

Oscillations of First Order Neutral Differential Equations with Positive and Negative Coefficients

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Abstract:

Oscillation criterion is investigated for all solutions of the first-order linear neutral differential equations with positive and negative coefficients. Some sufficient conditions are established so that every solution of eq.(1.1) oscillate. Generalizing of some results in [4] and [5] are given. Examples are given to illustrate our main results.

Key words: Neutral differential equations oscillations.

Introduction:

The study of neutral differential equations with positive and negative coefficients has been recently considered the attention of many authors all over the world for the last several years, see [1]-[6] a few of them have been investigated the case with variable mixed coefficients, that is the coefficients are variable positive and negative, see [1],[4] and [6]. The authors in [1] investigated the first order delay differential equations with positive and negative coefficients. While in [2],[5] and [6] the authors gave some sufficient conditions for the oscillation of neutral differential equation with positive and negative coefficients and constant delays. In this paper we give a generalization to some results in [4] and [5] where we have used a variable delays. Consider the linear neutral differential equation with positive and negative coefficients:

$$\begin{aligned} & [y(t) - P(t)y(\tau(t))] + \\ & Q(t)y(\sigma(t)) - R(t)y(\alpha(t)) = 0 \end{aligned} \quad (1.1)$$

Where $P, Q, R, \in C[[t_0, \infty); R^+]$, and τ, σ, α are continuous strictly increasing functions with $\lim_{t \rightarrow \infty} \tau(t) = \infty, \lim_{t \rightarrow \infty} \sigma(t) = \infty,$

$$\sigma(t) < \alpha(t) \text{ and } \tau(t), \sigma(t), \alpha(t) < t. \quad (1.2)$$

By a solution of eq.(1.1) we mean a function $y \in ([t_y, \infty), R)$ such that $y(t) - P(t)y(\tau(t))$ is continuously differentiable and $y(t)$ satisfies eq. (1.1),

$t_y = \max\{\tau(t), \sigma(t), \alpha(t)\}$ in the initial interval. A solution of eq.(1.1) is said to be oscillatory if it has arbitrarily large zeros, otherwise is said to be nonoscillatory. The purpose of this paper is to obtain sufficient conditions for the oscillation of all solutions of eq. (1.1).

Some Basic Lemmas:

The following lemmas will be useful in the proof of the main results:

Lemma 1 (Theorem 2.1.1 [4])

Suppose that $p, q \in C[R^+, R^+]$, $q(t) < t$ for $t \geq t_0$, $\lim_{t \rightarrow \infty} q(t) = \infty$ and

$$\liminf_{t \rightarrow \infty} \int_{q(t)}^t p(s) ds > \frac{1}{e} \quad (2.1)$$

Then the inequality $y'(t) + P(t)y(q(t)) \leq 0$ has no eventually positive solutions.

Lemma 2.

Let $y(t)$ be an eventually positive solution of (1.1) and set

$$W(t) = y(t) - P(t)y(\tau(t)) - \int_{\alpha^{-1}(\sigma(t))}^t R(s)y(\alpha(s))ds, \quad t \geq \alpha^{-1}(\sigma(t)) \quad (2.2)$$

and the following assumptions are hold.

$$H1: Q(t) - R(\alpha^{-1}(\sigma(t))[\alpha^{-1}(\sigma(t))]' \geq 0$$

$$H2: \limsup_{t \rightarrow \infty} [P(t) + \int_{\alpha^{-1}(\sigma(t))}^t R(s)ds] \leq 1$$

Then $W(t)$ is eventually positive and non increasing function.

Proof. Suppose that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\sigma(t)) > 0$ and $y(\alpha(t)) > 0$, $t \geq t_0$

Differentiate (2.2) and use (1.1) we get $W'(t)$

$$\begin{aligned} &= [y(t) - P(t)y(\tau(t))]' - R(t)y(\alpha(t)) \\ &+ R(\alpha^{-1}(\sigma(t)))y(\sigma(t))(\alpha^{-1}(\sigma(t)))' \\ &= -Q(t)y(\sigma(t)) + R(t)y(\alpha(t)) \\ &- R(t)y(\alpha(t)) \\ &+ R(\alpha^{-1}(\sigma(t)))y(\sigma(t))(\alpha^{-1}(\sigma(t)))' \\ &= -[Q(t) - R(\alpha^{-1}(\sigma(t))(\alpha^{-1}(\sigma(t)))']y(\sigma(t)) \leq 0 \end{aligned} \quad (2.3)$$

Hence $W(t)$ is monotone (nonincreasing) function then

$\lim_{t \rightarrow \infty} w(t) = l$, $-\infty \leq l < \infty$
we claim that $l \geq 0$ otherwise $l < 0$,
we have two cases for $y(t)$ to consider :

