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## Accurate Four-Step Hybrid Block Method for Solving Higher-Order Initial Value Problems

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### Abstract:

This paper focuses on developing a self-starting numerical approach that can be used for direct integration of higher-order initial value problems of Ordinary Differential Equations. The method is derived from power series approximation with the resulting equations discretized at the selected grid and off-grid points. The method is applied in a block-by-block approach as a numerical integrator of higher-order initial value problems. The basic properties of the block method are investigated to authenticate its performance and then implemented with some tested experiments to validate the accuracy and convergence of the method.

**Keywords:** Accuracy, Block Method, Convergence, Higher-Order, Initial Value Problems (IVPs), Ordinary Differential Equations, **AMS Group:** 65L05, 65L06, 65L10.

### Introduction:

The article considers higher-order initial value problems of the form:

$$y^{(\mu)} = f(x, y, y^{(1)}, \dots, y^{(\mu-1)}), y(a) = \eta_0, y^{(1)}(a) = \eta_1, \dots, y^{(\mu-1)}(a) = \eta_{\mu-1} \dots (1)$$

Equation (1) applies in applied science and engineering to model problems with linear and nonlinear IVPs. Most of the resulting problems of these phenomena have no analytical solutions. Thus, the quest is to provide numerical methods as numerical integrators to the resulting analytical solutions.

A computational solution to various differential equations creates the need to develop methods to solve such equations. These approaches provide more novel behaviour to the field of science and engineering. The authors considered developing and using the mathematical functions and terms to model phenomena. In the pursuit of using these concepts such as Hilbert Space, Fredholm operators, to solve fractional, Bagley–Torvik and Painlevé, Riccati and Bernoulli, Volterra integro-differential equations within the Atangana–Baleanu fractional approach<sup>1-4</sup>. The developed numerical methods were a powerful tool to generate virtuous universally charming numerical solutions proficient in treating various fractional equations. This has

helped in scientific forecasts through mathematical modelling and numerical simulations<sup>1-4</sup>

Conventionally, higher-order differential equations are reduced to a system of first-order equations, which are solved by an appropriate numerical method for first-order. However, this approach has some shortcomings as it requires evaluation of too many functions and much computational effort<sup>5-6</sup>. Therefore,<sup>7-8</sup> developed block methods for the direct integration of higher-order to reduce the computational burden experienced in the reduction approach with better accuracy and a lower convergence rate.

The block approach is a method that generates approximations at the different grid and off-grid points within the proposed space of integration without overlying the sub-intervals to overcome the difficulties encountered in the reduction process and the predictor-corrector approach as discussed by<sup>(9-16)</sup>. Likewise,<sup>17</sup> developed a block method with generalized equidistant points, which allow varying the choice of the points to produce a five-step block method. The resulting family of implicit schemes considered a general block form with the incorporation of Taylor series expansion to produce solutions to fourth-order initial value problems at all grid points.

The good of the block method is that the solution will be assessed at more than one point at each use of the method. The points vary on the construction of the block method, which improves the accuracy and efficiency of the method (18). In a recent publication, (19) developed a method that is capable of handling two different orders of differential equations. This was used to solve second and third-order differential equations.

Thus, our intension in this paper is to propose a four-step hybrid block method that can directly integrate three different higher-order (third, fourth and fifth-orders) ordinary differential equations to solve initial value problems with better accuracy when compared to existing methods and validated with their basic properties.

### Derivation of the Method

Consider the power series approximation of the form:

$$\sum_{j=1}^3 \alpha_j(t) y_{n+j} = h^3 \left( \beta_0(t) f_n + \beta_1(t) f_{n+\frac{1}{2}} + \beta_3(t) f_{n+\frac{3}{2}} + \beta_5(t) f_{n+\frac{5}{2}} + \beta_7(t) f_{n+\frac{7}{2}} \right) + h^4 (\gamma_0(t) g_n + \gamma_4(t) g_{n+4}) + h^5 (\omega_0(t) m_n + \omega_4(t) m_{n+4}) \quad (3)$$

where  $f$ ,  $g$  and  $m$  are the third, fourth and fifth derivatives of (2), respectively.  
For,

$$y(x) = \sum_{j=0}^{\tau+\mu} a_j x^j \quad (2)$$

as a solution to equation (1), where  $\tau$  and  $\mu$  are the collocation and interpolation points respectively. Interpolating equation (2) at  $x_{n+j} = x_n + jh$ ,  $j = 1, 2, 3$ , the third derivative of equation (2) is collocated at  $x_{n+j} = x_n + jh$ ,  $j = \left(0, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, 4\right)$ . Collocating the fourth and fifth derivative of equation (2) at  $x_{n+j} = x_n + jh$ ,  $j = (0, 4)$  which is the starting and the endpoints of the system to give a system of a linear equation.

The resulting equations are solved with Gaussian elimination to get the value of the unknown variables  $a$ 's, which are inserted into equation (2) to produce a continuous scheme of type:

$$t = \frac{x - x_n + 3h}{h}$$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} (t) = \begin{pmatrix} -3 & \frac{1}{2} & \frac{1}{2} \\ 3 & -2 & -1 \\ 0 & \frac{3}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} t^0 \\ t^1 \\ t^2 \end{pmatrix}, \quad \begin{pmatrix} \beta_0 \\ \beta_{\frac{1}{2}} \\ \beta_{\frac{3}{2}} \\ \beta_{\frac{5}{2}} \\ \beta_{\frac{7}{2}} \\ \beta_4 \end{pmatrix} (t) = \frac{h^3 (3+t)}{32859509760000} [A] \begin{pmatrix} t^0 \\ t^1 \\ t^2 \\ t^3 \\ t^4 \\ t^5 \\ t^6 \\ t^7 \\ t^8 \\ t^9 \\ t^{10} \\ t^{11} \end{pmatrix}$$

