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δ − Small submodule and Lifting property

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ABSTRACT

In this paper, T is a commutative ring with identity. We were interested in providing A new conclusion about δ-small submodule by clarifying the connection between the lifting module and δ-small submodule. Moreover, by employing the concept δ-projective cover we illustrate how to put any submodule of the module as δ-small. In addition, we state the definition of δ-lifting module and associate it with additional concepts such as finitely generated and the property of cyclic module known as principally δ-lifting to get δ-small. Lastly, we explained the relationship between p- δ- hollow and δ-small in a certain conditions.

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1- Introduction:

All rings in this article are commutative with identity and all T-modules are unitary. Any T-module M is called hollow if every none zero submodule A of M is small (A << M) where A is a small submodule means there exists

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another submodule B in M such that A+B≠M [8]. As a generalization of the small submodule, we denote A is δ-small if there exists a non-zero submodule B of M such that A+B≠M with M/B is a singular module (A$\ll_{\delta}$M) [11]. Note that any T-module M is called singular if Z(M)=M and is called non-singular if Z(M)=0, where Z(M)={x∈M:ann_T(x) $\leq_{ess}$T } [7]. A T-module M is said to be δ-hollow if A is a submodule of M is δ-small [4]. Any submodule A of M is called essential (A$\leq_{ess}$M) if there exists B≤M such that A∩B≠0 [9]. δ-lifting and lifting modules in [12].

2- The Main Results:

In this section, we present and study δ-small submodule. Different properties will be shown about the main relationship between the lifting concept and δ-small submodule.

Definition 2.1: Any submodule A of M is small if there exists 0≠B≤M such that A+B=M.

Definition 2.2: Any submodule A of M is said to be δ-small if there exists B≠0 such that A+B≠M and M/B singular module.

Remarks and Example 2.3:

1- The Hollow module implies M is δ-hollow module. The converse is true if M is an indecomposable module.

2- Any submodule A of Z-module Z_4= {0,1,2,3} is δ-small (because Z_4 is δ-hollow modules).

3- In general, Z_p has δ-small submodule (because Z_p is δ-hollow module).

4- The Z_{12}= {0,1,…,11} has no δ-small submodule (because <3> ⊕ <4> =Z_{12} , but $Z_{12}/< 4 > \not\ll_{\delta} Z_4$.

Definition 2.4: [12] Let M be a T-module. Then M is called the lifting module if, for all N≤M, there exists a decomposition M=A ⊕ B such that A≤N and N∩B << M, also it is called δ-lifting module if A≤M, so ∃ A_1, A_2 ≤M ∋ M=A_1⊕A_2, A_1<A and A∩A_2 is δ-small in M.

Remark 2.5: Every lifting module M is δ-lifting.

Definition 2.6: [5] An T-module M is called indecomposable if M=M ⊕ {0}. In other words, a T-module M is indecomposable if M≠0 and the only direct summand of M are < 0 > and M. Implies that M has no direct sum of two non-zero submodules.

Example 2.7: The simple module is indecomposable, but Z_6 as Z-module is not indecomposable.

Proposition 2.8: Let M be a T-module. If M is δ-lifting, then it has δ-small submodule.

Proof: Assume that M is a indecomposable T-module. So, by definition 2.6, M=M ⊕ {0} (M is a direct summand of {0} and itself only); suppose that M is a δ-lifting with A as a proper submodule of M.

Hence

$$M=A_1\oplus A_2 \ni A_1\leq A \text{ and } A\cap A_2 \ll_{\delta} A_2$$

Note that M is indecomposable. So, A_2=0 and M=A_1. Therefore, M≤A<M this is a contradiction.

Then A_2=M and hence A∩A_2 = A∩M = A Thus $A \ll_{\delta} M$ (M is δ-hollow module).

Proposition 2.9: Let M be the indecomposable module and δ-lifting module over the ring T. Then M has δ-small submodule.

Proof: Suppose that A is a submodule of a module M. Assume that M is δ-lifting module. So

$$M=A_1 \oplus A_2 \,;\, A_1 \leq A \;\wedge\; A \cap A_2 \ll_{\delta} M$$

M is indecomposable module . Then A_2=0 or A_1=0. If A_2=0, then A_1=M and hence M≤A this imposable. Therefore A_1=0 and then A_2=M with $A \cap A_2 = A \cap M = A \ll_{\delta} M$. Hence M is δ-hollow module. But M is an indecomposable module, so by remark [2.3 (1)], M is a hollow module. Then A is a small of M and thus is δ-small.

A T-module M is called an S-L-hollow module if M has a unique maximal submodule that contains each S-small submodule of M [1].

Remarks and Examples 2.10:

i) Each S-L-hollow module is a hollow module.

