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On δ^*-Supplemented Modules

Alaa Abbas alawi [1] *Tamadher Arif Ibrahiem* [2*]

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Abstract.

The main goal of this paper is to introduce and study a new concept named δ^*-supplemented which can be considered as a generalization of W- supplemented modules and δ-hollow module. Also, we introduce a δ^*-supplement submodule. Many relationships of δ^*-supplemented modules are studied. Especially, we give characterizations of δ^*-supplemented modules and relationship between this kind of modules and other kind modules for example every δ-hollow (δ-local) module is δ^*-supplemented and by an example we show that the converse is not true.

Key words: δ-hollow, δ -small submodule, δ^*- supplement, δ^*- supplemented, W-supplemented.

Introduction:

Throughout this paper all rings are commutative with identity and all modules are unitary left R-modules. A proper submodule N of M is called "small (N<<M), if N+K = M, for K≤ M implies K = M" (1). Equivalently a submodule N of a module M is called "small (N<<M), if N+K ≠ M, for every proper submodule N of M"(2). A module M is called "singular (nonsingular) if Z(M)=M,(Z(M)=(0)), where Z(M)={x∈M:ann(x) $\leq_e$ R}"(1). A submodule N of a module M is said to be "δ-small if N+K =M with $\frac{M}{K}$ is singular implies K = M" (1). A submodule N of a module M is called " supplement of a submodule N of M if N is a minimal element in the set of submodule L ≤M with N+L= M"(3). Equivalently, M = N+K and N∩K<< N "(3). And a module M is called a "supplemented module if every submodule of M has a supplemented in M" (4, p.348). An R-module M is called a "semisimple R-module if Soc(M) = M (where Soc(M) = Σ A, where A is simple submodule of M"(4). It is known that an R-module M is a "semisimple module if and only if every submodule of M is a direct summand" (4).

In this paper we introduce the concept of δ^*-supplemented module: "M is called δ^*-supplemented module if for every semisimple submodule N of M, there exist a submodule K such that M = N+K and $N\cap K \ll_{\delta} K$" and investigate characterizations and properties of δ^*-supplemented modules. Also the relationship between this kind of modules and some other modules is given.

[1] Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq
[2] Department of Mathematics, College of Science for Women, University of Baghdad, Baghdad, Iraq
[*] Corresponding author: tamadherai_math@csw.uobaghdad.edu.iq
[*] ORCID ID: 0000-0002-7615-735x

Preliminary

Definitions 1

1. A submodule N of an R-module M is said to be "δ- supplement of a submodule K of M if N+K = M and $N\cap K \ll_{\delta} N$" (1). And a module M is called a "δ-supplemented module if for every submodule of M has a δ-supplement in M" (1).
2. A submodule N of an R-module M is called "δ -small if N+K =M with $\frac{M}{K}$ is singular implies K = M" (1).
3. Let M be an R-module, then δ (M) = ∩{N≤M;M/N is singular simple} = $\sum_{N \ll_{\delta} M} N$ (5).

Remarks and examples 2

1. Obviously, every small submodule of an R-module M is δ-small, but the converse is not true integral, for example $\mathbb{Z}_2$ as $\mathbb{Z}$ -module is δ-small but not small(1)
2. If A is a supplement of B in an R- module M, then B need not to be a supplement of A in M. For example in the $\mathbb{Z}$ -module $\mathbb{Z}_4$, we have $\mathbb{Z}_4$ is a supplement of $\{\bar{0},\bar{2}\}$. It is clear that $\{\bar{0},\bar{2}\}$ is not a supplement of $\mathbb{Z}_4$.

