Convergence of the Generalized Homotopy Perturbation Method for Solving Fractional Order Integro-Differential Equations

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Abstract:

In this paper,the homtopy perturbation method (HPM) was applied to obtain the approximate solutions of the fractional order integro-differential equations . The fractional order derivatives and fractional order integral are described in the Caputo and Riemann-Liouville sense respectively. We can easily obtain the solution from convergent the infinite series of HPM . A theorem for convergence and error estimates of the HPM for solving fractional order integro-differential equations was given. Moreover, numerical results show that our theoretical analysis are accurate and the HPM can be considered as a powerful method for solving fractional order integro-diffrential equations.

Key words: Homotopy perturbation method, fractional calculas, integro-differential equations.

Introduction:

In recent years various analytical and numerical methods have been applied for approximating the solutions of fractional order differential equations (FDEs). Since exact solutions of most of fractional differential equations do not exist, approximation and numerical methods are used for the solutions of the FDEs, [1-3]. He [1,2,3] was first propose the homotopy perturbation method (HPM) for finding the solutions of linear and nonlinear problems. The (HPM) is the traditional perturbation method and homotopy in topology . This method has been successfully applied by many authors[4,2,5,6] for finding the approximate solutions as well as numerical solutions of functional equations which arise in scientific and engineering problems.Fractional order integro-differential equations arise in modeling processes in applied sciences like physics, engineering,chemistry, and other sciences [7],which can be described very successfully by models using mathematics toolsfrom fractional calculus, such as, frequency dependent damping behavior of materials, diffusion process and motion of alarge thin plate in a Newtonian fluid creeping, etc., [8].The fractional integro-differential equations are usually difficult to solve analytically so it is required to obtain an efficient approximate solution. And there are few techniques for solving fractional integro-differential equations,such as, the Adomian decomposition method, the collocation method and the fractional differential transform method, [9],[6]. The purpose of this paper is to extend the analysis of HPM to construct the approximate solutions of fractional order integro-differential equations.

$$D_*^{\alpha} y(x) = g(x) + I^{\beta} K\big(y(X)\big) \qquad (1)$$

where D_*^{α} indicates the fractional order differential operator in the Caputo sense andI^{β} is the fractional order integral operator in the

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Riemann-Liouville sense and $K(y(X))$ is any nonlinear continuous function, α, β are real constants and g are given and can be approximated by Taylor polynomials.

Basic definitions

In this section, we shall give some but not all, of the basic definitions and properties of fractional calculus theory which are further used in this paper.

Definition(1), [10]:

The Riemann-Liouville definition of the right side fractional integral which is:

$$ {}_aI_x^{\alpha}u(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}u(t)\,dt\,,\alpha>0 \qquad (2)$$

while the left hand sided integral:

$$ {}_xI_b^{\alpha}u(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}u(t)\,dt\,,\alpha>0 \qquad (3)$$

Definition (2), [2]:

The Caputo definition of fractional derivative is given by:

$$ {}_0D_x^{\alpha}u(x)=\frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-q-1}u^m(t)\,dt$$

where $m-1<\alpha\leq m$, $m\in N$, $x>0$

Now, some properties of fractional concerning differintegration are given next, [11],[10]:

1- If $m-1<\alpha\leq m$, $m\in N$ and f is any function, then:

$D_*^{\alpha}I^{\alpha}f(x)=f(x)$

$$I^{\alpha}D^{\alpha}f(x)=f(x)-\sum_{k=0}^{n-1}f^{(k)}(0^+)\frac{x^k}{k!},$$

$x>0$

Where 0^+ refers to the right hand side limit of $f^{(k)}$ at 0.

2- $I^0f(x)=D_*^0f(x)=f(x)$.

3- $I^{\alpha}I^{\beta}f(x)=I^{\beta}I^{\alpha}f(x)=I^{\alpha+\beta}f(x)$, $\forall\alpha,\beta\geq 0$.

4- $I^{\alpha}f(x)=D_*^{-\alpha}f(x)$, $\alpha>0$.

5-$I^{\alpha}(0)=D_*^{-\alpha}0)=0$, $\alpha>0$.

Analysis of HPM

To illustrate the basic concepts of HPM for fractional order integro-differential equations, consider the fractional order integro-differential equation (1).

In view of HPM [2,3,6], construct the following homotopy for equation (1):

$$(1-p)D_*^{\alpha}y(x)+p\left(D_*^{\alpha}y(x)-g(x)-I^{\beta}K(Y(X))\right)=0 \qquad (4)$$

or

$$D_*^{\alpha}\,y(x)=p\left(g(x)+I^{\beta}K(y(x))\right) \qquad (5)$$

where $p\in[0,1]$ is an embedding parameter. If $p=0$, then equation (5) becomes a linear equation

$$D_*^{\alpha}y(x)=0 \qquad (6)$$

and when $p=1$, then equation (5) turns out to be the original equation (1).