$$A = \begin{bmatrix} 1080780148756 & 2928651264000 & 7320132886528 & 2093767008256 & 1161667584000 & 853066493972 \\ -1060627332782 & 1905103872000 & 1217198710784 & 2073401171968 & 257025024000 & 245299102034 \\ -33431958711 & 105255936000 & 1423172395008 & 3470647025664 & -1010009088000 & -565931443383 \\ 463333554837 & 539716608000 & 766475329536 & 280628748288 & -2537339904000 & -1566618879339 \\ -302496158169 & 371506176000 & 372325588992 & 549043052544 & 79377408000 & 325104889383 \\ -127132487157 & 163565568000 & 6355894272 & 212222803968 & 950704128000 & 768558510603 \\ 189137726259 & 243993600000 & 112678035456 & 101264400384 & 304128000 & -43138110669 \\ -1310201577 & 6110208000 & 19447013376 & 49641725952 & -230224896000 & -195230177001 \\ -42409673096 & 52706304000 & 19326189568 & 18379268096 & -31223808000 & -21874098568 \\ -99200248 & 419328000 & 1780043776 & 429674928 & 25191936000 & 22155776776 \\ 4397647072 & 5419008000 & 1966899200 & 2281603072 & 8515584000 & 7179519200 \\ 6431755904 & 774144000 & 236027904 & 236027904 & 774144000 & 643175904 \end{bmatrix}$$

$$\begin{bmatrix} \gamma_0 \\ \gamma_4 \end{bmatrix}(t) = \frac{h^4(3+t)}{1564738560000} \begin{bmatrix} 1444788884 & 1391482708 \\ -1646447998 & 448682626 \\ -79976079 & -810528687 \\ 801294093 & -2403001971 \\ -517250241 & 625470687 \\ -220403373 & 1196963667 \\ 321775851 & -166900341 \\ 1690047 & -349892289 \\ -73466344 & -29034152 \\ -892472 & 42488264 \\ 7709408 & 13132000 \\ 1157856 & 1157856 \end{bmatrix} \begin{bmatrix} t^0 \\ t^1 \\ t^2 \\ t^3 \\ t^4 \\ t^5 \\ t^6 \\ t^7 \\ t^8 \\ t^9 \\ t^{10} \\ t^{11} \end{bmatrix}, \begin{bmatrix} \omega_0 \\ \omega_4 \end{bmatrix}(t) = \frac{h^5(3+t)}{745113600} \begin{bmatrix} 545468 & 698236 \\ -782746 & 256102 \\ -52893 & -345789 \\ 933431 & -1132137 \\ -276147 & 356589 \\ -120471 & 563049 \\ 171057 & -133167 \\ 3789 & -181683 \\ -39928 & -2744 \\ -1064 & 27608 \\ 4256 & 7840 \\ 672 & 672 \end{bmatrix} \begin{bmatrix} t^0 \\ t^1 \\ t^2 \\ t^3 \\ t^4 \\ t^5 \\ t^6 \\ t^7 \\ t^8 \\ t^9 \\ t^{10} \\ t^{11} \end{bmatrix}$$

Evaluating equation (3) at the non-interpolation points give discrete schemes of the form:

$$\begin{bmatrix} y_{n+4} \\ y_{n+\frac{7}{2}} \\ y_{n+\frac{5}{2}} \\ y_{n+\frac{3}{2}} \\ y_{n+\frac{1}{2}} \\ y_n \end{bmatrix} - (B) \begin{bmatrix} y_{n+3} \\ y_{n+2} \\ y_{n+1} \end{bmatrix} = h^5 (C) \begin{bmatrix} m_{n+4} \\ m_n \end{bmatrix} + h^4 (D) \begin{bmatrix} g_{n+4} \\ g_n \end{bmatrix} + h^3 (E) \begin{bmatrix} f_{n+4} \\ f_{n+\frac{7}{2}} \\ f_{n+\frac{5}{2}} \\ f_{n+\frac{3}{2}} \\ f_{n+\frac{1}{2}} \\ f_n \end{bmatrix} \quad (4)$$

where,

$$B = \begin{bmatrix} 3 & -3 & 1 \\ 15 & 5 & 3 \\ 8 & 4 & 8 \\ 3 & 3 & -1 \\ 8 & 4 & 8 \\ 1 & 3 & 3 \\ 8 & 8 & 4 \\ 3 & 15 & -5 \\ 8 & 8 & 4 \\ 1 & -3 & 3 \end{bmatrix}, C = \begin{bmatrix} \frac{78101100}{35562240000} & \frac{99974700}{35562240000} \\ \frac{1240753500}{7283146752000} & \frac{9954296100}{7283146752000} \\ \frac{25493901300}{36415733760000} & \frac{24909929100}{36415733760000} \\ \frac{36415733760000}{24909929100} & \frac{36415733760000}{25493901300} \\ \frac{36415733760000}{9954296100} & \frac{36415733760000}{1240753500} \\ \frac{7283146752000}{99974700} & \frac{7283146752000}{78101100} \\ \frac{35562240000}{35562240000} & \frac{35562240000}{35562240000} \end{bmatrix}, D = \begin{bmatrix} \frac{985083330}{35562240000} & \frac{948738210}{35562240000} \\ \frac{1424512950}{7283146752000} & \frac{93039543030}{7283146752000} \\ \frac{234627724590}{36415733760000} & \frac{234627724590}{36415733760000} \\ \frac{36415733760000}{228890867730} & \frac{36415733760000}{234627724590} \\ \frac{36415733760000}{93039543030} & \frac{36415733760000}{1424512950} \\ \frac{7283146752000}{948738210} & \frac{7283146752000}{985083330} \\ \frac{35562240000}{35562240000} & \frac{35562240000}{35562240000} \end{bmatrix}$$

