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Normal Self-injective Hyperrings

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Abstract:

In this paper normal self-injective hyperrings are introduced and studied. Some new relations between this concept and essential hyperideal, dense hyperideal, and divisible hyperring are studied.

Keywords: Baer's condition, Dense hyperideal, Divisible hyperring, Essential hyperideal, Normal self-injective hyperring.

Introduction.

Hypergroup was firstly defined by Marty in 1934 [1]. Later Krasner introduced the hyperrings (resp. hypermodules), as an extension of rings (resp. modules) named by Krasner hyperrings (resp. Krasner hypermodules) [1]. After that, many others defined other types of hyperrings, for example (multiplication[2], general[2], etc) hyperrings. All hyperrings in this paper are Krasner hyperrings, unless otherwise stated would like to point out that, in this paper, the Krasner hyperring will be denoted by $\mathcal{R}$, the set P*($\mathcal{R}$) ={$\mathcal{A} \subseteq \mathcal{R}.\mathcal{A} \neq \emptyset$} and normal self-injective hypering referred as (NSI). The main objection of this paper is to shed light on the normal self-injective hyperring and find some of its characterizations. Jeshraghani S.H and Ameri R[1] in 2020 defined projective and injective Krasner hypermodules. Then, in 2022 H. Bordbar and I. Cristea defined the normal injective hypermodules[2]. The following definitions were mentioned as simple retrench during the paper.

Definition 1[3] **.** The map $\dotplus: \mathcal{G} \times \mathcal{G} \longrightarrow$ P*($\mathcal{G}$), on a nonempty set $\mathcal{G}$, which is defined as $\dotplus(g_1, g_2) = g_1 \dotplus g_2$ is called "hyperoperation".

Definition 2[3] **.** The pair ($\mathcal{G}$,$\dotplus$) is called "semihypergroup" if for each g_1, g_2 *and* g_3 in $\mathcal{G}$ the following condition holds; $g_1 \dotplus (g_2 \dotplus g_3) = (g_1 \dotplus g_2) \dotplus g_3$. This means $\cup_{g \in g_2 \dotplus g_3} g_1 \dotplus g = \cup_{g' \in g_1 \dotplus g_2} g' \dotplus g_3$. If the semihypergroup satisfies the axiom $x \dotplus \mathcal{G} = \mathcal{G} = \mathcal{G} \dotplus x$, for all $x \in \mathcal{G}$, then it is called a "hypergroup".

Definition 3 [3]**.** The hyperstructure ($\mathcal{G}$,$\dotplus$) is called commutative hyperstructure if, for all $g_1, g_2 \in \mathcal{G}$; $g_1 \dotplus g_2 = g_2 \dotplus g_1$.

Definition 4 [4] **.** A commutative semihypergroup is called a canonical hypergroup if it satisfied the following axioms.

- There exists $0 \in \mathcal{G}$ *such that*, $0 \dotplus g_1 = \{g_1\} = g_1 \dotplus 0$, for each $g_1 \in \mathcal{G}$.
- For each $g_1 \in \mathcal{G}$, there is a unique element $-g_1$ which is the opposite of g_1 and will be denoted as g_1' s.t, $0 \in (g_1 \dotplus g_1')$, where $0 \in \mathcal{G}$. And $g_1 - g_2$ will write as recompense of $g_1 \dotplus (-g_2)$;
- $g_3 \in g_1 \dotplus g_2$ implies $g_1 \in g_3 - g_2$;

Definition 5 [4]**.** The non-empty set $\mathcal{R}$ equipped with the hyperoperation $\dotplus$: $\mathcal{R} \times \mathcal{R} \longrightarrow$P*($\mathcal{R}$), and the multiplication $\cdot : \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}$, such that;

i. ($\mathcal{R}$,$\dotplus$) satisfies the conditions of canonical hypergroup with unite element 0,
ii. ($\mathcal{R}$,$\cdot$) satisfies a condition of the semigroup, and $x.0 = 0 = 0.x$,
iii. $r_1 \cdot (r_2 \dotplus r_3)$ =($r_1 \cdot r_2$)$\dotplus$($r_1 \cdot r_3$). Also ($r_1 \dotplus r_2$)$\cdot r_3$=($r_1 \cdot r_3$)$\dotplus$($r_2 \cdot r_3$), where r_1, r_2 and r_3 are all in $\mathcal{R}$.