In view of basic assumption of homotopy perturbation method, solution of equation(1) can be expressed as a power series in p:

$y(x) = y_0(x) + py_1(x) + p^2y_2(x) + p^3y_3(x)\cdots$
(7)
setting $p = 1$, in (5) then we get an approximate solution of equation(1)
$y(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \cdots$
(8)
Substitution (7) into (5), then equating the terms with identical power of p, we obtain the following series of linear equations.
$D_*^\alpha(y_0(x) + py_1(x) + p^2y_2(x) + p^3y_3(x) + \cdots)$
$= p\left(g(x) + I^\beta K(y_0(x) + py_1(x) + p^2y_2(x) + \cdots)\right)$, implies to:

$D_*^\alpha y_0 + pD_*^\alpha y_1 + p^2D_*^\alpha\ \ y_2 + p^3D_*^\alpha\ \ y_3 + \cdots$
$= pg(x) + pI^\beta K_1y_0 + p^2I^\beta K_2y_1 + p^3I^\beta K_3y_2 + p^4I^\beta K_4y_2 + \cdots$,thus
$p^0: D_*^\alpha y_0 = 0$
(9)
$p^1: D_*^\alpha y_1 = g(x) + I^\beta K_1(y_0)$
(10)
$p^2: D_*^\alpha y_2 = I^\beta K_2(y_1)$
(11)
$p^3: D_*^\alpha y_3 = I^\beta K_3(y_2)$
(12) ⋮
where the functions $K_1, K_2, \cdots$satisfy the following condition:
$K(y_0(x) + py_1(x) + p^2y_2(x) + \cdots)$=$K_1(y_0(x)) + pK_2(y_1(x)) + p^2K_3(y_2(x)) + \cdots$),
x϶[0,T]

Equations (9)-(12) can be solved by applying the operator I^α, which is inverse of the operator D_*^α and then by simple computation, we approximate the series solution of HPM by the following n-term truncated series:
$\varphi_n(x)$=$y_0(x)$+$y_1(x)$+$y_2(x)$+$y_3(x)$+⋯+$y_n(x)$

(13)

Now , we show that the series defined by (13) with $y_0(x) = y_0$converges to the solution
of (1). To do this we state and prove the following theorem.

Theorem(1):
Let $y \in C^1[0,T]$ which defined with maximam norm $\|.\|_\infty$.
$D_*^\alpha y(x) = g(x) + I^\beta K(y(x))$, $y(0) = y_0$,

(14)
and $y_n \in (C^1[0,T], \|.\|_\infty)$be obtained solution of the sequence defined by
$\varphi_{n+1}(x) = \varphi_n(x) + I^{\alpha+\beta}K_{n+1}(y_n(x))$, n ≥ 1 .
(15)

$\varphi_1(x) = \varphi_0(x) + I^\alpha g(x)$+$I^{\alpha+\beta}K_1(y_0(x))$
.
$\varphi_0(x)$=$y_0(x)$.
If $E_n(x) = y_n(x) - y(x)$ and K in equation (15) satisfies Lipschitz condition with constant L_n,n ≥ 1such that $L = \max\{L_n\ \ , n \geq 1\}$ and $L < \Gamma(\alpha + \beta)$, then the sequence of approximate solution $\{\varphi_n\}, n = 0, 1, \cdots$, converges to the exact solution y.

Proof
Consider the fractional order integro-differential equation of fractional order
$D_*^\alpha y(x) = g(x) + I^\beta K(y(x))$, $y(0) = y_0$
, x ∈ [0,T]
where the approximate solution using HPMis given by:
$\varphi_{n+1}(x) = \varphi_n(x) + I^{\alpha+\beta}K_{n+1}(y_n(x))$, n ≥ 1 .

$\varphi_1(x) = \varphi_0(x) + I^\alpha g(x)$+$I^{\alpha+\beta}K\ \ (y_0(x))$
.
$\varphi_0(x)$=$y_0(x)$.

Since y is the exact solution of the integro-differential equation of fractional order, hence:

$y(x) = y(x) + I^{\alpha} g(x) + I^{\alpha+\beta} K(y(x))$

hence subtracting y from φ_{n+1} yields to:

$$\varphi_{n+1}(x) - y(x) = \varphi_n(x) - y(x) + I^{\alpha} g(x) - I^{\alpha} g(x) + I^{\alpha+\beta}(K \ (\varphi_n(x)) - K(y(x)))$$

$$\varphi_{n+1}(x) = \varphi_n(x) + I^{\alpha+\beta}((K \ (\varphi_n(x)) - K(y(x)))$$

$$\begin{aligned}\varphi_{n+1}(x) - \varphi_n(x) &= I^{\alpha+\beta}((K \ (\varphi_n(x)) - K(y(x))) \\ &= I^{\gamma}[K \ (\varphi_n(x)) - K(y(x))], \quad \text{set } \gamma = \alpha + \beta \\ &= \frac{1}{\Gamma(\gamma)} \int_0^x (x-s)^{\gamma-1} [K \ (\varphi_n(s)) - K(y(s))] ds\end{aligned}$$