$$E = \begin{bmatrix} \frac{3509026457}{35562240000} & \frac{9508608000}{35562240000} & \frac{23766665216}{35562240000} & \frac{6797944832}{35562240000} & \frac{3771648000}{35562240000} & \frac{3771648000}{35562240000} \\ \frac{39324419755}{7283146752000} & \frac{188037504000}{7283146752000} & \frac{1661551295488}{7283146752000} & \frac{5578068042880}{7283146752000} & \frac{3608513280}{7283146752000} & \frac{268763465387}{7283146752000} \\ \frac{67016488711}{36415733760000} & \frac{884610432000}{36415733760000} & \frac{1639532561408}{36415733760000} & \frac{1059985009664}{36415733760000} & \frac{8622593280000}{36415733760000} & \frac{653170860217}{36415733760000} \\ \frac{653170860217}{36415733760000} & \frac{862259328000}{36415733760000} & \frac{1059985009664}{36415733760000} & \frac{1639532561408}{36415733760000} & \frac{884610432000}{36415733760000} & \frac{67016488711}{36415733760000} \\ \frac{268763465387}{7283146752000} & \frac{360851328000}{7283146752000} & \frac{557806842880}{7283146752000} & \frac{1661551295488}{7283146752000} & \frac{188037504000}{7283146752000} & \frac{39324419755}{7283146752000} \\ \frac{7283146752000}{2769696409} & \frac{7283146752000}{3771648000} & \frac{7283146752000}{6797944832} & \frac{7283146752000}{3766665216} & \frac{7283146752000}{9508608000} & \frac{7283146752000}{3509026457} \\ \frac{35562240000}{35562240000} & \frac{35562240000}{35562240000} & \frac{35562240000}{35562240000} & \frac{35562240000}{35562240000} & \frac{35562240000}{35562240000} & \frac{35562240000}{35562240000} \end{bmatrix}$$

Using matrix inversion, the combination of the discrete schemes in (4) produces the block as follows:

$$[\hat{A}] \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \\ y_{n+\frac{5}{2}} \\ y_{n+3} \\ y_{n+\frac{7}{2}} \\ y_{n+4} \end{bmatrix} = [\hat{B}] \begin{bmatrix} y_{n-\frac{1}{2}} \\ y_{n-1} \\ y_{n-\frac{3}{2}} \\ y_{n-2} \\ y_{n-\frac{5}{2}} \\ y_{n-3} \\ y_{n-\frac{7}{2}} \\ y_n \end{bmatrix} + h[\hat{C}] \begin{bmatrix} y'_{n-\frac{1}{2}} \\ y'_{n-1} \\ y'_{n-\frac{3}{2}} \\ y'_{n-2} \\ y'_{n-\frac{5}{2}} \\ y'_{n-3} \\ y'_{n-\frac{7}{2}} \\ y'_n \end{bmatrix} + h^2[\hat{D}] \begin{bmatrix} y''_{n-\frac{1}{2}} \\ y''_{n-1} \\ y''_{n-\frac{3}{2}} \\ y''_{n-2} \\ y''_{n-\frac{5}{2}} \\ y''_{n-3} \\ y''_{n-\frac{7}{2}} \\ y''_n \end{bmatrix} + h^3[\hat{E}] \begin{bmatrix} f_{n-\frac{1}{2}} \\ f_{n-1} \\ f_{n-\frac{3}{2}} \\ f_{n-2} \\ f_{n-\frac{5}{2}} \\ f_{n-3} \\ f_{n-\frac{7}{2}} \\ f_n \end{bmatrix} + h^3[\hat{F}] \begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \\ f_{n+\frac{7}{2}} \\ f_n \end{bmatrix} + h^4[\hat{G}] \begin{bmatrix} g_{n-\frac{1}{2}} \\ g_{n-1} \\ g_{n-\frac{3}{2}} \\ g_{n-2} \\ g_{n-\frac{5}{2}} \\ g_{n-3} \\ g_{n-\frac{7}{2}} \\ g_n \end{bmatrix} + h^4[\hat{H}] \begin{bmatrix} g_{n+\frac{1}{2}} \\ g_{n+1} \\ g_{n+\frac{3}{2}} \\ g_{n+2} \\ g_{n+\frac{5}{2}} \\ g_{n+3} \\ g_{n+\frac{7}{2}} \\ g_{n+4} \end{bmatrix} + h^5[\hat{I}] \begin{bmatrix} m_{n-\frac{1}{2}} \\ m_{n-1} \\ m_{n-\frac{3}{2}} \\ m_{n-2} \\ m_{n-\frac{5}{2}} \\ m_{n-3} \\ m_{n-\frac{7}{2}} \\ m_n \end{bmatrix} + h^5[\hat{J}] \begin{bmatrix} m_{n+\frac{1}{2}} \\ m_{n+1} \\ m_{n+\frac{3}{2}} \\ m_{n+2} \\ m_{n+\frac{5}{2}} \\ m_{n+3} \\ m_{n+\frac{7}{2}} \\ m_{n+4} \end{bmatrix} \quad (5)$$

The first and second derivatives of the block gives the following:

$$[\hat{A}] \begin{bmatrix} y'_{n+\frac{1}{2}} \\ y'_{n+1} \\ y'_{n+\frac{3}{2}} \\ y'_{n+2} \\ y'_{n+\frac{5}{2}} \\ y'_{n+3} \\ y'_{n+\frac{7}{2}} \\ y'_{n+4} \end{bmatrix} = [\hat{B}] \begin{bmatrix} y'_{n-\frac{1}{2}} \\ y'_{n-1} \\ y'_{n-\frac{3}{2}} \\ y'_{n-2} \\ y'_{n-\frac{5}{2}} \\ y'_{n-3} \\ y'_{n-\frac{7}{2}} \\ y'_n \end{bmatrix} + h[\hat{C}] \begin{bmatrix} y''_{n-\frac{1}{2}} \\ y''_{n-1} \\ y''_{n-\frac{3}{2}} \\ y''_{n-2} \\ y''_{n-\frac{5}{2}} \\ y''_{n-3} \\ y''_{n-\frac{7}{2}} \\ y''_n \end{bmatrix} + h^2[\hat{D}] \begin{bmatrix} f_{n-\frac{1}{2}} \\ f_{n-1} \\ f_{n-\frac{3}{2}} \\ f_{n-2} \\ f_{n-\frac{5}{2}} \\ f_{n-3} \\ f_{n-\frac{7}{2}} \\ f_n \end{bmatrix} + h^2[\hat{E}] \begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \\ f_{n+\frac{7}{2}} \\ f_n \end{bmatrix} + h^3[\hat{F}] \begin{bmatrix} g_{n-\frac{1}{2}} \\ g_{n-1} \\ g_{n-\frac{3}{2}} \\ g_{n-2} \\ g_{n-\frac{5}{2}} \\ g_{n-3} \\ g_{n-\frac{7}{2}} \\ g_n \end{bmatrix} + h^3[\hat{G}] \begin{bmatrix} g_{n+\frac{1}{2}} \\ g_{n+1} \\ g_{n+\frac{3}{2}} \\ g_{n+2} \\ g_{n+\frac{5}{2}} \\ g_{n+3} \\ g_{n+\frac{7}{2}} \\ g_{n+4} \end{bmatrix} + h^4[\hat{H}] \begin{bmatrix} m_{n-\frac{1}{2}} \\ m_{n-1} \\ m_{n-\frac{3}{2}} \\ m_{n-2} \\ m_{n-\frac{5}{2}} \\ m_{n-3} \\ m_{n-\frac{7}{2}} \\ m_n \end{bmatrix} + h^4[\hat{I}] \begin{bmatrix} m_{n+\frac{1}{2}} \\ m_{n+1} \\ m_{n+\frac{3}{2}} \\ m_{n+2} \\ m_{n+\frac{5}{2}} \\ m_{n+3} \\ m_{n+\frac{7}{2}} \\ m_{n+4} \end{bmatrix} \quad (6)$$

$$[\hat{A}] \begin{bmatrix} y''_{n+\frac{1}{2}} \\ y''_{n+1} \\ y''_{n+\frac{3}{2}} \\ y''_{n+2} \\ y''_{n+\frac{5}{2}} \\ y''_{n+3} \\ y''_{n+\frac{7}{2}} \\ y''_{n+4} \end{bmatrix} = [\hat{B}] \begin{bmatrix} y''_{n-\frac{1}{2}} \\ y''_{n-1} \\ y''_{n-\frac{3}{2}} \\ y''_{n-2} \\ y''_{n-\frac{5}{2}} \\ y''_{n-3} \\ y''_{n-\frac{7}{2}} \\ y''_n \end{bmatrix} + h[\hat{C}] \begin{bmatrix} f_{n-\frac{1}{2}} \\ f_{n-1} \\ f_{n-\frac{3}{2}} \\ f_{n-2} \\ f_{n-\frac{5}{2}} \\ f_{n-3} \\ f_{n-\frac{7}{2}} \\ f_n \end{bmatrix} + h[\hat{D}] \begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \\ f_{n+\frac{7}{2}} \\ f_n \end{bmatrix} + h^2[\hat{E}] \begin{bmatrix} g_{n-\frac{1}{2}} \\ g_{n-1} \\ g_{n-\frac{3}{2}} \\ g_{n-2} \\ g_{n-\frac{5}{2}} \\ g_{n-3} \\ g_{n-\frac{7}{2}} \\ g_n \end{bmatrix} + h^2[\hat{F}] \begin{bmatrix} g_{n+\frac{1}{2}} \\ g_{n+1} \\ g_{n+\frac{3}{2}} \\ g_{n+2} \\ g_{n+\frac{5}{2}} \\ g_{n+3} \\ g_{n+\frac{7}{2}} \\ g_{n+4} \end{bmatrix} + h^3[\hat{G}] \begin{bmatrix} m_{n-\frac{1}{2}} \\ m_{n-1} \\ m_{n-\frac{3}{2}} \\ m_{n-2} \\ m_{n-\frac{5}{2}} \\ m_{n-3} \\ m_{n-\frac{7}{2}} \\ m_n \end{bmatrix} + h^3[\hat{H}] \begin{bmatrix} m_{n+\frac{1}{2}} \\ m_{n+1} \\ m_{n+\frac{3}{2}} \\ m_{n+2} \\ m_{n+\frac{5}{2}} \\ m_{n+3} \\ m_{n+\frac{7}{2}} \\ m_{n+4} \end{bmatrix} \quad (7)$$

Check appendix for the values of the block coefficients

$(\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{E}, \hat{F}, \hat{G}, \hat{H}, \hat{I}, \hat{J}), (\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{E}, \hat{F}, \hat{G}, \hat{H}, \hat{I})$  and  $(\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{E}, \hat{F}, \hat{G}, \hat{H})$  respectively.

### Analysis of the Properties of the Block Method

The properties of the Block Method are analyzed as follows:

### Order and Error Constant

According to (20), our discrete scheme is a Linear Multistep Method (LMM) of the form:

$$\sum_{j=0}^{\tau+\mu} \alpha_j y_{n+j} = h^3 \sum_{j=0}^{\tau+\mu} \beta_j f_{n+j} + h^4 \sum_{j=0}^{\tau+\mu} \gamma_j g_{n+j} + h^5 \sum_{j=0}^{\tau+\mu} \omega_j m_{n+j} \dots (8)$$

Let define the Local Truncation Error (LTE) with equation (8) above as:

$$L[y(x):h] = \sum_{j=0}^{\tau+\mu} [\alpha_j y(x_n + jh) - h^3 \beta_j f(x_n + jh) - h^4 \gamma_j g(x_n + jh) - h^5 \omega_j m(x_n + jh)] \quad (9)$$

where,  $y(x)$  is an arbitrary function, continuously differentiable on  $[a, b]$ . Expanding (4) by Taylor series about the point  $x$  to obtain the expression

$$L[y(x):h] = C_0 y(x) + C_1 h y'(x) + \dots + C_p h^p y^{(p)}(x) + C_{p+1} h^{p+1} y^{(p+1)}(x) + C_{p+2} h^{p+2} y^{(p+2)}(x) + \dots$$

where,  $C_0, C_1, C_2, C_p, \dots, C_{p+2}$  are obtained as

$$C_0 = \sum_{j=0}^{\tau+\mu} \alpha_j, C_1 = \sum_{j=0}^{\tau+\mu} j \alpha_j, C_2 = \frac{1}{2!} \sum_{j=1}^{\tau+\mu} j^2 \alpha_j,$$