It is called "Krasner hyperring". A Krasner hyperring is said to be "commutative" with unite element, if ($\mathcal{R}$,$\cdot$) is a "commutative semigroup" with unite element.

Definition 6 [4]**.** The nonempty subset I of the Krasner hyperring $\mathcal{R}$ is called right (resp. left) hyperideal in $\mathcal{R}$, if the following two conditions are satisfied;

- If $\mathscr{k}, \mathscr{b} \in I$, then $\mathscr{k}$-$\mathscr{b} \subseteq I$;
- If $\mathscr{k} \in I$, $t \in \mathcal{R}$, then $t.\mathscr{k} \in I$, $(resp. \mathscr{k}.t \in I)$.

If I is right and left hyperideal, then it is said to be "hyperideal". Also, a proper hyperideal I of $\mathcal{R}$ is called maximal hyperideal if for any hyperideal J of $\mathcal{R}$ with $I \subsetneq$ J $\subseteq \mathcal{R}$, then J = $\mathcal{R}$.

Definition 7[4]**.** If $(\mathcal{R} \setminus \{0\},\cdot)$ is a group, then the Krasner hyperring $(\mathcal{R},\dotplus,\cdot)$ is called a hyperfeild.

Remarks 1 [5]**.**

1. Any hyperfield is a hyperring.
2. The only hyperideals in any hyperfeild $F$are 0 and F itself.

Definition 8 [2]**.** A non-zero element x in $\mathcal{R}$ where $\mathcal{R}$ is a commutative hyperring is said to be a zero divisor if there is $0 \neq y \in \mathcal{R}$ such that $\cdot\, y = \{0\}$.

Definition 9 [4]**.** A commutative Krasner hyperring $(\mathcal{R},\dotplus,\cdot)$ with unite element is said to be a hyperdomain if whenever $a.\mathscr{b}$ = 0, then either $a = 0$ or $\mathscr{b} = 0$ for each $a, \mathscr{b} \in \mathcal{R}$.

Definition 10 [6]**.** A commutative Krasner hyperring with a unite element is called "principal hyperideal hyprdomain" if any hyperideal of $\mathcal{R}$ is generated by a single element and $\mathcal{R}$ has no zero divisor elements.

Definition 11 [2]**.** The hyperstructuer $(\mathcal{M},+)$, over a hyperring $(\mathcal{R},\dotplus,\cdot)$ with unite element "1" is said to be a "left Krasner hypermodule" over a hyperring $\mathcal{R}$ if $(\mathcal{M},+)$ is a canonical hypergroup with a map $\cdot : \mathcal{R} \times \mathcal{M} \longrightarrow \mathcal{M}$, defined as;

$$\cdot(s,m) \mapsto s.m = sm \in \mathcal{M}$$

such that for all $\mathscr{k}, \mathscr{b} \in \mathcal{R}$ and $m, m` \in \mathcal{M}$, we have

1. $(\mathscr{k} \dotplus \mathscr{b})m = \mathscr{k}m \dotplus \mathscr{b}m$
2. $\mathscr{k}(m + m`)=\mathscr{k}m + \mathscr{k}m`$
3. $(\mathscr{k}\mathscr{b})m=\mathscr{k}(\mathscr{b}m)$
4. $0_{\mathcal{R}}.m = 0_{\mathcal{M}}$, where $0_{\mathcal{R}}$ is a zero of $\mathcal{R}$, $0_{\mathcal{M}}$ the secular identity of $\mathcal{M}$.