Now, taking the maximum- norm of the two sides of $\varphi_{n+1}(x) - \varphi_n(x)$ will give:

$$\|\varphi_{n+1}(x) - \varphi_n(x)\| = \left\| \frac{1}{\Gamma(\gamma)} \int_0^x (x-s)^{\gamma-1} [K \ (\varphi_n(s)) - K(y(s))] ds \ \right\|_{\infty}$$

$$\leq \frac{1}{\Gamma(\gamma)} \int_0^x \|x-s\|_{\infty}^{\gamma-1} \|K \ (\varphi_n(s)) - K(y(s))\|_{\infty} \, ds$$

$$\leq \frac{1}{\Gamma(\gamma)} \int_0^x \max_{s\in[0,x]} |x-s|^{\gamma-1} L \|\varphi_n(s) - y(s)\|_{\infty} \, ds$$

$$\leq \frac{1}{\Gamma(\gamma)} \int_0^x x^{\gamma-1} L \|E_n(s)\|_{\infty} \, ds, \qquad \forall n = 0, 1, \cdots$$

From (9-12), we have that

$$\|E_{n+1}(s)\|_{\infty} = \|y_{n+1}(x) - y(x)\|_{\infty} = \left\|I^{\alpha+\beta}[K \ (y_n(x)) - K \ (y(x))]\right\|_{\infty} \leq \left\|I^{\alpha+\beta}(y_n(x) - y(x))\right\|_{\infty}$$

$$\leq \frac{1}{\Gamma(\gamma)} \int_0^x \max_{s\in[0,x]} |x-s|^{\gamma-1} L \|y_n(s) - y(s)\|_{\infty} \, ds \leq \frac{1}{\Gamma(\gamma)} \int_0^x x^{\gamma-1} L \|E_n(s)\|_{\infty} \, ds,$$

Hence

$$\|E_{n+1}(s)\|_{\infty} \leq \frac{1}{\Gamma(\gamma)} \int_0^x x^{\gamma-1} L \|E_n(s)\|_{\infty} \, ds$$

Now if $n = 0$, then:

$$\|\varphi_1(x) - \varphi_0(x)\|_{\infty} = \frac{1}{\Gamma(\gamma)} x^{\gamma-1} \int_0^x \|E_0(s)\|_{\infty} ds$$

$$\leq \frac{L}{\Gamma(\gamma)} x^{\gamma-1} \int_0^x \max_{s\in[0,x]} |E_0(s)|^{\gamma-1} ds \ \leq \frac{L}{\Gamma(\gamma)} x^{\gamma-1} \max_{s\in[0,x]} |E_0(s)| \int_0^x ds$$

$$\leq \frac{L}{\Gamma(\gamma)} x^{\gamma} \max_{s\in[0,x]} |E_0(s)| \quad ,$$

also, for $n = 1$, we have

$$\|\varphi_2(x) - \varphi_1(x)\|_{\infty} \leq \frac{1}{\Gamma(\gamma)} x^{\gamma-1} \int_0^x \|E_1(s)\|_{\infty} ds$$

$$\leq \frac{L}{\Gamma(\gamma)} x^{\gamma-1} \int_0^x \left[\frac{L}{\Gamma(\gamma)} x^{\gamma} \max_{s\in[0,x]} |E_0(s)|\right] ds$$

$$\leq \left(\frac{L}{\Gamma(\gamma)}\right)^2 x^{2\gamma-1} \max_{s\in[0,x]} |E_0(s)| \int_0^x ds \ \leq \left(\frac{L}{\Gamma(\gamma)}\right)^2 \frac{x^{2\gamma}}{\gamma+1} \max_{s\in[0,x]} |E_0(s)|$$

Similarly, for $n = 2$, then

$$\|\varphi_3(x)-\varphi_2(x)\|_\infty \le \frac{1}{\Gamma(\gamma)}x^{\gamma-1}\int_0^x \|E_2(s)\|_\infty ds$$
$$\le \frac{L}{\Gamma(\gamma)}x^{\gamma-1}\int_0^x\left[\left(\frac{L}{\Gamma(\gamma)}\right)^2\frac{s^{2\gamma}}{\gamma+1}\max_{s\in[0,x]}|E_0(s)|\right]ds$$
$$=\left(\frac{L}{\Gamma(\gamma)}\right)^3 x^{\gamma-1}\max_{s\in[0,x]}|E_0(s)|\int_0^x\left[\frac{s^{2\gamma}}{\gamma+1}\right]ds$$
$$=\left(\frac{L}{\Gamma(\gamma)}\right)^3 x^{\gamma-1}\max_{s\in[0,x]}|E_0(s)|\frac{x^{2\gamma}}{(\gamma+1)(2\gamma+1)}$$
$$=\left(\frac{L}{\Gamma(\gamma)}\right)^3\frac{x^{3\gamma}}{(\gamma+1)(2\gamma+1)}\max_{s\in[0,x]}|E_0(s)|$$