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3. Supplement needs not to be existing for example the module 2 $\mathbb{Z}$ of the module $\mathbb{Z}$ as $\mathbb{Z}$ -module has no supplement (since the only small submodule of $\mathbb{Z}$ are {0}, 2$\mathbb{Z}$ and $\mathbb{Z}$ is an indecomposable.
4. If N, K are two a submodule of an R-module M such that K is a supplement of N, then:

a) If W+K = M, for some W submodule of N, then K is a supplement of W(4, 4.41-1, p.348).

b) For L≤ N, $\frac{K+L}{L}$ is a supplement of $\frac{N}{L}$ in $\frac{M}{L}$ (4, 4.41-7, p.348)

Definition 3 (6) Let M be an R-module. M is said to be W-supplemented if every semisimple submodule of M has a supplement in M.

Definition 4 (7)((1)) An R-module M is called lifting (δ-lifting if and only if for every submodule N of M there exists submodule K, $K^{/}$≤M such that M=K⊕ $K^{/}$ with K≤N and $N\cap K^{/} << K^{/}$($N\cap K^{/} \underset{\delta}{<<} K^{/}$).

Lemma 5 (6) Let M = N+L, L is a submodule of an R-module M and N is semisimple submodule of M. Then M = $N^{/}$⊕ L for some $N^{/}$≤ N.

Characterization of δ^*-supplemented

Definition 6 An R-module M is called δ^*-supplemented module if for every semisimple submodule N of M, there exists a submodule K such that M = N+K and $N\cap K \underset{\delta}{<<} K$.

Definition 7 A submodule N of an R-module M is called δ^*-supplement of a submodule L of M means that N is semisimple submodule of M such that M = N+L with $N\cap L \underset{\delta}{<<} N$.

Examples and Remarks 8

1. Every supplemented module is δ^*-supplemented module. But the converse is not true in general for example: Q as $\mathbb{Z}$ -module is δ^*- supplemented since Q has no semisimple submodule, but Q is not supplemented module.
2. Every W-supplemented module is δ^*-supplemented.

Proof The proof is clear since every small submodule is δ-small (1).

3. If M is singular module then M is δ^*-supplemented iff M is W- supplemented.

Proof Since M is singular then $N\cap K \underset{\delta}{<<} K$ iff $N\cap K << K$ (where K and N are two submodules of M.

4. Every direct summand of is δ^*-supplemented module is is δ^*-supplemented

Proof Suppose that M is δ^*-supplemented module and M = $M_1 \oplus M_2$ .Let N semisimple submodule of M_1. So, there exists a submodule K of M such that M= N+K with $N\cap K \underset{\delta}{<<} K$. Thus $M_1 = N\oplus(M_1\cap K)$ (by modular law). Therefore $M_1 = L\oplus(M_1\cap K)$,for some L≤ N (by lemma 5). Hence $M_1\cap K \leq^{\oplus} M_1$. Now, $N\cap K \underset{\delta}{<<} K \leq M$, then by (7, proposition(1.2.10)) $N\cap K \underset{\delta}{<<} M$. Since $N\cap K \leq M_1\cap K \leq^{\oplus} M$, therefore $N\cap(M_1\cap K) = M_1\cap(N\cap K) = N\cap K \underset{\delta}{<<} M_1\cap K$ (7, proposition1.2.10).

5. If A is δ^*-supplement of B in a module M, then B needs not to be δ^*-supplemented of A in M. For example: $\mathbb{Z}_2$ is δ^*-supplement of $\mathbb{Z}_6$ in $\mathbb{Z}_6$ but $\mathbb{Z}_6$ is not δ^*-supplemented of $\mathbb{Z}_2$ in $\mathbb{Z}_6$, since $\mathbb{Z}_2 \oplus \mathbb{Z}_6 = \mathbb{Z}_6$ and $\mathbb{Z}_2 \cap \mathbb{Z}_6 = \mathbb{Z}_2$, $\mathbb{Z}_2 \underset{\delta}{<<} \mathbb{Z}_2$ but $\mathbb{Z}_2$ is not δ-small in $\mathbb{Z}_6$.
6. Let M be an R-module with Rad(M) = 0, then M is semisimple supplemented module iff M is δ^*- supplemented .