And denoted to it by "left $\mathcal{R}$-hypermodule". In the same way one can define the right $\mathcal{R}$-hypermodule. The $\mathcal{R}$-hypermodule $\mathcal{M}$ is called unitary if $1.m=m$, for all $m \in \mathcal{M}$ where 1 is the identity of $\mathcal{R}$.

Definition 12 [2]**.** The function ℓ from the hyperring $(\mathcal{R}_1 \dotplus,\cdot)$ into the hyperring $(\mathcal{R}_2,\dotplus',\cdot')$, with unite elements $1_{\mathcal{R}_1}$, $1_{\mathcal{R}_2}$ respectively is called a hyperring homomorphism if the following hold.

1. For each $\mathscr{k}, \mathscr{b} \in \mathcal{R}_1$, $\ell(\mathscr{k} \dotplus \mathscr{b}) = \ell(\mathscr{k}) \dotplus' \ell(\mathscr{b})$.
2. For each $\mathscr{k}, \mathscr{b} \in \mathcal{R}_1$, $\ell(\mathscr{k}.\mathscr{b})= \ell(\mathscr{k}) \cdot' \ell(\mathscr{b})$.
3. $\ell(1_{\mathcal{R}_1})= 1_{\mathcal{R}_2}$.

Bordbar H and Cristea I. in their paper "Divisible hypermodules" [2] considered $\mathcal{M}$ and $\mathcal{N}$ as an $\mathcal{R}$-hypermodules. The single-value function $\ell: \mathcal{M} \longrightarrow \mathcal{N}$ is called strict $\mathcal{R}$-homomorphism if

1. for each m and m' $\in \mathcal{M}$, $\ell(\text{m}+_{\mathcal{M}}\text{m}') \subseteq \ell(\text{m}) \dotplus_{\mathcal{N}} \ell(\text{m}')$;
2. for each m $\in \mathcal{M}$ and each $r \in \mathcal{R}$, $\ell(r \cdot_{\mathcal{M}} \text{m}) = r \cdot_{\mathcal{N}} \ell(\text{m})$.

And it is called normal $\mathcal{R}$-homomorphism if it satisfies the condition (2) and the following point

3. for each m, m' $\in \mathcal{M}$, $\ell(\text{m}+_{\mathcal{M}}\text{m}') = \ell(\text{m}) +_{\mathcal{N}} \ell(\text{m}')$

Where the set of all normal $\mathcal{R}$-homomrphism from $\mathcal{M}$ into $\mathcal{N}$ is denoted by $Hom_R^n(\mathcal{M},\mathcal{N})$.

Main Results:

Recall that a hypermodule $\mathcal{M}$ over a Krasner hyperring $\mathcal{R}$ is called normal injective if and only if for each hyperideal I of $\mathcal{R}$, the inclusion i : $I \longrightarrow \mathcal{R}$, and a normal $\mathcal{R}$-homomorphism d : $I \longrightarrow \mathcal{M}$, there exists a normal $\mathcal{R}$-homomorphism j : $\mathcal{R} \longrightarrow \mathcal{M}$, such that ji = d [7]

Definition 13. A Krasner hyperring $\mathcal{R}$ is called a "Normal Self-Injective" (NSI) hyperring if $\mathcal{R}$ is a normal injective hypermodule over itself.

In the definition of a normal self-injective Krasner hyperring $\mathcal{R}$, we will use the same definition of normal $\mathcal{R}$-homomorphism which is defined by Bordbar H and Cristea I in their paper "Divisible hypermodules" [2]

Examples 1.

1. Every hyperfield F is (NSI) hyperring, since the only hyperideals of F are 0 and F.
2. $(\mathbb{Z}_p,\dotplus,\cdot)$ is a hyperfield so it is an (NSI) hyperring. The hyperoperation $\dotplus$ defined as $\text{a} \dotplus \text{z} = \{\text{a}, \text{z}, \text{a} + \text{z}\}$.