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$$\|\varphi_n(x)-\varphi_{n-1}(x)\|_\infty \le \left(\frac{L}{\Gamma(\gamma)}\right)^n\frac{x^{n\gamma}}{(\gamma+1)(2\gamma+1)\ldots((n-1)\gamma+1)}\max_{s\in[0,x]}|E_0(s)|$$
$$\le \left(\frac{L}{\Gamma(\gamma)}\right)^n\frac{T^{n\gamma}}{(\gamma+1)(2\gamma+1)\ldots((n-1)\gamma+1)}\max_{s\in[0,x]}|E_0(s)| \quad \text{for } n\ge 1,$$

and since $L<\Gamma(\gamma)$, so as $n\to\infty$, we have $\|\varphi_{n+1}(x)-\varphi_n(x)\|\to 0$.
Now, we have that
$y(x)=\varphi_0(x)+\sum_{n=0}^{\infty}(\varphi_{n+1}(x)-\varphi_n(x))$
Now we have that $\varphi_j(x)=\varphi_0(x)+\sum_{m=1}^{j-1}(\varphi_{m+1}(x)-\varphi_m(x))$
$y(x)-\varphi_j(x)=\sum_{n=j}^{\infty}(\varphi_{n+1}(x)-\varphi_n(x))$

$$\|y(x)-\varphi_j(x)\|_\infty \le \sum_{n=j}^{\infty}\|\varphi_{n+1}(x)-\varphi_n(x)\|_\infty$$
$$\le \sum_{n=j}^{\infty}\left(\frac{L}{\Gamma(\gamma)}\right)^n\frac{T^{n\gamma}}{(\gamma+1)(2\gamma+1)\ldots((n-1)\gamma+1)}\max_{s\in[0,x]}|E_0(s)|$$
$$\le T^j\left(\frac{L}{\Gamma(\gamma)}\right)^j\max_{s\in[0,x]}|E_0(s)|$$

$$\sum_{n=0}^{\infty}\left(\frac{L}{\Gamma(\gamma)}\right)^n\frac{T^{n\gamma}}{(\gamma+1)(2\gamma+1)\cdots((n-1)\gamma+1)}$$

Since $L<\Gamma(\gamma)$ therefore $T^{j\gamma}\left(\frac{L}{\Gamma(\gamma)}\right)^j\to 0$ as $j\to\infty$. Hence $\varphi_j\to y$.

Applications

In this section the (HPM) has bon applied to linear and nonlinear fractional order integro-differential equations in order to illustrate the validity of the proposed method.

The following examples had been studied and discussed by using iteration variational method in [6].

Example(1)

Consider the following linear fractional orderintegro-differential equation

$$D_*^{0.75}y(x)=\frac{6}{\Gamma(3.25)}x^{2.25}-\frac{6}{\Gamma(4.75)}x^{3.75}+I^{0.75}y(x)$$

(16) $y(0)=0, x\in[0,2]$

for comparison purpose, the exact solution of equation(16) is given by $y(x)=x^3$.

According to HPM, we construct the following homotopy:

$$D_*^{0.75}y(x) = p\left(\frac{6}{\Gamma(3.25)}x^{2.25} - \frac{6}{\Gamma(4.75)}x^{3.75} + I^{0.75}y(x)\right)$$

Substitution of (5) into (14) and then equating the terms with same powers of p yield the following series of linear equations:

$$p^0: D_*^{0.75}y_0 = 0 \tag{17}$$

$$p^1: D_*^{0.75}y_1 = \frac{6}{\Gamma(3.25)}x^{2.25} - \frac{6}{\Gamma(4.75)}x^{3.75} + I^{0.75}y_0(x) \tag{18}$$

$$p^2: D_*^{0.75}y_2 = I^{0.75}y_1(x) \tag{19}$$

$$p^3: D_*^{0.75}y_3 = I^{0.75}y_2(x) \tag{20}$$

$\vdots$

Applying the operator $I^{0.75}$ to the above series of linear equations, we get

$y_0(x) = y(0) = 0$

$$I^{0.75}D_*^{0.75}y_1(x) = I^{0.75}\left[\frac{6}{\Gamma(3.25)}x^{2.25} - \frac{6}{\Gamma(4.75)}x^{3.75} + I^{0.75}y_0(x)\right]$$