Proof: Suppose that M is an S-L-hollow module. Then there exist a unique maximal submodule that contains every S-small submodule say A in M. And since A is a submodule of M. Then each S-small contains in M. By definition hollow module [3] so; A is an S-small submodule of M, wich implies that M is a hollow module.

While the converse of Remark (i) is not true (in general), for example: z_p^{∞} is a hollow module; but z_p^{∞} is not the semi-local hollow module.

ii) The Z-module Z_4 is semi-local hollow module, while the Z-module Z_6 is not an S-L-hollow module.

Proposition 2.11: If M is semi-local hollow module (S-L-hollow module) with δ-lifting property, then M has δ-small submodule.

Proof: Let M be an S-L-hollow module. Then there exists a unique max-submodule A such that contains each S-small submodule of M. Suppose that M≠{0}+M, so there are a proper submodule B and C ∋ B, M are submodules of A and B ⊕ M. But M is semi-local hollow module then either M is an S-small submodule of M with M is a submodule of A; this implies that B=M. Or, B is an S-small submodule of M with B as a submodule of A, which implies that M=M. Which is a contradiction. Then M is indecomposable. But M satisfies δ-lifting property. Thus M has δ-small submodule.

Definition 2.12: [5] Let M be a T-module. Then M is called finitely generated if $M=\sum t_i\, x_i$, t_i ∈T , x_i∈M .

Recall that T is called an artinian ring if T has (O.C.C) i.e. $I_1 \supset I_2 \supset \ldots \supset I_n \ldots$.

Example 2.13: Let M = Z_4 = {0,1,2,3} as a Z-module and X = { $\bar{0}$ $\bar{1}$, $\bar{2}$ } . Since $1 +_4 2 = 3$, so

M= <X>= {0,1,2,3} = Z_4

Then Z_4 is the f-generated module.

Lemma 2.14: Every cyclic module M is f-generated, but the converse is not true.

Proof: Let M be a T-module such that M is cyclic. Then there exists x an element in M such that <x> = M. Since {x} is a singleton set, {x} is finite subset of M and <{x}> = M. Hence M is f-generated.

Proposition 2.15: Let M be an f-generated module over Artinian ring T. if M is δ-lifting then any submodule A of M is δ-small in M.

Proof: Since M is a finitely generated module over the artinian ring R, then M is the Notherian module and Artinian module. Suppose that M is cannot be decomposed into a direct sum of indecomposable submodules. So M = $A_0 \oplus Á_0$, $Á_0$ not decomposed into a direct sum of indecomposable submodules. Let $Á_0 = A_1 \oplus Á_1$ such that $Á_1$ not decomposed into the direct sum of indecomposable submodules. Hence we get infinite (D.C.C) of submodules of M and then M is the indecomposable module. But M is δ-lifting module, thus by proposition 2.9 , $A \ll_\delta M$.

Proposition 2.16: Let M be an f-generated module over Artinian ring T. If M is a projective and δ-lifting module, then M has δ-small submodule of M.

Proof: From the above proposition, M can be written as a direct sum of indecomposable submodules. Since M is projective, hence every direct summand of M is projective. So M is a direct sum of indecomposable projective submodules. Moreover, M is an indecomposable module. But M is δ-lifting module. Thus any submodule A of M is δ-small (Proposition 2.9).

Corollary 2.17: If End_T(M) is local with M is δ-lifting, then any submodule A of M is δ-small.

Proof: We must show that M is the indecomposable module. If M is not indecomposable, so M = $A_1 \oplus A_2 \ni A_1$ and A_2 are proper submodules. Note that the projection onto A_1 and onto A_2 are orthogonal idempotents which not invertible and not nilpotent. Hence End_T(M) is not local, this contradiction. Therefore M is the indecomposable module. But M is δ-lifting module. Thus any submodule A of M is δ-small.

Corollary 2.18: Let M be δ-liftng module over Artinian ring T. If A≤M with A_i any set such that M = $\sum Ai$, i∈I , then $A \ll_\delta M$.

Proof: We consider the set { xT: x∈M }. So ∃{ x_1T , x_2T ,… ,x_nT} ∋

$$x_1T + x_2T + \ldots + x_nT = M$$

So M is a finitely generated module. But we have T as Artinian ring and M as δ-lifting module, thus $A \ll_\delta M$.

Definition 2.19: [10] Let g : M_1→M_2 be an epimorphism and let a kernel of (g) is a δ-small in M_1 where M_1 is a projective module[2]. Then we say the pair (M_1,g) is a δ-projective cover of M_2.

Now we use definition 2.19; to explain how can obtain any submodule of M as a δ-small.

Proposition 2.20: For a projective cover (M_1,g) of M; if the module M_1 has δ-small submodule, then M_2 also has δ-small submodule.