Definition 14. A hyperring $\mathcal{R}$ is said to be satisfying the Baer's condition if for every family $\mathcal{F}$ of hyperideals in $\mathcal{R}$, for every hyperideal I in $\mathcal{F}$, and every normal homomorphism $f: I \longrightarrow \mathcal{R}$, there exists $r \in \mathcal{R}$ such that $f(\text{x}) = r\text{x}$, for each x $\in I$.

The following proposition shows that normal self-injectivity is equivalent to Baer's condition.

Proposition 1. A hyperring $\mathcal{R}$ with unite element is a normal self-injective hyperring if and only if $\mathcal{R}$ satisfies Baer's condition.

Proof. Let $\mathcal{R}$ be a (NSI) hyperring, I be any hyperideal in $\mathcal{R}$, and ℓ: $I \longrightarrow \mathcal{R}$ be an $\mathcal{R}$-homomorphism. By the concept of (NSI) of $\mathcal{R}$, ℓ can be extended to a normal $\mathcal{R}$-homomorphism g. $\mathcal{R} \longrightarrow \mathcal{R}$. Now $\ell(x)= \ell(1.x)= g(1.x)= g(1).x$. Put r = $g(1)$ thus $\ell(x)$ =rx, $\forall\, x \in I$.

Conversely, suppose that Baer's condition holds for $\mathcal{R}$, and $\ell : I \longrightarrow \mathcal{R}$ is an $\mathcal{R}$-homomorphism, let $\mathcal{H}$ be the set of all pairs (I', ℓ'), for a hyperideal I' of $\mathcal{R}$ with $I \subseteq I'$, $\ell'|_I = \ell$. Firstly, $\mathcal{H} \neq \emptyset$, since $(I, \ell) \in \mathcal{H}$. The elements of $\mathcal{H}$ are partially ordered as $(I', \ell') \leq (I'', \ell'')$ if and only if $I' \subseteq I''$ and $\ell''|_{I'} = \ell'$.

Let $\mathfrak{S} = \{(I_\alpha, \ell_\alpha) : \alpha \in \wedge\}$ be a chain in $\mathcal{H}$. $I_{\mathscr{k}} = U_{\alpha\in\wedge} I_\alpha$ is a hyperideal of $\mathcal{R}$ which contain I [4]. Now $\ell_{\mathscr{k}}: I_{\mathscr{k}} \longrightarrow \mathcal{R}$, is defined as follows, if a $\in I_{\mathscr{k}}$, then a $\in I_\alpha$ for some $\alpha \in \wedge$. Put $\ell_{\mathscr{k}}(\text{a}) = \ell_\alpha(\text{a})$, since $\ell_\beta(\text{a}) = \ell_\alpha(\text{a})$ for all $\alpha \geq \beta$ and a $\in I_{\mathscr{k}}$ so $\ell_{\mathscr{k}}$ is well defined. Hence $(\ell_{\mathscr{k}}, I_{\mathscr{k}})$ is an upper bound of $\mathfrak{S}$. By Zorn's lemma, $\mathcal{H}$ contains maximal element denoted by (J, g). For that: suppose $J = \mathcal{R}$. If not, there is an element y $\in \mathcal{R}$ - J.

Put $C = J + y\mathcal{R}$, C is a hyperideal of $\mathcal{R}$ which properly contains J, consider the hyperideal K ={r ∈ $\mathcal{R}$: yr ∈ J}. Define h . $K \longrightarrow \mathcal{R}$ by h(r) = g(yr) for each r ∈ K. It is clear that h is an $\mathcal{R}$-homomorphism. Hence there is an element t ∈ $\mathcal{R}$ such that $h(r) = g(\text{yr}) = \text{tr}$, for r ∈ K. h is well-defined. Let $g^{'}$: C ⟶ $\mathcal{R}$ be defined as $g^{'}(\text{a} + \text{yr}) = g(\text{a}) + \text{tr}$ for each a + yr ∈ C. If $\text{a} + \text{y}r_1 = \text{b} + \text{y}r_2$, then $\text{y}(r_1 - r_2) = \text{b} - \text{a}$, and b − a ∈ J. Hence $r_1 - r_2 \in K$. Therefore $g(\text{y}(r_1 - r_2)) = \text{t}(r_1 - r_2) = \text{t}r_1 - \text{t}r_2$. But $g(\text{y}(r_1 - r_2)) = g(\text{b} - \text{a}) = g(b) - g(\text{a})$, then $g(\text{a}) + \text{t}r_1 = g(\text{b}) + \text{t}r_2$. Thus $g^{'}$ is well-defined, since $g^{'}$ is an extended to g then $g^{'}$ is an $\mathcal{R}$-homomorphism, this contradiction with the maximality of (J, g). Therefore $\mathcal{R}$ is a normal self-injective hyperring. ■