$$y_1(x) - y_1^{(0)}(0^+)\frac{x^0}{0!} = I^{0.75}\left[\frac{6}{\Gamma(3.25)}x^{2.25} - \frac{6}{\Gamma(4.75)}x^{3.75} + I^{0.75}y_0(x)\right]$$

$$y_1(x) = I^{0.75}\left[\frac{6}{\Gamma(3.25)}x^{2.25} - \frac{6}{\Gamma(4.75)}x^{3.75} + I^{0.75}y_0(x)\right]$$

$$y_2(x) = I^{0.75}I^{0.75}y_1(x)$$

$$y_3(x) = I^{0.75}I^{0.75}y_2(x)$$

$\vdots$

Hence we find that

$$y_1(x) = x^3 - \frac{6x^{4.5}}{\Gamma(5.5)},\; y_2(x) = \frac{\Gamma(4)}{\Gamma(5.5)}x^{4.5} - \frac{x^6}{120} \quad ,y_3(x) = \frac{\Gamma(4)}{\Gamma(7)}x^6 - \frac{\Gamma(7)x^{7.5}}{120\Gamma(8.5)}$$

$$y_4(x) = \frac{\Gamma(4)}{\Gamma(8.5)}x^{7.5} - \frac{6}{\Gamma(10)}x^9 \quad ,y_5(x) = \frac{\Gamma(4)}{\Gamma(10)}x^9 - \frac{6}{\Gamma(11.5)}x^{10.5}$$

and

$$y_6(x) = \frac{\Gamma(4)}{\Gamma(11.5)}x^{10.5} - \frac{6}{\Gamma(13)}x^{12},$$

$\vdots$

and according to equation (8) the approximate solution of equation (16) can be written as

$\varphi_n(x) = y1(x) + y2(x) + y3(x) + \cdots + y_n(x)$

.

Thus therefer the approximate solution up to seven terms given as

$$\varphi_7(x) = x^3 + 1.10^{-23}x^{7.5} - \frac{1}{79833600}x^{12}$$

Follows table (1) represents a comparison between the approximate solution and the exact solution

Table(1):The absolute error between the exact and approximate solutions of example (1).

x	$y(x)$	$\varphi_7(x)$	$\lvert y(x)-\varphi_7(x)\rvert$
0	0	0.000000000000000	0
0.2	$8\cdot10^{-3}$	7.999999999999950e-3	0
0.4	0.064	0.063999999999790	$2.102\cdot10^{-13}$
0.6	0.216	0.215999999972734	$2.727\cdot10^{-11}$
0.8	0.512	0.511999999139216	$8.608\cdot10^{-10}$
1	1	0.999999987473946	$1.253\cdot10^{-8}$
1.2	1.728	1.727999888316442	$1.117\cdot10^{-7}$
1.4	2.744	2.743999289848980	$7.102\cdot10^{-7}$
1.6	4.096	4.095996474229187	$3.526\cdot10^{-6}$
1.8	5.832	5.831985509467422	$1.449\cdot10^{-5}$
2	8	7.999948693282024	$5.131\cdot10^{-5}$

Example(2):

Consider the following linear integro-differential equation

$$D_*^{0.5}y(x)=\frac{2}{\Gamma(2.5)}x^{1.5}-\frac{\Gamma(5)}{\Gamma(5.5)}x^{4.5}+I^{0.5}(y(x))^2 \quad (21)$$

$y(0)=0, x\in[0,1]$

for comparison purpose, the exact solution of equation (21) is given by$y(x)=x^2$.

According to homotopy perturbation method, we construct the following homotopy:

$$D_*^{0.5}y(x)=p\left(\frac{2}{\Gamma(2.5)}x^{1.5}-\frac{\Gamma(5)}{\Gamma(5.5)}x^{4.5}+I^{0.5}(y(x))^2\right)$$

Substitution (7) into (21) and then equating the terms with same powers of p yield the following series of linear equations:

$$p^0: D_*^{0.5}y_0=0 \quad (22)$$

$$p^1: D_*^{0.5}y_1=\frac{2}{\Gamma(2.5)}x^{1.5}-\frac{\Gamma(5)}{\Gamma(5.5)}x^{4.5}+I^{0.5}(y(x))^2 \quad (23)$$

$$p^2: D_*^{0.5}y_2=2I^{0.5}y_0(x)y_1(x) \quad (24)$$

$$p^3: D_*^{0.5}y_3=I^{0.5}\left(2y_0(x)y_2(x)+(y_1(x))^2\right) \quad (25)$$

$$p^4: D_*^{0.5}y_4=2I^{0.5}(y_1(x)y_2(x)+(y_0(x)y_3(x)) \quad (27)$$

$$p^5: D_*^{0.5}y_5=I^{0.5}\left(2y_0(x)y_4(x)+2(y_1(x)y_3(x)+(y_2(x))^2\right) \quad (28)$$

$$p^6: D_*^{0.5}y_6=0 \quad (29)$$

$$p^7: D_*^{0.5}y_7=I^{0.5}\left(2y_1(x)y_5(x)+(y_3(x))^2\right) \quad (30)$$

$$P^8: D_*^{0.5\,.5}y_8=0 \quad (31)$$

$$P^9: D_*^{0.5}y_9=I^{0.5}\,2(y_1(x)\,y_7(x)+y_3(x)y_5(x)) \quad (32)$$

⋮

Applying the operator $I^{0.5}$ to the above series of linear equations:

$y_0(x)=0$

$y_1(x)=I^{0.5}I^{0.5}y_0^2(x)$, in general we obtain that,

$$y_{2n}(x) = I^{\alpha}\left(I^{\beta}\left(2y_0(x)y_{2n-0}(x) + 2y_1(x)y_{2n-1}(x) + \cdots + 2y_{n-1}(x)y_{2n-(n-1)}(x) + y_n^2(x)\right)\right)$$

$n = 1,2,3,4,\cdots$ (33)

$$y_{2n+1}(x) = I^{\alpha}\left(I^{\beta}\left(2y_0(x)y_{2n+1}(x) + y_1(x)y_{(2n+1)-1}(x) + \cdots + y_n(x)y_{2n+1-n}(x)\right)\right)$$

(34)

$n = 1,2,3,4,\cdots$

Thus, by solving equations (22)-(32), we obtain $y_1, y_2 \cdots$ as follows:

$y_1(x) = x^2 - .20000000000000000001x^5$, $y_2(x) = 0$
$y_3(x) = 3.6363636363636363640 \cdot 10^{-3}x^{11} - 5.0000000000000000003 \cdot 10^{-2}x^8 + .20000000000000000000x^5$
, $y_4(x) = 0$
$y_5(x) = 8.5561497326203208569 \cdot 10^{-5}x^{17} + 1.9480519480519480522 \cdot 10^{-3}x^{14} - 1.6363636363636363637 \cdot 10^{-2}x^{11} + 5.0000000000000000 \cdot 10^{-2}x^8$
$y_6(x) = 0$,
and hence the approximate solution of example (2) up to 6-terms may be given as

$$\varphi_6(x) = y_1(x) + y_2(x) + y_3(x) + y_4(x) + \cdots + y_6(x)$$

The comparison between the exact and approximate solution up to 6-terms of example (2) is given in table (2):

Table (2):The absolute error between the exact and approximate solutions of example (2).

x	$y(x)$	$\varphi(x)$	$\lvert y(x) - \varphi(x)\rvert$
0	0	0.000000000000000	0
0.1	0.01	0.010000000000000	0
0.2	0.04	0.039999999999999	$1.082 \cdot 10^{-15}$
0.3	0.09	0.089999999998939	$1.061 \cdot 10^{-12}$
0.4	0.16	0.159999999860521	$1.395 \cdot 10^{-10}$
0.5	0.25	0.249999993924831	$6.075 \cdot 10^{-9}$
0.6	0.36	0.359999869075625	$1.309 \cdot 10^{-7}$
0.7	0.49	0.489998272615760	$1.727 \cdot 10^{-6}$
0.8	0.64	0.639984171065769	$1.583 \cdot 10^{-5}$
0.9	0.81	0.809890823887024	$1.092 \cdot 10^{-4}$
1	1	0.999401830014784	$5.982 \cdot 10^{-4}$

In order of the remarks in [6], we referred that the other types of equations may be consider as a special case of the Homtopy perturbation method formula given by eq. (3)

Concluding remarks (1):

Recall the fractional order integro-differential equation (1) and the Homotopy perturbation method formula (3), then the following special cases may be derived from fractional integro-differential equations:

1. If $\alpha = 0$, then eq. (1) will be reduced to:

$$u(x) = g(x) + I^{\beta}k(u(x)) \qquad (35)$$

which is known as the fractional integral equation.

2. If $\beta = 0$, then eq. (1) will be reduced to

$$D^{\alpha}u(x) = g(x) + k(u(x)) \qquad (36)$$

which is known as fractional order differential equation.

3. If $\alpha = 1$, $\beta = 1$ then eq. (1) will be reduced to

$$u'(x) = g(x) + \int_a^x k[x,t;u(t)]dt \qquad (37)$$

which is known as integro-differential equation.

4. If $\alpha = 0$, $\beta = 1$ then eq. (1) will be reduced to

$$u(x) = g(x) + \int_a^x k[x,t;u(t)]dt \qquad (38)$$

which is the usual volterra integral equation.

5. If $\alpha = 1$, $\beta = 0$ then eq. (1) will be reduced to

$$u'(x) = g(x) + k(u(x)) \qquad (39)$$

which is a first order ODE.

