An Algorithm for nth Order Intgro-Differential Equations by Using Hermite Wavelets Functions

*Asmaa A. Abdalrehman**

Received 21, May, 2013
Accepted 26, September, 2013

Abstract:

In this paper, the construction of Hermite wavelets functions and their operational matrix of integration is presented. The Hermite wavelets method is applied to solve nth order Volterra integro diferential equations (VIDE) by expanding the unknown functions, as series in terms of Hermite wavelets with unknown coefficients. Finally, two examples are given

Key words: Hermite wavelets, integro-differential equation, operational matrix of integrations.

Introduction:

The solution of integral and integro-differential equations has a major role in the fields of science and engineering when a physical system is modeled under the differential sense, it finally gives a differential equation, an integral equation or an integro-differential equations mostly appear in the last equation [1,2].

Wavelets permit the accurate representation of a variaty of functions and operators. Special attention has been given to application of the Chebyshev wavelets [3-5] the Sin and Cosin wavelets [6] and the Legendre wavelets [7,8] .

In this paper the operational matrix of integration for Hermite wavelets is derived and used it for obtaining approximate solution of the following nth order VIDE.

$u^{(n)}(x)=g(x)+\int_0^x k(x,t)u^{(s)}(t)dt$… (1)

where k(x,t) and g(x) are known functions, and u(x) is an unknown function.

Hermite Polynomials and Their Properties:

An important equation wich appears in problems of physics is called Hermite's differential equation; it is given by [9]

$y'' - 2xy' + 2ny = 0$ … (2)

where n=0,1,2,3…

Eq (2) has polynomial solutions called Hermite polynomials given by Rodrigue's formula

$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$… (3)

- The first few Hermite polynomials are $H_0 = 1, H_1 = 2x, H_2 = 4x^2 - 2, H_3 = 8x^3 - 12x$
- The generating function for Hermite polynomials is given by

$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n}{n!} t^n$

This result is useful in obtaining many properties of $H_n(x)$. The Hermite polynomials satisfy the recurrence formulas

$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$ … (4)

$H_n'(x) = 2nH_{n-1}(x)$

Starting with $H_0 = 1, H_1 = 2x$.

- Orthgonality of Hermite polynomials [9]

*University of Technology, Applied Science Department

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = \begin{cases} 0 & m \neq n \\ 2^n n! \sqrt{\pi} & m = n \end{cases} \quad \ldots(5)$$

So that the Hermite polynomials are mutually orthogonal with respect to the weight function or density function e^{-x^2} and if m=n we can normalize the Hermite polynomial so as to obtain an orthonormals set.

Hermite Wavelets:

Hermite wavelets, $h_{nm}(t)$ have four arguments l, m, k, t, $l = 1,2,3, \ldots, 2k$, k any non-negative integer, m is the degree of Hermite polynomial and t independent variable in [0,1], Here we can define Hermite wavelets as follows:

$$h_{nm}(t) = \begin{cases} 2^{\frac{k}{2}} H_m^*(2^{k+1}t - 2l + 1) & t \in \left[\frac{l-1}{2^k}, \frac{l}{2^k}\right] \\ 0 & o.w \end{cases} \quad \ldots \quad (6)$$

where

$$H_m^* = \frac{1}{2^m l! \sqrt{\pi}} H_m \quad \ldots (7)$$

m=0,1,2,...,M-1 $\quad l = 0,1,2, \ldots, 2k$

we should note that Hermite wavelets are orthonormal set with respect to the weight function

$$W_k^*(t) = \begin{cases} W_{1,k}(t) & 0 \leq t < \frac{1}{2^k} \\ W_{2,k}(t) & \frac{1}{2^k} \leq t < \frac{2}{2^k} \\ \vdots & \vdots \\ W_{2^k,k}(t) & \frac{2^k - 1}{2^k} \leq t < 1 \end{cases} \quad \ldots \quad (8)$$

where $W_{l,k} = W(2^{k-1}t - l + 1)$.