Definition 15 [2]**.** The element x in the commutative hyperring $\mathcal{R}$ is called divisible if, for any nonzero divisor element say r ∈ $\mathcal{R}$, there is y ∈ $\mathcal{R}$ such that x= r.y.

Definition 16. A commutative hyperring $\mathcal{R}$ is called divisible if every element in $\mathcal{R}$ is divisible, and it can be written as $\mathcal{R}$ =x$\mathcal{R}$.

Proposition 2. If Baer's condition holds for all principal hyperideals of a hyperring $\mathcal{R}$, then $\mathcal{R}$ is divisible.

Proof. Clearly that x$\mathcal{R}$ ⊆ $\mathcal{R}$, where x$\mathcal{R}$ is a hyperideal of $\mathcal{R}$ and (0≠x) ∈ $\mathcal{R}$ is a nonzero divisor. To show that $\mathcal{R}$ ⊆ x$\mathcal{R}$, let y ∈ $\mathcal{R}$, and consider the mapping f: x$\mathcal{R}$ ⟶ $\mathcal{R}$, defined by f(xr) = yr for each xr ∈ x$\mathcal{R}$. If xr = xt, then 0 ∈ x(r − t) thus there is y ∈ (r − t) such that 0 = xy. Since x is a nonzero divisor, so $y = 0$, then 0 ∈ r − t implies t ∈ r + 0 = {t}. Since $\mathcal{R}$ is a canonical hypergroup under +, thus r = t, and hence f is well-defined and f is an $\mathcal{R}$-homomorphism. By hypothesis, there exists an element s ∈ $\mathcal{R}$ such that $f(w) = \text{s}w$, $w \in$ x$\mathcal{R}$. Since $\mathcal{R}$ has a unite element, $y = f(\text{x}) = \text{sx}$ and $y \in$ x$\mathcal{R}$. Hence $\mathcal{R}$ =x$\mathcal{R}$. ■

Theorem 1. Let $\mathcal{R}$ be a principal hyperideal hyperdomain. If $\mathcal{R}$ is divisible, then $\mathcal{R}$ satisfies Baer's condition.

Proof. Let I be any hyperideal of $\mathcal{R}$ and f : I ⟶ $\mathcal{R}$ be an $\mathcal{R}$-homomorphism. If I is the zero hyperideal, then it is done. Consider the case I ≠0. Since $\mathcal{R}$ is the principal hyperideal hyperdomain, therefore I=s$\mathcal{R}$ for some nonzero element s ∈ I. Now f(s) in $\mathcal{R}$, hence there is a nonzero element t ∈ $\mathcal{R}$ such that f(s) = ts by divisibility of $\mathcal{R}$. Thus $\mathcal{R}$ is satisfy Baer's condition. Hence by Proposition 1, $\mathcal{R}$ is (NSI) hyperring.■

Example 2[2]**.** Let (Z,+) be a canonical hypergroup, where the hyperoperation "$\dotplus$" defined as a$\dotplus$b = {0, a+b}\{a,b} and the hyperstructuer (Z,$\dotplus$,·) is commutative Krasner hyperring with unit element 1. So the hyperring Z is not divisible hyperring. Therefore, it is not a normal self-injective hyperring.