Case 1:-

If $y(t)$ is unbounded then there exist a sequence $\{t_n\}$ such that

$$\lim_{n \rightarrow \infty} t_n = \infty, \lim_{n \rightarrow \infty} y(t_n) = \infty, \text{ and}$$

$$y(t_n) = \max\{y(t): t_2 \leq t \leq t_n\}$$

from (1.2) we get

$$\begin{aligned} W(t_n) &= y(t_n) - P(t_n)y(\tau(t_n)) \\ &- \int_{\alpha^{-1}(\sigma(t_n))}^{t_n} R(s)y(\alpha(s))ds \end{aligned}$$

$$\begin{aligned} &\geq y(t_n) - P(t_n)y(\tau(t_n)) \\ &- y(t_n) \int_{\alpha^{-1}(\sigma(t_n))}^{t_n} R(s)ds \\ &\geq y(t_n) - P(t_n)y(t_n) \\ &- y(t_n) \int_{\alpha^{-1}(\sigma(t_n))}^{t_n} R(s)ds \\ &= y(t_n)[1 - P(t_n) \\ &- \int_{\alpha^{-1}(\sigma(t_n))}^{t_n} R(s)ds] \\ &\text{as } n \rightarrow \infty, \lim_{n \rightarrow \infty} W(t_n) = l \geq 0. \end{aligned}$$

Case 2:-

Let $y(t)$ be bounded, that is $\limsup_{t \rightarrow \infty} y(t) = k < \infty$ then there exist a sequence $\{s_n\}$ such that, $\lim_{n \rightarrow \infty} s_n = k$, and

$$y(\eta_n) = \max\{y(t): \lambda_1(s_n) \leq t \leq \lambda_2(s_n)\}$$

Where

$$\lambda_1(t) = \min\{\tau(t), \sigma(t)\}, \lambda_2(t) = \max\{\tau(t), \alpha(t)\}.$$

$$\begin{aligned} W(s_n) &= y(s_n) - P(s_n)y(\tau(s_n)) \\ &- \int_{\alpha^{-1}(\sigma(s_n))}^{s_n} R(s)y(\alpha(s))ds \\ &= y(s_n) - W(s_n) \\ &= P(s_n)y(\tau(s_n)) \\ &+ \int_{\alpha^{-1}(\sigma(s_n))}^{s_n} R(s)y(\alpha(s))ds \\ &\leq P(s_n)y(\eta_n) \\ &+ y(\eta_n) \int_{\alpha^{-1}(\sigma(s_n))}^{s_n} R(s)ds \end{aligned}$$

$$= y(\eta_n)[P(s_n) + \int_{\alpha^{-1}(\sigma(s_n))}^{s_n} R(s)ds]$$

as $n \rightarrow \infty$ then $k - l \leq k$ which implies that $l \geq 0$, this is a contradiction

The proof of lemma is complete. ■

The proof of the following lemma is similar to the proof of lemma 2, so we state it without proof.

Lemma 3.

Let $y(t)$ be an eventually positive solution of eq. (1.1) and set

$$W(t) = y(t) - P(t)y(\tau(t)) - \int_{\sigma(t)}^{\alpha(t)} R(\alpha^{-1}(s))y(s)ds \quad (2.4)$$

and the following assumptions are hold.

$$H1': Q(t) - R(\alpha^{-1}(\sigma(t))) \sigma'(t) \geq 0$$

$$H2': \limsup_{t \rightarrow \infty} [P(t) + \int_{\sigma(t)}^{\alpha(t)} R(\alpha^{-1}(s))ds] \leq 1,$$

$$H3: \alpha'(t) \geq 1$$

Then $W(t)$ is eventually positive and nonincreasing function.

Main results:

The next result provides a sufficient conditions for the oscillation of all solutions of eq. (1.1)

Theorem 1.

Let $W(t)$ defined as in (2.2) and the assumptions H1 - H2 hold, in addition to the condition

$$\liminf_{t \rightarrow \infty} \int_{\sigma(t)}^t [Q(s) - R(\alpha^{-1}(\sigma(s))) (\alpha^{-1}(\sigma(s)))'] [1 + P(\sigma(s)) + \int_{\alpha^{-1}(\sigma(\sigma(s)))}^s R(r)dr] ds > \frac{1}{e} \quad (3.1)$$

Then every solution of eq.(1.1) oscillates.