Hermit wavelets method for VIDE with mth order:

In this section the introduced Hermite wavelets will be applied to solve VIDE with mth order,

$$u_i^{(n)}(x) = g_i(x) + \int_0^x K_{i,j}(x,t) u_i^{(s)}(t) dt, n \geq s \quad \ldots(9)$$

With the following conditions

$u_i^s(0) = a_{is}$ $\quad i = 1,2, \ldots, l \quad s = 0,1,2, \ldots, n-1$

Afunction $u_i^n(x)$ which is defined on the interval $x \in [0,1]$ can be expanded into the Hermite wavelet series

$$u_i^n(x) = \sum_{i=1}^{M} c_i h_i(t) \quad \ldots(10)$$

Where c_i are the wavelet coefficients.

Integrate eq.(10) m times,yields

$$u(x) = \sum_{i=0}^{M} c_i \int_0^x \ldots \int_0^x h_i(t) dt + \sum_{j=0}^{m-1} \frac{x^j}{j!} a_{m-j} \quad \ldots (11)$$

Using the following formula

$$\int_0^x \ldots \int_0^x h_i(t) dt = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} h_i(t) dt$$

therefore eq.(11) becomes

$$u(x) = \sum_{i=0}^{M} c_i \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} h_i(t) dt + \sum_{j=0}^{n-1} \frac{x^j}{j!} a_{n-j} \quad \ldots (12)$$

Let $K_n(x,t) = \frac{(x-t)^{n-1}}{(n-1)!}$ and $L_i^n = \int_0^x K_n(x,t) h_i(t) dt$

i=0,1,…,M

This leads to

$$u(x) = \sum_{i=0}^{M} c_i L_i^n + \sum_{j=0}^{n-1} \frac{x^j}{j!} a_{n-j}$$

In similar way, we can get

$$u^{(s)}(x) = \sum_{i=0}^{M} c_i L_i^{n-s} + \sum_{j=0}^{n-s-1} \frac{x^j}{j!} a_{n-s-j} \ldots (13)$$

Substituting eqs (11) and (13) in (9), yield

$$\sum_{i=1}^{M} c_i h_i(t) = g_i(x) + \int_0^x K_{i,j}(x,t) \left[\sum_{i=0}^{M} c_i L_i^{n-s} + \sum_{j=0}^{n-s-1} \frac{x^j}{j!} a_{n-s-j} \right] dt \ldots \quad (14)$$

or

$\sum_{i=1}^{M} c_i h_i(t) - A_i(x) = g_i(x) + \sum_{j=0}^{n-s-1} \frac{a_{n-s-j}}{j!} B_j(x)$ …(15)

where $A_i(x) = \int_0^x K_n(x,t) L_i^{n-s}(t)dt$ i=0,1,2,…,M

$B_j(x) = \int_0^x K_n(x,t) t^j dt$ j=0,1,2,…,n-s-1 … (16)

Next the interval $x \in [0,1]$ is devided in to l $\Delta x = \frac{1}{l}$ and introduce the collocation points $x_k = \frac{k-1}{l}$, k=1,2,…,l eq(19) is satisfied only at the collocation points we get asystem of linear equations

$\sum_{i=1}^{M} c_i [h_i(x) - A_i(x)] = g_i(x) + \sum_{j=0}^{n-s-1} \frac{a_{n-s-j}}{j!} B_j(x)$ … (17)

The matrix form of this system is C F=G+ $\sum_{j=0}^{n-s-1} \frac{a_{n-s-j}}{j!} B_j(x)$ where F=h(x), G=g(x)

1.Design of the matrix A:-

When Hermite wavelets are integrated m times, the following integral must be evaluated.