Example 3[8]**.** The hyperoperation "$\dotplus$" and the multiplication "·" which are defined in Tables 1 and 2 on the set $\mathcal{R}$ = {0, 1, 2}.

Table 1. Additive hyperoperation

$\dotplus$	0	1	2
0	{0}	{1}	{2}
1	{1}	{1,2}	{1}
2	{2}	{1}	{0,2}

Table 2. Multiplicative operation

.	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Then $\mathcal{R}$ is a hyperring and the only hyperideals of it are {0} and $\mathcal{R}$ it-self, where {0} is generated by 0 and $\mathcal{R}$ by 1, thus $\mathcal{R}$ is a principal hyperideal hyperdomain and every nonzero element is divisible, thus $\mathcal{R}$ is divisible hyperring and by Theorem 1, $\mathcal{R}$ is (NSI).

Definition 17. The hyperring $\mathcal{R}$ is said to be the direct sum of two hyperrings $\mathcal{R}_1$ and $\mathcal{R}_2$, if $\mathcal{R}_1 + \mathcal{R}_2 = \mathcal{R}$ and $\mathcal{R}_1 \cap \mathcal{R}_2 = \{0\}$. For each element in $\mathcal{R}$, say r, there are unique elements $r_1 \in \mathcal{R}_1$ and $r_2 \in \mathcal{R}_2$, such that $r \in r_1 + r_2$, and denoted by $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$.

Theorem 2. If $\mathcal{R}$ is a direct sum of two hyperrings $\mathcal{R}_1$ and $\mathcal{R}_2$, then $\mathcal{R}$ is (NSI) if and only if each of $\mathcal{R}_1$ and $\mathcal{R}_2$ is (NSI).

Proof. Assume that each of $\mathcal{R}_1$ and $\mathcal{R}_2$ is (NSI) hyperring. Let I be a hyperideal of $\mathcal{R}$, and f an $\mathcal{R}$-homomorphism of I into $\mathcal{R}$. I can write as $I = I_1 + I_2$, where I_1, I_2 are hyperideals in $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively, [8] and f can be represented as a pair (f_1, f_2) where f_i is an $\mathcal{R}_i$-homomorphism of I_i into $\mathcal{R}_i$, for i=1,2. Then for each x ∈ I we have x = (x_1, x_2), and f(x) = $(f(x_1), f(x_2))$. Since each $\mathcal{R}_i$ is a normal self-injective hyperring, i=1, 2, there is $k_i \in \mathcal{R}_i$ such that $f_i(x_i) = k_i x_i$ for all $x_i \in I_i$, i =1,2. by Proposition 1. Then for any x ∈ I, f(x)=$(f_1(x_1), f_2(x_2))$=$(k_1x_1, k_2x_2) = (k_1, k_2)(x_1, x_2) =$ kx ,where k=$(k_1, k_2) \in \mathcal{R}$. Hence Baer's condition holds for $\mathcal{R}$. Therefore by Proposition 1, $\mathcal{R}$ is a normal self-injective.

For the converse, suppose that $\mathcal{R}$ is an (NSI) hyperring. Let I_1 be a hyperideal in $\mathcal{R}_1$ and f_1 be an $\mathcal{R}_1$-homomorphism of I_1 into $\mathcal{R}_1$. $I_1 + \mathcal{R}_2$ is a hypeideal in $\mathcal{R}_1 + \mathcal{R}_2 = \mathcal{R}$.

Define f: $I_1 + \mathcal{R}_2 \longrightarrow \mathcal{R}$ by f=(f_1, i_2) where i_2 is the identity mapping of $\mathcal{R}_2$. It is clearly that f is an R-homomorphism. Since $\mathcal{R}$ is (NSI) hyperring, therefore by Proposition 1, there is an element $s = (s_1, s_2) \in \mathcal{R}$ such that $f((x_1, x_2)) = (f_1(x_1),\ I_2(x_2)) = (s_1, s_2)(x_1, x_2) = (s_1x_1, s_2x_2)$ for each $(x_1, x_2) \in I_1 + \mathcal{R}_2$ in fact s_2 is the unite element in $\mathcal{R}_2$. Hence $(f_1(x_1)) = s_1x_1$ for each $x_1 \in I_1$, thus Baer's condition holds for $\mathcal{R}_1$. By Proposition 1 $\mathcal{R}_1$ is (NSI) hyperring. Similar proved that $\mathcal{R}_2$ is (NSI) hyperring. ■