The following examples are designed to illustrate the above concluding remark

Example(3):

Consider the following linear fractional order integral equation of fractional order

$$y(x) = x - \frac{2}{\Gamma(4.5)}x^{3.5} + I^{1.5}(y(x))^2 \quad , \quad y(0) = 0, \; x \in [0,1] \qquad (40)$$

for comparison purpose, the exact solution of equation(40) is given by $y(x) = x$.

According to HPM, we construct the following homotopy :

$$y(x) = p\left(x - \frac{2}{\Gamma(4.5)}x^{3.5} + I^{1.5}(y(x))^2\right)$$

Substitution of (5) into (40) and then equating the terms with same powers of p yield the following series of linear equations:

$$p^0 = y_0(x) = 0 \qquad (41)$$

$$p^1 : y_1 = x - \frac{2}{\Gamma(4.5)}x^{3.5} + I^{1.5}(y_0(x))^2 \qquad (42)$$

$$p^2 : y_2 = 2I^{1.5}y_0(x)y_1(x) \qquad (43)$$

$$p^3 : y_3 = I^{1.5}\left(2y_0(x)y_2(x) + (y_1(x))^2\right) \qquad (44)$$

$$p^4 : y_4 = 2I^{1.5}\left(y_1(x)y_2(x) + (y_0(x)y_3(x)\right) \qquad (45)$$

$$p^5 : y_5 = I^{1.5}\left(y_0(x)y_4(x) + 2(y_1(x)y_3(x) + (y_2(x))^2\right) \qquad (46)$$

$$p^6 : y_6 = 0 \qquad (47)$$

$$p^7 : y_7 = I^{1.5}\left(y_1(x)y_5(x) + (y_3(x))^2\right) \qquad (48)$$

⋮

Hence,

$y_0(x) = y(0) = 0,$

$y_1(x) = x - \frac{2}{\Gamma(4.5)}x^{3.5}$

$y_2(x) = 0$

,

$y_3(x) = -2.5.10^{-2}x^6 + 0.1719.x^{\frac{7}{2}} + 1.249.10^{-3}x^{\frac{17}{2}}$

$, y_4(x) = 0$

⋮

and hence the approximate solution of example (2) up to 4-terms may be given as

$$\varphi_4(x) = y_1(x) + y_2(x) + y_3(x) + y_4(x)$$

The comparison between the exact and approximate solution up to 4-terms of example (3) is given in table (3):

Table(3):The absolute error between the exact and approximate solutions of example (3).

x	$y(x)$	$\varphi(x)$	$\lvert y(x)-\varphi(x)\rvert$
0.000	0.000	0.000000000000000	0.000
0.100	0.100	0.099999987540024	1.246e-8
0.200	0.200	0.199999210062249	7.899e-7
0.300	0.300	0.299991141535752	8.858e-6
0.400	0.400	0.399951302058240	4.870e-5
0.500	0.500	0.499819391869387	1.806e-4
0.600	0.600	0.599479087530889	5.209e-4
0.700	0.700	0.698739660702434	1.260e-3
0.800	0.800	0.797323641026904	2.676e-3
0.900	0.900	0.894863519789006	5.136e-3
1.000	1.000	0.990907308909784	9.093e-3

Example(4):

Consider the following linear integro-differential equation

$$D_*^{0.75}y(x)=x^5-\frac{2}{\Gamma(2.25)}x^{1.25}-\left(y(t)\right)^2 \quad , \quad y(0)=0 \qquad (49)$$

for comparison purpose, the exact solution of equation (49) is given by $y(x)=x^2$.

According to homotopy perturbation method, we construct the following homotopy:

$$D_*^{0.75}y/(x)=p\left(x^5+\frac{2}{\Gamma(2.25)}x^{1.25}-x\left(y(x)\right)^2\right)$$

Substitution (49) into (7) and then equating the terms with same powers of p yield the following series of linear equations:

$$p^0: D_*^{0.75}y_0=0$$

$$p^1: D_*^{0.75}y_1=x^5+\frac{2}{\Gamma(2.25)}x^{1.25}-\left(y_0(x)\right)^2$$

$$p^2: D_*^{0.75}y_2=-2xy_0(x)y_1(x) \qquad (50)$$

$$p^3: D_*^{0.75}y_3=-x\left(2y_0(x)y_2(x)+\left(y_1(x)\right)^2\right) \qquad (51)$$

$$p^4: D_*^{0.75}y_4=-2x\left(y_1(x)y_2(x)+\left(y_0(x)y_3(x)\right)\right) \qquad (52)$$

$$p^5: D_*^{0.75}y_5=-x\left(y_0(x)y_4(x)+2\left(y_1(x)y_3(x)+\left(y_2(x)\right)^2\right)\right) \qquad (53)$$

$$p^6: D_*^{0.75}y_6=0 \qquad (54)$$

⋮

Applying the operator $I^{0.75}$ to the above series of equations:

$y_0(x)=0$

,

$y_1(x)=I^{0.75}\left(x^5+\frac{2}{\Gamma(2.25)}x^{1.25}-\left(y_0(x)\right)^2\right)$

, $y_2(x)=I^{0.75}\left(-2xy_0(x)y_1(x)\right)$

$y_3(x)=I^{0.75}\left(-x\left(2y_0(x)y_2(x)+\left(y_1(x)\right)^2\right)\right)$

⋮

hence

$y_1(x)=x^2+0.26489x^{5.75}$,

$y_2(x)=0$,

$y_3(x)=-0.26489x^{5.75}-9.693\cdot10^{-2}x^{9.5}-1.00321\cdot10^{-2}x^{13.25}$

⋮

and hence the approximate solution of example (2) up to 3-terms may be given as

$\varphi_3(x)=y_1(x)+y_2(x)+y_3(x)$

The comparison between the exact and approximate solutions up to 3-term of example (4) is given in table (4):

Table (4):The absolute error between the exact and approximate solutions of example (4).

x	$y(x)$	$\varphi(x)$	$\lvert y(x)-\varphi(x)\rvert$
0	0	0.000000000000000	0.000000000000000
0.1	0.01	9.999999984673746e-3	1.532625588940295e-11
0.2	0.04	0.0399999889096390	1.109036124741225e-8
0.3	0.09	0.089999479112133	5.208878669227257e-7
0.4	0.16	0.159992038914931	7.961085068536322e-6
0.5	0.25	0.249934517057493	6.548294250710574e-5
0.6	0.36	0.359638304475523	3.616955244774567e-4
0.7	0.49	0.488496457657771	1.503542342229325e-3
0.8	0.64	0.635000214445637	4.999785554362868e-3
0.9	0.81	0.796355168210930	0.013644831789070
1	1	0.969904622919535	0.030095377080465

Conclusion:

In this paper, homotopy perturbation method (HPM) has been successfully applied to integro-differential equations. Two examples are presented toillustratethe accuracy of the present schemes of HPM and the efficiency of the methods.

References:

1. He J.H.2003.Homotopy Perturbation method: a new nonlinear Analytic technique, Applied Mathematics Computation 135:73-79.

2. HeJ.H.1999.Homotopy Perturbation method. Computer Methods in Applied Mechanics and Engineering 178: 257-262.

3. He J.H.2000. A coupling method of homtopy technique and Perturbation technique for nonlinear problems .International Journal of Nonlinear Mechanics 35(1):37- 43.

4. Ahmet Yildirim.2008. Solution of BVPs for fourth-order integro-differentials by using homtopy perturbation method .Computer and Mathematics applications 56:3175-3180.

5.Momani S., OdibatZ. 2007. Application of homtopy – perturbation method to fractional IVPs, Journal of Computational and Applied Mathematics 207(1):96.

6. Wadea M.H. Variational Iteration Method for Solving Fractional Order Integro-Diffrential Equations. M.Sc.Thesis college of eduction, Al-Mustansiriya University.2012.

7. Mittal R. C., Nigam R.2008. Solution of fractional integro-diffrential equations by A domain decomposition method. International Journal of Appll.M. and Mech.,4(2): 87-94.

8. Wang W. H.2009. An Effective Method for solving fractional integro-diffrential Equations. Acta Universitatis Apulensis. 20:229-235.

9. Kurulay M.,Secer A.2011.Varational Itration method for solving nonlinear factional integro-diffrential equations, International Journal of Computer Science an Emerging Technologies. 2:18-20.

10. Hilfer R.2000.Fractional Calculus in Physics, First Edition, World Scientific Publishing, Singapore, New Jersey. London and Hong Kong,.

11.Yasir Nawaz.2011.Variational iteration method and homotopy perturbation method for fourth-order fractional integro-diffrential equations. Journal of computer and mathematics with applications 61:2330-2341.

تقارب طريقة المقلقلة الهوموتوبية المعممة لحل المعادلات التكاملية-التفاضلية ذات الرتب الكسرية

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الخلاصة:

في هذا البحث ، تم تطبيق الطريقة المقلقلة الهوموتوبية (HPM) للحصول على الحلول التقريبية للمعادلات التكاملية-التفاضلية. فتم وصف المشتقات الكسرية والتكاملات الكسرية بصيغة كابوتو وريمان ليوفيل على التوالي . فاستطعنا بسهولة الحصول على الحل من خلال متسلسلة منتهية تمثل الحل التقريبي مستخدما فيها تطبيق طريقة المقلقلة الهوموتوبية (HPM). كذلك تم اعطاء نظرية التقارب و تقديرات الخطأ لطريقة (HPM) لحل المعادلات التكاملية-التفاضلية. علاوة على ذلك ،لقد بينت النتائج العددية دقة الجانب النظري التحليلي و قوة طريقة (HPM) في حل المعادلات التكاملية-التفاضلية.