Proof: Suppose that A≤M. And suppose g:M_1→M_2 is a δ-projective cover. So

$$g^{-1}(\mathrm{A}) \leq \mathrm{M}_1$$

But M_1 has a δ-small submodule. Then g^{-1}(A) is a δ-small in M_1 and hence $g\, g^{-1}$(A) is a δ-small in M_2 (because if g :M_1→M_2 is a homomorphism between two modules M_1 and M_2 and A≤M_1 ∋ A is a δ-small in M_1, imply g (A) is a δ-small of M_2). But we have $g\, g^{-1}$(A) =A. Therefore A is a δ-small in M_1.

Definition 2.21: Any T-module M is called δ-lifting if A_1≤M, such that M=A_2⊕A_3 with A_2≤A_1 and A∩A_3 is a δ-small in A_3 [12]. Therefore M is called δ-hollow when A is a δ-small in M.

Remark 2.22: Since from [Remark 2.5] every lifting T-module M is a hollow module, then every δ-lifting module is δ-hollow module and hence M has δ-small submodule.

Definition 2.23: We say M is f-δ-lifting if for A≤M is finitely generated has δ-lifting, so M=A_1⊕ B with A_1≤A , A∩B$\ll_\delta$B . Hence A∩B $\ll_\delta$B ⇔A∩B$\ll_\delta$M. Therefore M is a principally δ-lifting module (p-δ-lifting) if every cyclic submodule has p-δ-lifting property. So ∀x∈M, then M=A⊕ B , A≤xT and xT∩B $\ll_\delta$B [6].

Example 2.24: Let A≤M where M is a semi-simple module. So A is p-δ-lifting.

Example 2.25: Suppose that M=Z/Z_{p^n} is a Z-module. So M is p-δ-lifting module, n∈ Z^+ , P is prime.

Recall that M is said to be δ-hollow if A≤M is δ-small inside M so M is p-δ-hollow module if A≤M is cyclic and δ-small in M.

Proposition 2.26: Let M be an R-module if M={0}⊕ M and p-δ-hollow module, then A≤M is δ-small in M.

Proof: Suppose that x∈M. So xT can be written as

$$\mathrm{xT}= (\mathrm{xT}) \oplus (0)$$

But M is p-δ-hollow module. Then A=xT $\ll_\delta$M with (0) is a direct summand in M. So M is p-δ-lifting. Hence xT $\ll_\delta$M.

Proposition 2.27: Let M be a T-module and let A submodule of M with M/A is cyclic and A$\ll_\delta$M. Then M is p-δ-hollow module.

Proof: We need to show that A is cyclic and δ-small in M. Assume that x∈M with xT+A=M, M/A is singular. So M/A is also cyclic and A$\ll_{\delta}$M. There exists B≤A is projective and semi-simple with

$$M=(xT) \oplus B$$

Suppose that B=$\oplus A_i$; A_i is a simple submodule.

M=((xT) $\oplus A_j$) $\oplus A$. So M/K is the cyclic module and $M/K \cong A_i$. Then K is δ-small in M. Thus M is p-δ-hollow module.

Proposition 2.28: If M is an S-L-hollow module, then M has δ-small submodule.

Proof: Let M be an S-L-hollow module, then there exists a unique maximal submodule A of M contains all S-small submodule, then M=M⊕ {0}; where {0} is a submodule of A, and since M is an S-L-hollow module, therefore A∩M=A is S-small submodule of M. Hence M is lifting module. Then M is δ-lifting module and thus A is δ-small in M.

Corollary 2.29: If M is an S-L-hollow module then each non-zero co closed submodule of the maximal submodule of M is the semi-local hollow module.

Proof: Suppose that M is an S-L-hollow module and to consider A be a unique maximal submodule of M. Let N be a non-zero co closed submodule of A [3]. Suppose that M is a proper submodule of N. Since M is the S-L-hollow module, thus M is the S-small submodule of M contained in A. And hence N is the co closed submodule of M. Thus, M is the S-small submodule of N. Hence N is the S-L-hollow module.

Corollary 2.30: To consider B S-small submodule of module M, if M/B is the S-L-hollow module, then M is the S-L-hollow module.

Proof: Suppose that M/B is a semi-local hollow module, with B as semi-small submodule of M; then there exists a unique maximal submodule A/B of M/B with N+C=M where C is a submodule of M and N is a proper submodule of M then N+C B=M/B. Implies that (N+B B) + (C+B B) = M/B, since $(N+B)/B$ is a proper submodule of A/B and M/B is S-L-hollow module then $(N+B)/B$ is an S-small submodule of M/B . Thus C+B B=M/B, so C+B=M. Since B is an S-small submodule of M then C=M. Therefore M is the S-L-hollow module.

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