2.20. □

Corollary 2.22. For a ring R, the following conditions are equivalent:

(1) $Sa(R_R) \cap J(R)$ is a Noetherian right R-module.
(2) All direct sums of SAS-injective right R-modules are SAS-injective.
(3) All direct sums of small-injective right R-modules are SAS-injective.
(4) All direct sums of injective right R-modules are SAS-injective.

(5) If M is an injective right R-module, then $M^{(L)}$ is SAS-injective, for any index set L.
(6) If M is a small-injective right R-module, then $M^{(L)}$ is SAS-injective, for any index set L.
(7) $M^{(L)}$ is SAS-injective, for any SAS-injective right R-module M and for an index set L.
(8) $M^{(\mathbb{N})}$ is SAS-injective, for any injective right R-module M.
(9) $M^{(\mathbb{N})}$ is SAS-injective, for any small-injective right R-module M.
(10) $M^{(\mathbb{N})}$ is SAS-injective, for any SAS-injective right R-module M.

Proof. By applying Theorem 2.20 and Corollary 2.21. □

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