Proof . Suppose $y(t)$ be eventually positive solution of eq.(1.1) then by lemma 2 it follows that $W(t)$ is positive nonincreasing function, differentiate (2.2) and use eq.(1.1) we get (2.3) and from (2.2) we obtain $W(t) \leq y(t)$, hence

$$y(t) = W(t) + P(t)y(\tau(t)) + \int_{\alpha^{-1}(\sigma(t))}^t R(s)y(\alpha(s))ds$$

$$\begin{aligned} &\geq W(t) + P(t)W(\tau(t)) \\ &+ \int_{\alpha^{-1}(\sigma(t))}^t R(s)W(\alpha(s))ds \\ &\geq W(t) + P(t)W(\tau(t)) \\ &+ W(\alpha(t)) \int_{\alpha^{-1}(\sigma(t))}^t R(s)ds \\ &\geq W(t) + P(t)W(t) \\ &+ W(t) \int_{\alpha^{-1}(\sigma(t))}^t R(s)ds \\ &= W(t) [1 + P(t) + \int_{\alpha^{-1}(\sigma(t))}^t R(s)ds] \\ &y(\sigma(t)) \geq W(\sigma(t)) [1 + P(\sigma(t)) + \int_{\alpha^{-1}(\sigma(t))}^t R(s)ds] \\ &W'(t) + \left[Q(t) - R(\alpha^{-1}(\sigma(t))) \frac{d}{dt} \alpha^{-1}(\sigma(t)) \right] [1 + P(\sigma(t)) + \int_{\alpha^{-1}(\sigma(t))}^{\sigma(t)} R(s)ds] W(\sigma(t)) \leq 0 \end{aligned}$$

by lemma 1, and the condition (3.1) the last inequality cannot has eventually positive solution, which is a contradiction. ■

Example 1.

Consider the neutral differential equation:

$$\left[y(t) - P y \left(t - \frac{\pi}{2} \right) \right]' + \left(e^{-\pi} + 2P e^{-\frac{3\pi}{4}} \right) y(t - \pi) - \left(P - 2e^{-\frac{3\pi}{4}} \right) y \left(t - \frac{\pi}{4} \right) = 0 \quad (E1)$$

$$P(t) = P, \text{ where } 0 \leq P \leq 1.178082.$$

$$Q(t) = e^{-\pi} + 2P e^{-\frac{3\pi}{4}}, R(t) = P - 2e^{-\frac{3\pi}{4}}$$

We can see that:

• $\sigma(t) < \alpha(t)$, let $P=0.92$ then

$$Q(t) - R(\alpha^{-1}(\sigma(t))) (\alpha^{-1}(\sigma(t)))' = 2.8552 \geq 0$$

$$\begin{aligned}
& \bullet \limsup_{t \rightarrow \infty} [P(t) + \\
& \int_{\alpha^{-1}(\sigma(t))}^t R(s) ds] = 0.94 < 1 \\
& \bullet \liminf_{t \rightarrow \infty} \int_{\sigma(t)}^t [Q(s) - \\
& R(\alpha^{-1}(\sigma(s))) (\alpha^{-1}(\sigma(s)))'] [1 + \\
& P(\sigma(s)) + \\
& \int_{\alpha^{-1}(\sigma(\sigma(s)))}^s R(r) dr] ds = \\
& 17.39 > \frac{1}{e}
\end{aligned}$$

all the conditions of theorem 1 are hold, so according to theorem 1 every solution of eq.(E1) is oscillatory, for instance the solution $y(t) = e^{-t} \sin t$ is such a solution.

Theorem 2.

Let $W(t)$ defined as in (2.4) and the assumptions H1 - H2 hold and

$$\begin{aligned}
& \liminf_{t \rightarrow \infty} \int_{\sigma(t)}^t [Q(s) - \\
& R(\alpha^{-1}(\sigma(s))) \sigma'(s)] ds > \frac{1}{e} \\
& (3.2)
\end{aligned}$$

Then every solution of equation (1.1) oscillates.

Proof. Assume for the sake of contradiction that $y(t)$ is eventually positive solution of eq.(1.1). then by lemma 2 it follows that $W(t)$ is eventually positive and non increasing function, differentiate (2.2) and use (1.1) we get

$$\begin{aligned}
& W'(t) \\
& = -[Q(t) \\
& - R(\alpha^{-1}(\sigma(t))) \sigma'(t)] y(\sigma(t)) \leq 0 \\
& \leq -[Q(t) \\
& - R(\alpha^{-1}(\sigma(t))) \sigma'(t)] W(\sigma(t)) \\
& \leq 0
\end{aligned}$$

then

$$\begin{aligned}
& W'(t) \\
& + [Q(t) \\
& - R(\alpha^{-1}(\sigma(t))) \sigma'(t)] W(\sigma(t)) \\
& \leq 0
\end{aligned}$$

it follows from lemma1 and condition (3.2) that the last inequality has no eventually positive solution. The proof is complete .