$L_i^n = \int_0^x K_n(x,t) h_i(t)dt$, i=0, 1, 2, …, M

$$L_i^n(x) = \frac{(x-t)^n}{2^k(n-1)!} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 & \vdots & 1 & 0 & \dots & 0 \\ \frac{-1}{8} & 0 & \frac{1}{2} & \dots & 0 & \vdots & 0 & 0 & 0 & 0 \\ \frac{-1}{24} & 0 & 0 & \dots & 0 & \vdots & \frac{-1}{3} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \frac{1}{2} & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{-1}{M2^M} & 0 & 0 & \dots & 0 & \vdots & -\frac{1}{M} & 0 & \dots & 0 \end{bmatrix} \qquad \frac{l-1}{2^k} \le x < \frac{l}{2^k}$$

Therefore the matrix $A_i(x)$ can be constructed as follows

Since $A_i(x) = \int_0^x K_n(x,t) L_i^{n-s}(t)dt$ i=0,1,2,…,M

$$A_i(x) = \begin{bmatrix} \int_0^{x_0} K_n(x_0,t) L_i^{n-s}(t)dt & i=0 \\ \int_0^{x_n} K_n(x_i,t) L_i^{n-s}(t)dt & i>0 \end{bmatrix}$$

2. Hermite Wavelets Method for VIDE with nth Order:

For solving VIDE with mth order the matrix $L_i^n(x)$ in section(4.1) will be followed to get

$\sum_{i=1}^{M} c_i [h_i(x_L - A_L)] = g(x_L) + \sum_{j=0}^{n-s-1} \frac{a_{n-s-j}}{j!} B_j(x_L)$ $L \in [a,b]$

But $A_i(x_L) = \int_0^{x_L} K_n(x_L,t) L_i^{n-s}(t)dt$ where i=0,…,M

$B_j(x_L) = \int_0^{x_L} K_n(x_L,t) t^{n-s}dt$ where $L_i^{n-s}(t)$ as in eq(17),(18) that is $A_i(x_L) = A_L$, $F_i(x_L) = h_i(x_L) - A_i(x_L) = F_L$

Numerical Results:

In this section VIDE is considered and solved by the introduced method. parameters k and M are considered to be 1 and 3 respectively.

Example 1: Consider the following VIDE:

$U''(x) = e^{2x} - \int_0^x e^{2(x-t)} U'(t)dt$

Initial conditions $U(0) = 0, U'(0) = 0$.

The exact solution $U(x) = xe^x - e^x + 1$. Table 1 shows the numerical results for this example with k=1, M=3 with error $=10^{-3}$ and k=1, M=4, with error $=10^{-4}$ are compared with exact solution graphically in fig.

Table 1:some numerical results for example 1

x	Exact solution	Approximat solution k=1,M=3	Approximat solution k=1,M=4
0	0.00000000	0.00000001	0.00000001
0.2	0.02287779	0.02280000	0.02287000
0.4	0.10940518	0.10945544	0.10940544
0.6	0.27115248	0.25826756	0.27826756
0.8	0.55489181	0.54330957	0.55330957
1	1.00000000	0.99999995	0.99999998

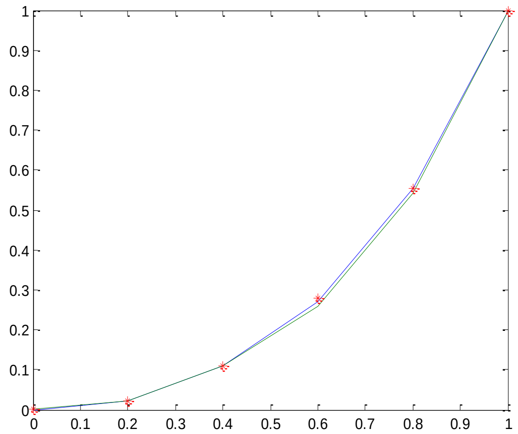

Fig 1:Approximate solution for example 1

Example 2: Consider the following VIDE :

$U^{(5)}(x) = -2\sin x + 2\cos x - x + \int_0^x (x-t)U^{(3)}(t)dt$

Initial conditions $U(0) = 1, U'(0) = 0, U''(0) = -1, U^3(0) = 0, U^3(0) = 1$,.

The exact solution $U(x) = \cos x$. Table 2 shows the numerical results for this example with k=1, M=3 with error$=10^{-3}$ and k=1, M=4, with error $=10^{-4}$ are compared with exact solution graphically in fig, 2.