⋮

$$C_q = \frac{1}{q!} \left[ \sum_{j=1}^{\tau+\mu} j^q \alpha_j - q(q-1)(q-2) \sum_{j=1}^{\tau+\mu} \beta_j j^{q-3} - q(q-1)(q-2)(q-3) \sum_{j=1}^{\tau+\mu} \gamma_j j^{q-4} - q(q-1)(q-2)(q-3)(q-4) \sum_{j=1}^{\tau+\mu} \omega_j j^{q-5} \right]$$

Thus, the block in equation (4) is of order  $p$  if  $C_0 = C_1 = C_2 = \dots = C_p = C_{p+2} = 0$  ( $C_{p+r} \neq 0$  ( $r$  is the order of the differential equation) called the error constant and  $C_{p+3} h^{p+3} y^{(p+3)}(x_n) + O(h^{p+3})$  is

$$C_{p+3} = \left[ -\frac{39121}{53648179200}, -\frac{26141}{58179453100}, -\frac{17291}{53634509200}, -\frac{22341}{8172494600}, -\frac{23451}{817965200} \right]^T$$

### Zero Stability

A block method is said to be stable if as  $h \rightarrow 0$ , the roots  $r_j, j = 1(1)k$  of the first characteristics polynomial  $\rho(r) = 0$  that

is  $\rho(r) = \det \left[ \sum A^{(i)} r^{k-1} \right] = 0$ , satisfying

$|r| \leq 1$  and for those roots with  $|r| \leq 1$ , having multiplicity equal to unity. Hence,

$$\rho(r) = r \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = 0$$

$$\rho(r) = r^5 (r-1) = 0, r = 0, 0, 0, 0, 1$$

Hence, the block method is zero stable since the root  $|r|$  is simple.

### Consistency

A block method is said to be consistent if it has an order  $p \geq 1$  (20). Hence, the block method is consistent, since  $p = 10$ .

the Local Truncation Error (LTE) at the point  $x_n$ . Thus, the method is of uniform order  $p = [10, 10, 10, 10]^T$  and error constant of

### Convergence of the Block Method

According to (20), a method is said to be convergent, if  $p \geq 1$  and zero stable. Hence, the method is convergent since it satisfies these conditions. It follows that a necessary requirement for a linear multistep method to be convergent is that the solution  $y_n$  generated by the method converges to the exact solution  $y(x)$  as the step length  $h$  tends to zero. Thus, the Order of Accuracy measures the rate of convergence of the numerical approximation produced from the developed method to the exact solution. This is exemplified by the error generated from the computation. In this wise, the method is said to be accurate since the error produced, which is  $E(h) = \|u - u_h\|$  where  $E(h)$  is the error generated,  $u$  is the exact solution and  $u_h$  is the computed solution. (see (21) for details).

### Region of Absolute Stability of the Block Method

The Region of Absolute Stability (RAS) of a linear multistep method is the set of points  $z$  in the complex plane for which the polynomial  $\pi(\zeta; z)$  satisfies the root condition. It follows that if  $z$  is on the boundary of the stability

region, then  $\pi(\zeta; z)$  it must have at least one root  $\zeta_j$  with magnitude exactly equal to 1 (20). This region is established by applying the boundary locus method given as:

$$z(\theta) = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})} = \frac{\rho(r)}{\sigma(r)} \quad 0 \leq \theta \leq 2\pi \quad (10)$$

For  $z(\theta)$  is the polynomial equation,  $\rho(r)$  and  $\sigma(r)$  are the first and second characteristics polynomial with  $r = e^{i\theta} = \cos \theta + i \sin \theta$  respectively. Applying the first and second characteristics polynomials of equation (5) to mean that  $\rho(r)$  are the values of  $y$ 's and  $\sigma(r)$  as the values of

$f$ 's,  $g$ 's and  $m$ 's respectively. Equation (10) is applied to (5) and simplified in the form  $z(\theta) = x(\theta) + iy(\theta)$  for  $x(\theta)$  is the region of periodicity and  $iy(\theta)$  is the imaginary part of the complex plane. The resulting values are used to find the locus of all points, which is mapped in the MATLAB environment to produce the required stability region of the block method, as shown in Fig. 1.

**Definition:** A numerical method is said to be A-stable if its region of absolute stability contains the whole of the left-hand half-plane  $\text{Re } h\lambda < 0$  (see (20) for details).

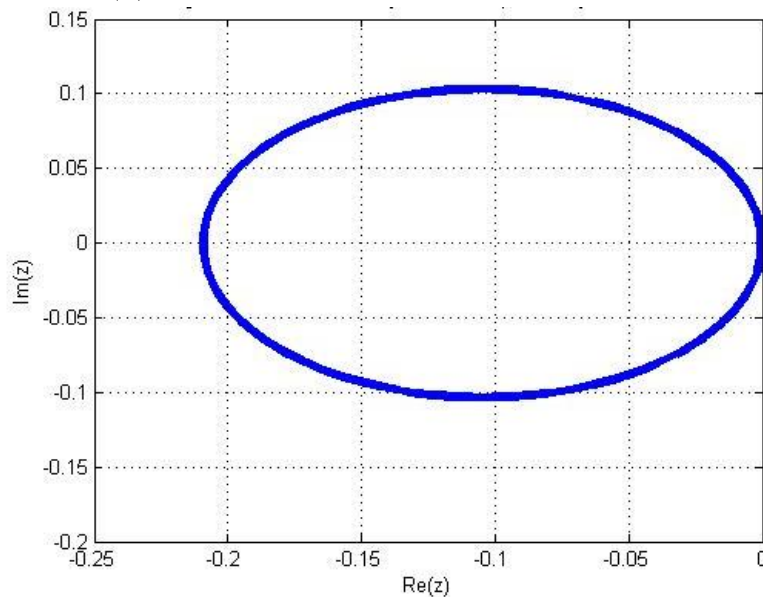


Figure 1. RAS of the four-point Hybrid Block Approach

From the definition, the figure above shows that the method is A-Stable since its area of periodicity lies within  $x(\theta) = (-\infty, 0)$ .