Proof ⇒)By (1) every supplemented module is δ^*-supplemented

⇐) Let K be a submodule of M, thus K is semisimple, but M is δ^*- supplemented ,thus there exists N≤M such that N+K =M and $N\cap K \underset{\delta}{<<} K \leq \delta(M)$. But δ(M) ≤ Rad(M) = 0, thus $N\cap K = 0 << K$ and hence M is a supplemented module.

Now, we have the following

Remark 9 Let M be an R-module, then M is δ^*-supplemented iff every semisimple submodule N of M, there exists a submodule $N^{/}$ such that M= $N^{/}$⊕L and $N\cap L \underset{\delta}{<<} L$.

Proof Let N be a semisimple submodule of M, since M is δ^*- supplemented and M = N+L and $N\cap L \underset{\delta}{<<} L$, then by lemma 5 there exists $N^{/}$≤N such that M= $N^{/}$⊕L and $N\cap L \underset{\delta}{<<} L$. Conversely, let N be a semisimple submodule of M. Then by assumption, there exists $N^{/}$≤N such that M= $N^{/}$⊕L and $N\cap L \underset{\delta}{<<} L$ implies M=N+L. Hence M is δ^*- supplemented.

Proposition 10 Let M be a δ^*- supplemented, then every semisimple submodule of $\frac{M}{\delta(M)}$ is a direct summand.

Proof. Suppose that M is a δ^*- supplemented. Then every very semisimple submodule of $\frac{M}{\delta(M)}$has the form $\frac{N}{\delta(M)}$for semisimple submodule N of M and $\delta(M) \subseteq N$. So there exist a submodule L of M such that M=N+L and $N\cap L \ll_\delta L$ implies $N\cap L \subseteq \delta(M)$. Now, $N\cap(L+\delta(M))=(N\cap L)+\delta(M) = \delta(M)$. So, $\frac{M}{\delta(M)} = \frac{N+L}{\delta(M)} = \frac{N}{\delta(M)} \oplus \frac{L+\delta(M)}{\delta(M)}$. Thus $\frac{N}{\delta(M)}$ is a direct summand of $\frac{M}{\delta(M)}$.

The following theorem gives a characterization for δ^*- supplemented module.

Theorem 11 Let M be an R-Module, then the following statements are equivalent:-

1. M is δ^*- supplemented.
2. For every semisimple submodule N of M, there is a decomposition $M= M_1\oplus M_2$ such that $M_1 \leq N$ and $N\cap M_2 \ll_\delta M_2$.
3. Every semisimple submodule N of M can be written as $N= A\oplus B$, where A is a direct summand of M and $B \ll_\delta M$.

Proof (1)→(2) Following Remark (9)

(2)→(3) Let N be a semisimple submodule of M, then by(2), $M= A\oplus B$ for some $A\leq N$ and $N\cap B \ll_\delta B$. By Modular Law, $N= A\oplus(N\cap B)$ (where $A\cap N\cap B = 0$).Since $N\cap B \ll_\delta B \subseteq M$, then by (7, proposition 2.1.10), we have and $N\cap B \ll_\delta M$.

(3)→(1) Let N be a semisimple submodule of M, thus by the hypothesis $N= A\oplus B$, where A is a direct summand of M and $B \ll_\delta M$. Now, $M= A\oplus K$, for $K \leq M$ and $A \leq N$, then $M= N+K= (A\oplus B)+K= A\oplus(B+K)$ and $(A\oplus B)\cap K= (A\cap K)\oplus(B\cap K) = 0 \oplus (B\cap K) = B\cap K$. Since $B \ll_\delta M$ and $K \ll_\delta K$, then $B\cap K \ll_\delta M\cap K= K$. That is $M = N+K$ and $N\cap K = B\cap K \ll_\delta K \leq M$. Therefore, M is δ^*-supplemented module.

The following proposition is similar to (5, proposition 2.3).