Table 2:some numerical results for example 2

x	Exact solution	Approximat solution k=1,M=3	Approximat solution k=1,M=4
0	1.00000000	0.99812235	0.99999875
0.2	0.98006658	0.98024711	0.98005541
0.4	0.92106099	0.92158990	0.92104326
0.6	0.82533561	0.82479820	0.82535367
0.8	0.69670671	0.69689632	0.69678976
1	0.54030231	0.54032879	0.54035879

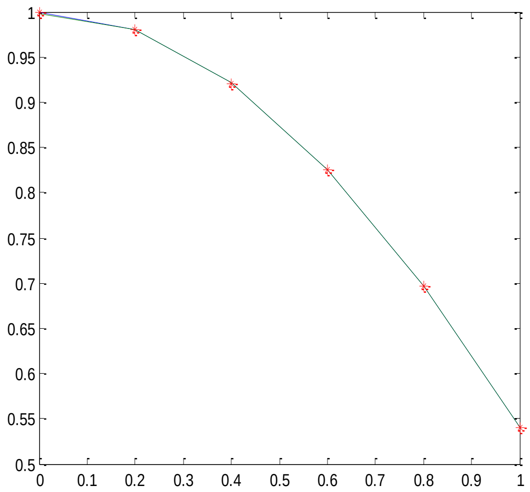

Fig 2:Approximate solution for example

Conclusion:

In this work, VIDE has been solved by using Hermite wavelets in collocation method. Comparison of the approximate solutions and the exact solutions shows that the proposed method is efficient tool. Illustrative examples are included to demonstrate the validity and applicability of the technique.

References

1. Shihab. S. N. and Mohammed. A. 2012. An Efficient Algorithm for n^{th}Order Integro- Differential Equations Using New Haar Wavelets Matrix Designation, International Journal of Emerging. Technologies in Computational and Applied Sciences (IJETCAS). 12(209): 32-35.

2. Elayaraja. A. and Jumat. S. 2010. Numerical Solution of Second-Order Linear Fredholm Integro-Differential Equation Using Generalized Minimal Residual Method, (AJAppSci). 7(6):780-783.

3. Branch. A. and Azad. I. 2011. Numerical Solution of Integral Equations with Legendre Basis, Int. J. Contemp. Math. Sciences. 6(23):1131-1135.

4. Shihab. S. N. and Abdalelah. A. 2012. Numerical Solution of Calculus of Variations by using the Second Chebyshev Wavelets, Eng. & Tech. Journal. 30(18): 3219-3229.

5. Fariborzi. M. A. and Daliri. S. 2012. Numerical Solution of Integro-Differential Equation by using Chebyshev Wavelets Operational Matrix of Integration, Int. J. Math. Mod & Comput. 2(2):127-136.

6. Kajani. M. and hasem. M. 2006. Numerical Solution of Linear Integro Differential Equation by Using Sine-Cosine Wavelets, Appl. Math. Comput.1(8): 569-574.

7. Rahbar. S. 2007. A Numerical Solution to the Linear and nonlinear Fredholm integral equations using Legendre Wavelet functions, PAMM. Proc. Appl.Math. Mech. 7(71): 2020149-2020150.

8. Tao. X. and Yuan. L. 2012. Numerical Solution of Fredholm Integral Equation of Second kind by General Legendre Wavelets, Int. J. Inn. Comp and Cont. 8(1): 799-805.

9. Habibullah. G. M. 2013. A Generalization of Hermite Polynomials, Int. Math Forum. 8(15): 701-706.

خوارزمية لحل المعادلات التكاملية التفاضلية من الرتبة n باستخدام هرمت الموجية

أسماء عبد الاله عبد الرحمن*

*الجامعة التكنولوجية، قسم العلوم التطبيقية

الخلاصة:

في هذا البحث تم بناء دوال هرمت الموجية ومصفوفة العمليات للتكاملات ومن ثم تم تطبيقها في حل المعادلات التكاملية التفاضلية من نوع فولتيرا من الرتبة n التي تم تطبيقها في بعض الامثلة.