### Numerical Experiments

To validate the proficiency of the developed method, three test experiments were considered. The ATFF Block method was applied to solve the following experiments.

#### Experiment 1

$$y''' + 4y = x, \quad y(0) = y'(0), \quad y''(0) = 1,$$

$$h = 0.1, \quad 0 \leq x \leq 1$$

Theoretical Solution:

$$y(x) = \frac{3}{16}(1 - \cos 2x) + \frac{1}{8}x^2$$

Experiment 1 can be found in (13) and (16).

#### Experiment 2

$$y^{iv} + y'' = 0,$$

$$y(0) = 0, y'(0) = \frac{1.1}{72 - 50\pi}, y''(0) = \frac{1}{144 - 100\pi}, y'''(0) = \frac{1.2}{144 - 100\pi}, h = 0.1$$

$$\text{Theoretical Solution: } y(x) = \frac{1 - x - \cos x - 1.2 \sin x}{144 - 100\pi}$$

Experiment 2 can be found in (5), (9) and (11).

#### Experiment 3

$$y' = 2y'y'' - yy^{iv} - yy''' - 8x + (x^2 - 2x - 3)e^x, \quad x \in [0, 2]$$

$$y(0) = 1, y'(0) = 1, y''(0) = 3, y'''(0) = 1, y^{iv}(0) = 1,$$

$$h = 0.01$$

$$\text{Theoretical Solution: } y(x) = e^x + x^2$$

Experiment 3 can be found in (6) and (15).

Maple 2016 Software Package was used to generate the numerical results of the above experiments with the Algorithm given as follows:

**Algorithm**

Steps of Solutions for Higher-Order Hybrid Block Method for Experiments (1 – 3).

**Stage I**

For  $0 \leq n \leq 50$  set the following:

Subroutine 1:  $h = 0.1$

Subroutine 2:  $y_n = 0$

Subroutine 3:  $y'_n = 0$

Subroutine 4:  $y''_n = 1$

Evaluate  $f_n = h(n) - 4y'_n$ ;  $g_n = h(n) - 4y''_n$  and  $m_n = 4(f_n)$

Output: Display  $f_n$ ,  $g_n$ , and  $m_n$

**Stage II**

Supply the Higher equations for  $f_n$ ,  $g_n$ , and  $m_n$

Input the block variables

Output: Block Value Coefficients

**Stage III**

For  $0 \leq n \leq 50$  perform the following:

1. Set digits = 13
2. Evaluate the block equations
3. Evaluate the block coefficients
4. Evaluate the exact equation

Output:

1. The Computed Solution
2. The Exact Solution
3. Error which is the absolute value of the Computed Solution from the Exact Solution

**Numerical Results and Comparison (Table 1-3)**

**Table 1. Numerical Result for Experiment 1 with  $h = 0.1$**

$x$	Approximate Results	Exact Results	Error in New Method $k = 4, p = 10$	Error in (13) $k = 2, p = 5$	Error in (16) $k = 5, p = 5$
0.1	0.0049875164	0.0049875167	3.000E-10	1.531E-10	2.095E-09
0.2	0.0198012792	0.0198010636	2.156E-10	1.540E-10	1.637E-08
0.3	0.0439999703	0.0439995722	3.981E-10	1.280E-09	1.115E-07
0.4	0.0768682206	0.0768674920	7.286E-09	9.146E-08	9.880E-07
0.5	0.1174479650	0.1174433176	4.647E-09	2.432E-08	3.040E-06
0.6	0.1645678253	0.1645579210	9.904E-09	2.631E-08	9.012E-06
0.7	0.2168984887	0.2168811607	1.732E-08	2.631E-08	1.696E-05
0.8	0.2730015548	0.2729749104	2.664E-08	4.660E-07	2.677 E-05
0.9	0.3313933617	0.3313503928	4.296E-08	4.660E-07	3.813 E-05
1.0	0.3905903268	0.3905275318	6.279E-08	4.660E-07	5.059 E-05

**Table 2. Numerical Result for Experiment 2 with  $h = 0.1$**

$x$	Approximate Results	Exact Results	Error in New Method $k = 4, p = 10$	Error in (5) $k = 7, p = 8$	Error in (9) $k = 1, p = 7$	Error in (11) $k = 6, p = 7$
0.1	0.0000403745631	0.0000403745930	2.988E-19	4.607E-20	0.381E-18	6.505E-19
0.2	0.0000806913409	0.0000806915800	2.391E-19	5.421E-20	0.371E-17	1.301E-18
0.3	0.0001209499397	0.0001209507467	8.070E-18	2.710E-19	0.268E-16	4.770E-18
0.4	0.0001611499662	0.0001611518793	1.913E-18	1.084E-19	0.293E-16	1.734E-17
0.5	0.0002012910276	0.0002012947644	3.736E-17	4.336E-19	0.418E-15	4.336E-17
0.6	0.0002413727325	0.0002413791895	6.456E-17	8.673E-19	0.387E-15	9.540E-17
0.7	0.0002813946880	0.0002814049424	1.025E-17	1.084E-18	0.287E-15	1.812E-16
0.8	0.0003213565051	0.0003213718116	1.530E-16	1.734E-18	0.867E-14	3.157E-16
0.9	0.0003612577930	0.0003612795861	2.179E-16	2.818E-18	0.708E-14	5.186E-16
1.0	0.0004010981618	0.0004011280556	2.989E-16	3.469E-18	0.351E-14	8.049E-16