Proposition 12 Let M be an R-module A and B are submodules of M such that $A \leq B$. Then:

1. If B is a δ^*- supplement submodule in M, then $\frac{B}{A}$ is a δ^*- supplement submodule in $\frac{M}{A}$.
2. If B is a δ^*- supplement summand of C in M, then $\frac{C+A}{A}$ is a δ^*- supplement of $\frac{B}{A}$ in $\frac{M}{A}$

Proof 1 Suppose that B is a δ^*- supplemented of N in M, so B is semisimple submodule of M with M = B+N such that $B\cap N \ll_\delta N$. Now, $\frac{B}{A}$ is semisimple submodule in $\frac{M}{A}$. Then $\frac{M}{A} = \frac{B}{A} + \frac{N+A}{A}$ and $\frac{B}{A} \cap \frac{N+A}{A} = \frac{B\cap(N+A)}{A} = \frac{A+(B\cap N)}{A} \ll_\delta \frac{N+A}{A}$ and so $\frac{B}{A}$ is a δ^*- supplemented in$\frac{M}{A}$, where $f:A\to\frac{N+A}{A}$ is a homomorphism and $A\cap B \ll_\delta N$, implies $f(A\cap B) = \frac{(A\cap B)+A}{A} \ll_\delta \frac{N+A}{A}$

2. It is similar to proof 1.

Corollary 13 Every factor of δ^*- supplemented module is δ^*- supplemented module.

Remark 14 An inverse image of δ^*- supplemented module needed not δ^*- supplemented. For example: Let $f: \mathbb{Z} \to \mathbb{Z}_6$ be an epimorphism $\frac{\mathbb{Z}}{(6)} \approx \mathbb{Z}_6$ and $\mathbb{Z}_6$ is semisimple but $f^{-1}(\mathbb{Z}_6) = \mathbb{Z}$ and $\mathbb{Z}$ is not semisimple.

Proposition 15 Let M be an R-module and $A\leq B\leq M$. If $\frac{B}{A}$ is δ^*-supplement in $\frac{M}{A}$ and A is a δ^*- supplement in M. Then B is a δ^*- supplement in M.

Proof Suppose that A is A δ^*- supplemented of L in M and $\frac{B}{A}$ is δ^*- supplement of $\frac{N}{A}$ in$\frac{M}{A}$. Thus, $\frac{M}{A} = \frac{B}{A} + \frac{N}{A}$ and $\frac{B}{A}\cap\frac{N}{A} \ll_\delta \frac{N}{A}$, also $M = A+L$ and $A\cap L \ll_\delta L$ with each of A and $\frac{B}{A}$ is semisimple. Since $B = B\cap(A+L) = A+(B\cap L)$ and $\frac{B}{A} \cap \frac{N}{A} = \frac{B\cap N}{A} \ll_\delta \frac{N}{A}$,that is $\frac{B\cap N\cap L}{A\cap L} \ll_\delta \frac{N}{A\cap L}$. Notice that, $A\cap L \ll_\delta L \leq N$ and hence $A\cap L \ll_\delta N$ .Furthermore $B\cap(N\cap L) \ll_\delta N$ (7, proposition 2.1.10). By Modular Law, $N = A+(N\cap L)$, but B=B+A, then $M = B+N = B+(N\cap L)$. Therefore B is δ^*- supplement in M.

Now, we have the following proposition

Proposition 16 Let $M = M_1\oplus M_2$ if A is δ^*- supplement of A_1 in M_1 and B is δ^*- supplement of B_1 in M_2, then $A\oplus B$ is δ^*- supplement of $A_1\oplus B_1$ in M.