Theorem 3.

Let $W(t)$ be defined as in (2.2) and the assumptions H1', H2' and H3 hold and suppose that

$$\begin{aligned}
& \liminf_{t \rightarrow \infty} \int_{\sigma(t)}^t [Q(s) - \\
& R(\alpha^{-1}(\sigma(s))) \sigma'(s)] [1 + P(\sigma(s)) + \\
& \int_{\sigma(t)}^{\alpha(t)} R(\alpha^{-1}(u)) du] ds > \frac{1}{e} \quad (3.3)
\end{aligned}$$

Then every solution of equation (1.1) oscillates.

Proof.

The proof is similar to the proof of theorem 1 and we omitted it.

Example 2.

Consider the neutral differential equation;

$$\begin{aligned}
& \left[y(t) - \left(\frac{1}{4} - \frac{1}{4} \sin 2t \right) y(t - 2\pi) \right]' + \\
& \frac{4}{5} \left(\frac{3}{4} - \frac{1}{4} \sin 2t \right) y \left(t - \frac{5\pi}{2} \right) - \\
& \frac{1}{5} \left(\frac{3}{4} - \frac{1}{4} \sin 2t \right) y \left(t - \frac{3\pi}{2} \right) = 0, \quad t \geq \\
& t_0 \quad (E2)
\end{aligned}$$

We can see that

$$\begin{aligned}
& \bullet Q(t) - R(\alpha^{-1}(\sigma(t))) (\alpha^{-1}(\sigma(t)))' = \\
& \frac{3}{10} \geq 0 \\
& \bullet P(t) + \int_{\sigma((\alpha^{-1}(\sigma(t)))t)}^t R(s) ds = \\
& 0.971238 < 1
\end{aligned}$$

$$\bullet \liminf_{t \rightarrow \infty} \int_{\sigma(t)}^t [Q(s) - R(\alpha^{-1}(\sigma(s))) \sigma'(s)] ds = \frac{15\pi}{20} > \frac{1}{e}$$

all the conditions of theorem 2 or Theorem 3 are hold and so according to Theorem 2 or Theorem 3 every solution of eq.(E2) are oscillatory, for instance the solution $y(t) = \frac{\cos t}{\frac{3}{4} - \frac{1}{4} \sin 2t}$

is such solution. ■

Theorem 4.

Let $W(t)$ defined as in (2.4) and the assumptions H1', H2', H3 hold, suppose that

$$\liminf_{t \rightarrow \infty} \int_{\sigma(t)}^t [Q(s) - R(\alpha^{-1}(\sigma(s))) \sigma'(s)] ds > \frac{1}{e} \quad (3.4)$$

Then every solution of equation (1.1) oscillates.

Proof. The proof is similar to the proof of theorem 2. and will be omitted.

Example 3.

Consider the neutral differential equation:

$$\left[y(t) - \frac{1}{2\sqrt{2}} y\left(t - \frac{5\pi}{2}\right) \right]' + \frac{3}{2} y\left(t - \frac{\pi}{2}\right) - \frac{1}{2\sqrt{2}} y\left(t - \frac{\pi}{4}\right) = 0 \quad (E3)$$

Solution: We can see that

$$\bullet Q(t) - R(\alpha^{-1}(\sigma(t))) (\alpha^{-1}(\sigma(t)))' \geq 0$$

$$\bullet P(t) + \int_{\sigma((\alpha^{-1}(\sigma(t)))t)}^t R(s) ds = 0.63123 < 1$$

$$\bullet \liminf_{t \rightarrow \infty} \int_{\sigma(t)}^t [Q(s) - R(\alpha^{-1}(\sigma(s))) \sigma'(s)] ds = 1.785 > \frac{1}{e}.$$

all the conditions of theorem 4 hold and so according to theorem 4 every solution of eq.(E3) are oscillatory for

instance the solution $y(t) = \sin t$ is such a solution. ■

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تذبذب المعادلات التفاضلية المحايدة من الرتبة الأولى ذات المعاملات الموجبة والسالبة

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الخلاصة:

في هذا البحث تم استخراج بعض الشروط الضرورية والكافية لتذبذب جميع حلول المعادلات التفاضلية المحايدة من الرتبة الأولى ذات المعاملات الموجبة والسالبة بالشكل

$$(1.1) \quad [y(t) - P(t)y(\tau(t))] + Q(t)y(\sigma(t)) - R(t)y(\alpha(t)) = 0$$

وقد أعطينا بعض الأمثلة لتوضيح النتائج المستخلصة.