**Table 3. Numerical Result for Experiment 3 with  $h = 0.01$**

$x$	Approximate Results	Exact Results	Error in New Method $k = 4, p = 10$	Error in [6] $k = 4, p = 9$	Error in [15] $k = 2, p = 5$
0.1	0.01000000000083333334	0.01000000000083333333	1.0000E-19	3.443E-12	1.592 E-12
0.2	0.0200000000266666667	0.0200000000266666667	0.0000E-10	2.619E-12	1.509 E-10
0.3	0.0300000002025000001	0.0300000002025000000	1.0000E-19	3.401E-11	7.517 E-09
0.4	0.0400000008533333335	0.0400000008533333333	2.0000E-19	1.092E-11	1.432 E-08
0.5	0.0500000026041666698	0.0500000026041666667	3.1000E-18	1.611E-11	4.728 E-07
0.6	0.0600000064800000058	0.0600000064800000000	5.8000E-18	1.023E-10	1.962 E-06
0.7	0.0700000140058333425	0.0700000140058333333	9.2000E-18	1.838E-10	4.034 E-06
0.8	0.0800000273066666800	0.0800000273066666667	1.3300E-17	7.059E-10	4.385 E-06
0.9	0.0900000492075000307	0.0900000492075000000	3.0700E-17	1.009E-09	1.732 E-06
1.0	0.1000008333333333330	0.1000008333333333333	4.3000E-17	9.031E-09	4.417 E-04

**Discussion of Results:**

The numerical results displayed in Tables (1 – 3) show the results generated for an experiment (1 – 3) using the proposed method to solve third, fourth, and fifth-order problems. The new method in terms of errors outperforms the errors in the methods of (13) and (16). (13), is a non-hybrid block method with  $k = 2, p = 5$ , while [16] is a hybrid block method with  $k = 5, p = 5$  for solving experiment 1 (see Table 1 for details). In the same vein, the proposed method generates more accurate results when compared to the methods of (9) and (11) as observed in Table 2 for experiment 2. (9), is a hybrid block method with six off-steps and  $k = 1, p = 7$ . Whereas, (11) is a non-hybrid block method of  $k = 6, p = 7$ . Likewise, the better performance of the method compared with (6) which is a non-hybrid block method with  $k = 4, p = 9$  and (15) which is a hybrid block method with  $k = 2, p = 5$  for solving experiment 3 (see Table 3). Despite the performance of the proposed method to (5), as seen in table 2, the proposed method has an advantage over (5) because of its abilities to handle three different orders of differential equations which [5] is incapable of handling. Method (5) was designed to solve only

fourth-order IVPs, but the proposed method is designed to solve the third, fourth, and fifth-order IVPs.

**Conclusion:**

The proposed method, which is a four-point hybrid block approach, was used to integrate higher-order problems with initial values. The convergence and the accuracy of the method, as reflected in the results and errors presented to show the efficiency of the block with enhanced precision when matched with some previous works. Thus, the new method can be considered as novel and a good substitute to solve Higher-Order differential equations with initial values.

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Appendix

$$\hat{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \hat{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \hat{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix},$$

$$\hat{D} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{25}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{49}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix}, \hat{E} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{53501924101099}{2804011499520000} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{498997194119}{4694215680000} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{6839998362027}{34617425920000} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9940652611}{42785820000} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3335394099425}{13459255197696} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{37785934539}{135224320000} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{32941887041}{106168320000} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1305525208}{4011170625} \end{bmatrix},$$

$$\hat{F} = \begin{bmatrix} \frac{1480439}{811345920} & 0 & -\frac{819817}{7983360000} & 0 & \frac{152231}{2661120000} & 0 & -\frac{316649}{2434037760} & \frac{8815314293}{84970045440000} \\ \frac{42821}{713097} & 0 & \frac{3968}{11694375} & 0 & \frac{193}{11694375} & 0 & -\frac{34}{509355} & \frac{1999122767}{32859509760000} \\ \frac{92035521}{270448640} & 0 & \frac{2981547}{98560000} & 0 & \frac{958311}{98560000} & 0 & \frac{5163507}{270448640} & \frac{519797092779}{34617425920000} \\ \frac{223744}{237699} & 0 & \frac{712192}{3898125} & 0 & \frac{51712}{1299375} & 0 & \frac{31232}{396165} & \frac{884705089}{14261940000} \\ \frac{2648921875}{1460422656} & 0 & \frac{21599125}{38320128} & 0 & \frac{1837025}{38320128} & 0 & \frac{193390625}{1460422656} & \frac{1421084262625}{13459255197696} \\ \frac{54918}{18865} & 0 & \frac{58491}{48125} & 0 & \frac{3456}{48125} & 0 & \frac{17253}{132055} & \frac{2093991453}{19317760000} \\ \frac{862645}{202752} & 0 & \frac{2426040029}{1140480000} & 0 & \frac{156826117}{380160000} & 0 & \frac{174587}{1013760} & \frac{51027560857}{389283840000} \\ \frac{1900544}{324135} & 0 & \frac{38469632}{11694375} & 0 & \frac{11730944}{11694375} & 0 & \frac{1507328}{3565485} & \frac{953478776}{4011170625} \end{bmatrix}$$