Proof Since each of A and B is semisimple, then so is $A \oplus B$. Now, $M_1 = A+A_1$ with $A\cap A_1 \ll_\delta A_1$ and $M_2 = B+B_1$ with $B\cap B_1 \ll_\delta B_1$, then $M = (A+A_1) \oplus (B+B_1) = (A \oplus B) + (A_1\oplus B_1)$. Since $(A\cap A_1) \oplus (B\cap B_1) \ll_\delta A_1\oplus B_1$. So, $(A\oplus B) + (A_1\oplus B_1) \ll_\delta$

$A_1\oplus B_1$. That is A⊕B is a δ^*- supplement of $A_1\oplus B_1$. So that $(A\oplus B)\cap(A_1\oplus B_1)\underset{\delta}{<<}A_1\oplus B_1$. Therefore A⊕B is δ^*- supplement of $A_1\oplus B_1$.

Proposition 17 Let M_1 and M_2 are two submodules of M with M_1 is δ^*- supplemented and M_1+M_2 has δ^*- supplement in M, then M_2 has δ^*- supplement in M.

Proof Since M_1+M_2 has a δ^*- supplement in M, so there exist a submodule L of M such that $M = (M_1+M_2) + L$ and $(M_1+M_2)\cap L\underset{\delta}{<<}L$. Furthermore M_1 is δ^*-supplement with the submodule $M_2+L\cap M_1$ of M_1+M_2, hence $(M_2+L)\cap M_1$ is semisimple submodule of M_1+M_2 . But $M_1\le M_1+M_2$, so $M_1$is also semisimple (since (M_1+M_2) is semisimple). Hence $(M_2+L)\cap M_1$ is semisimple in M_1. This means that there exists a submodule K of M_1 such that $((M_2+L)\cap M_1)+K=M_1$ with $((M_2+L)\cap M_1)\cap K\underset{\delta}{<<}K$. That is $(M_2+L)\cap K\underset{\delta}{<<}K$. Now, $M=M_1+M_2+L = (((M_2+L)\cap M_1)+K)+M_2+L= M_2+(K+L)$. Since $M_2\cap(K+L) \le ((M_2+ M_1)\cap L)+((M_2+L)\cap K)$. So $M_2\cap(K+L)\underset{\delta}{<<}K+L$. But M_2 is semisimple. Thus M_2 is δ^*- supplement of K+L in M.

The following proposition gives some properties of δ^*- supplemented modules

Proposition 18 Let M be an R-module, N and K be submodules of M such that K is δ^*- supplement of N then

1. If W+K = M for some submodule W of N then K is δ^*- supplement of W.
2. If K is δ^*- supplement of L≤ M, then K is δ^*- supplement of N+L.
3. If L≤ N, then $\frac{K+L}{L}$ is δ^*- supplement of $\frac{N}{L}$ in $\frac{M}{L}$.

Proof

1. Since K is δ^*- supplement of N thus K is semisimple submodule of M and N+K = M, $N\cap K\underset{\delta}{<<}K$. But W≤N and $W\cap K\le N\cap K\underset{\delta}{<<}K$, so $W\cap K\underset{\delta}{<<}K$(8,lemma1.3-1). Therefore K is δ^*- supplement of W.
2. Since K is δ^*- supplement of N thus K is semisimple submodule of M, N+K = M and $N\cap K\underset{\delta}{<<}K$. Also K is δ^*- supplement of L, thus K+L = M and $K\cap L\underset{\delta}{<<}K$. Therefore M= N+L+K and by Modular law $(N+L)\cap K=(N\cap K)+(L\cap K)\underset{\delta}{<<}K$ (8, lemma1.3-1). Hence K is δ^*- supplement of N+L.
3. Suppose that N+K = M and $N\cap K\underset{\delta}{<<}K$ (K is δ^*- supplement of N). Now, L≤ N ,$N\cap(K+L) = (N\cap K)+L$ (modularity) and $\frac{N}{L}\cap\frac{K+L}{L} = \frac{(N\cap K)+L}{L}$. Since $N\cap K\underset{\delta}{<<}K$ thus $\frac{(N\cap K)+L}{L}\underset{\delta}{<<}\frac{K+L}{L}$ (8, lemma1.3-1). Now, the assertion follows from $\frac{N}{L}+\frac{K+L}{L}=\frac{M}{L}$.