$$\begin{aligned}
 \dot{i} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 11905 \\ & & & & & & & & & & 317915136 \\ & & & & & & & & & & 22741 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 93139200 \\ & & & & & & & & & & 2253501 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 883097600 \\ & & & & & & & & & & 1103 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 242550 \\ & & & & & & & & & & 2100625 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 953745408 \\ & & & & & & & & & & 81 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 689920 \\ & & & & & & & & & & 218491 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 54067200 \\ & & & & & & & & & & 3232 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 363825 \end{bmatrix} \\
 \dot{j} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 274567 \\ & & & & & & & & & & 63583027200 \\ & & & & & & & & & & 2341 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 745113600 \\ & & & & & & & & & & 4298913 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7064780800 \\ & & & & & & & & & & 2441 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 970200 \\ & & & & & & & & & & 33321875 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7629963264 \\ & & & & & & & & & & 243 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 51200 \\ & & & & & & & & & & 7243817 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1297612800 \\ & & & & & & & & & & 3232 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 363825 \end{bmatrix} \\
 \ddot{A} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
 \ddot{B} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
 \ddot{C} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2310262769827 \\ & & & & & & & & & & 6827950080000 \\ & & & & & & & & & & 505049509 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 53343360000 \\ & & & & & & & & & & 1525396707 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5619712000 \\ & & & & & & & & & & 59720737 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 833490000 \\ & & & & & & & & & & 971830355 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10924720128 \\ & & & & & & & & & & 1817337 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 219520000 \\ & & & & & & & & & & 1355618411 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 19906560000 \\ & & & & & & & & & & 658558 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10418625 \end{bmatrix} \\
 \ddot{D} &= \begin{bmatrix} 652829 & 0 & 1374761 & 0 & 742921 & 0 & 24253 & 33384544093 \\ 3951360 & 0 & 272160000 & 0 & 272160000 & 0 & 3951360 & 6827950080000 \\ 14197 & 0 & 96119 & 0 & 36559 & 0 & 215 & 2930204069 \\ 15435 & 0 & 1063125 & 0 & 1063125 & 0 & 3087 & 53343360000 \\ 12069 & 0 & 108489 & 0 & 22761 & 0 & 85131 & 2493099 \\ 8960 & 0 & 224000 & 0 & 224000 & 0 & 439040 & 16384000 \\ 3328 & 0 & 1000192 & 0 & 64768 & 0 & 256 & 7036097 \\ 3087 & 0 & 1063125 & 0 & 1063125 & 0 & 15435 & 833490000 \\ 685625 & 0 & 480455 & 0 & 225305 & 0 & 225625 & 2274823085 \\ 790272 & 435456 & 435456 & 435456 & 790272 & 10924720128 \\ 243 & 0 & 4533 & 0 & 3987 & 0 & 243 & 2279169 \\ 245 & 0 & 4375 & 0 & 4375 & 0 & 1715 & 31360000 \\ 12299 & 0 & 38840977 & 0 & 39143503 & 0 & 2065 & 7993747349 \\ 11520 & 38880000 & 38880000 & 38880000 & 2304 & 19906560000 \\ 16384 & 0 & 212992 & 0 & 212992 & 0 & 16384 & 658558 \\ 15435 & 0 & 212625 & 0 & 212625 & 0 & 15435 & 10418625 \end{bmatrix} \\
 \ddot{E} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1523148203 \\ & & & & & & & & & & 32514048000 \\ & & & & & & & & & & 13560199 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 254016000 \\ & & & & & & & & & & 11688129 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 80281600 \\ & & & & & & & & & & 302993 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3969000 \\ & & & & & & & & & & 5143225 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 260112384 \\ & & & & & & & & & & 171279 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3136000 \\ & & & & & & & & & & 50558053 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 663552000 \\ & & & & & & & & & & 7384 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 99225 \end{bmatrix} \\
 \ddot{F} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 57755477 \\ & & & & & & & & & & 32514048000 \\ & & & & & & & & & & 5029441 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 254016000 \\ & & & & & & & & & & 626697 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 11468800 \\ & & & & & & & & & & 7633 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3969000 \\ & & & & & & & & & & 18512825 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 260112384 \\ & & & & & & & & & & 9423 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 448000 \\ & & & & & & & & & & 80464027 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 663552000 \\ & & & & & & & & & & 7384 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 99225 \end{bmatrix}
 \end{aligned}$$

$$\ddot{G} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{420401}{154828800} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{8053}{1209600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{18297}{1146880} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{23}{2700} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{14375}{6193152} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{279}{44800} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{191737}{22118400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{8}{945} \end{bmatrix}, \ddot{H} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{31439}{154828800} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2707}{1209600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{7047}{1146880} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{18900} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{6625}{884736} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{81}{44800} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{247303}{22118400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{8}{945} \end{bmatrix}$$

**Authors' declaration:**

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in Federal University.

**Authors' contributions:**

Adeyefa E.O. conceived the idea of the method presented. Olanegan O. O. developed the method and perform analysis on the method. Adeyefa E. O. validated the analysis of the method. Olanegan O. O. conducted some numerical experiments and discussed the result generated from the experiment performed. Olanegan O. O. write the original draft of the manuscript with the supervision of Adeyefa E. O. All authors contributed to the final draft of the manuscript.

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## طريقة البلوك الهجينة الدقيقة ذات الأربع خطوات لحل مسائل القيمة الابتدائية ذات الرتب الأعلى

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### الخلاصة:

يركز هذا البحث على التعامل مع طريقة عددية ذاتية البدء والتي يمكن استخدامها لإيجاد التكامل بشكل مباشر لمسائل القيم الابتدائية للمعادلات التفاضلية الاعتيادية ذات الرتب العالية. هذه الطريقة مشتقة من تقريب متسلسلة القوى مع المعادلات الناتجة التي تم تقديرها في الشبكة المحددة ونقاط خارج الشبكة تم تطبيق الطريقة باستخدام طريقة البلوك كمعامل عددي لمسألة القيمة الابتدائية ذات الرتب الأعلى. تم التحقق من الخصائص الأساسية لطريقة البلوك والتحقق من أدائها ثم تنفيذها ببعض التجارب الاختبارية للتحقق من دقة الطريقة وتقاربها.

الكلمات المفتاحية: الدقة، طريقة الكتلة، التقارب، الرتب الأعلى، مسألة القيمة الابتدائية (IVPs)، المعادلات التفاضلية الاعتيادية.

مجموعة AMS: 65L05, 65L06, 65L10.