An R-module is called "δ-hollow if every proper submodule of M is δ-small (1)". The following proposition shows that the classes of δ-hollow modules is an embedding in the classes of δ^*-supplemented modules.

Proposition 19 Every δ-hollow module is δ^*-supplemented module.

Proof

Let N be a semisimple submodule of δ-hollow module M. Thus by (8), M= N⊕K, for K≤M . Then by N= N⊕(N∩K). Since M is δ-hollow, thus $N\underset{\delta}{<<}M$ and N∩K≤N, hence $N\cap K\underset{\delta}{<<}M$. Therefore by theorem 11 M is δ^*- supplemented.

Examples 20

1. The $\mathbb{Z}_6$-module $\mathbb{Z}_6$ and the $\mathbb{Z}$ -modules $\mathbb{Z}_4$, $\mathbb{Z}_{p^\infty}$, $\mathbb{Z}_8$ are δ-hollow(1), and hence they are δ^*- supplemented.
2. The converse of the last Proposition is not true as the following example:
   The $\mathbb{Z}$ -module, $\mathbb{Z}_{12}$ is δ^*-supplemented since $\mathbb{Z}_{12}$ has only one semisimple submodule which is $<\bar{2}>$, and $\mathbb{Z}_{12} = <\bar{4}>\oplus<\bar{3}>$, $<\bar{4}>\le<\bar{2}>$ and $<\bar{2}>\cap<\bar{3}> = <\bar{6}>\underset{\delta}{<<}<\bar{3}>$ by Theorem11, $\mathbb{Z}_{12}$ as $\mathbb{Z}$ -module is δ^*- supplemented but $\mathbb{Z}_{12}$ as $\mathbb{Z}$ -module is not δ-hollow(1).

Corollary 21 Every hollow module is δ^*-supplemented module.

Proof It's an obvious, since every hollow is δ-hollow, and by proposition 19 the proof is omitted.

The converse is not true for example, Q as $\mathbb{Z}$ -module is δ^*- supplemented module but not hollow module.

Corollary 22 Every indecomposable and δ-lifting module is δ^*- supplemented.

Proof By (1, proposition 3.8) every indecomposable and δ-lifting is δ-hollow and by proposition 19, M is δ^*- supplemented module.

Following (5), a module M is "δ-local if, δ(M) $\ll_{\delta}$ M and δ (M) is a maximal submodule of M".

Corollary 23 Every local (δ-local) module is δ^*-supplemented module.

Remark 24 The converse of corollary 23 is not true in general: $\mathbb{Z}_{12}$ as $\mathbb{Z}$ -module is δ^*- supplemented module but not local module.

Conflicts of Interest: None.

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المقاسات التكميلية من النمط – δ^*

الاء عباس عليوي[1] تماضر عارف ابراهيم[2]

[1] قسم الرياضيات، كلية العلوم ، جامعة بغداد، بغداد، العراق
[2] قسم الرياضيات، كلية العلوم للبنات، جامعة بغداد، بغداد، العراق

الخلاصة:

الهدف الرئيسي من هذا البحث هو تقديم ودراسة مفهوم جديد اسميناه مقاسات تكميلية من النمط - δ^* و التي يمكن اعتبارها إعمام للمقاسات التكميلية من النمط –W والمقاسات المجوفة من النمط- δ . كذلك قدمنا مفهوم المقاس الجزئي التكميلي من النمط- δ^*. تم مناقشة الكثير من العلاقات لهذا المفهوم مع مفاهيم أخرى حيث تم البرهنة على كل مقاس مجوف من النمط-δ (محلي من النمط- δ^*) هو مقاس تكميلي من النمط - δ^* ومن خلال اعطاء الامثلة الداحظة برهنا بأن العكس غير صحيح.

الكلمات المفتاحية : المقاسات المجوفة من النمط - δ ، المقاسات الجزئية الصغيرة من النمط δ، المقاسات التكميلية من النمط -δ^* ، المقاسات التكميلية من النمط –W.