Corollary 1. If $\mathcal{R}$ is a direct sum of a finite family of hyperrings $(\mathcal{R}_i)_{i\in I}$, then $\mathcal{R}$ is (NSI) hyperring if and only if each of $\mathcal{R}_i$ is (NSI).

Theorem 3. Let $\mathcal{R}$ be a hyperring considered as an $\mathcal{R}$-hypermodule. Then $\mathcal{R}$ is (NSI) if and only if $\mathcal{R}$ is a direct summand of every extension of $\mathcal{R}$

Proof. Let $\mathcal{R}$ be (NSI) hyperring, and $\mathcal{R}'$ be any extension of $\mathcal{R}$ as an $\mathcal{R}$-hypermodule. Thus there is an $\mathcal{R}$-homomorphism $h: \mathcal{R}' \longrightarrow \mathcal{R}$ such that $hi = I_{\mathcal{R}}$, where $i: \mathcal{R} \longrightarrow \mathcal{R}'$ is the inclusion map and $I_{\mathcal{R}} : \mathcal{R} \longrightarrow \mathcal{R}$ is the Identity map.

Let $r' \in \mathcal{R}'$, then $h(r') \in \mathcal{R}$. Hence $h(r' - h(r')) = h(r') - h(h(r')) = h(r') - h(i(h(r'))) = h(r') - hi(h(r')) = h(r') - I_{\mathcal{R}}(h(r')) = h(r') - h(r') = 0$. Hence $r' - h(r') \in \ker(h)$, and so there exists an element $x \in \ker(h)$ such that $r' - h(r') = x$, hence $r' = h(r') + x$. Therefore $\mathcal{R}' = \mathcal{R} + \ker(h)$. But $\mathcal{R} \cap \ker(h) = \{0\}$, then $\mathcal{R}' = \mathcal{R} \oplus \ker(h)$.

Conversely, assume that $\mathcal{R}$ is a direct sum and of every extension of $\mathcal{R}$. Since every hypermodule can be embedded in an injective hypermodule [9] (9, Prop. 2.5, in proof part$(4 \to 5)$). Thus $\mathcal{R}$ has an injective extension say S. Therefore S = $\mathcal{R}$+N for some subhypermodule N in S. By Theorem 2, got $\mathcal{R}$ is (NSI) hyperring. ■

One of the main purposes of this paper is to introduce Baer's condition to get normal self injectivity. First need some definitions and propositions.

Proposition 3. If $\mathcal{R}$ is a hyperring and I is a hyperideal in $\mathcal{R}$, then there exists a hyperideal J in $\mathcal{R}$ which is maximal among all hyperideals in $\mathcal{R}$ with zero intersection with I.

Proof. Let $\mathcal{H} = \{J: J \cap I = \{0\}$, J is a hyperideal in $\mathcal{R}\}$. Since $\{0\} \in \mathcal{H}$, so $\mathcal{H} \neq \emptyset$. The elements of $\mathcal{H}$ is partially ordered by inclusion. Let $S = \{I_\alpha : \alpha \in A\}$ be any linearly ordered subset of $\mathcal{H}$. Clearly $I_o = \cup I_\alpha$ is a hyperideal in $\mathcal{R}$, and $I_o \cap I = \{0\}$, it can be proved that I_o is an upper bound of S. Therefore, by Zorn's lemma, $\mathcal{H}$ has a maximal element say J this completes the proof. ■

This hyperideal J in the above proposition is called a complement hyperideal of I in $\mathcal{R}$, and it is denoted by I^c.

Definition 18 [10]**.** A right hyperideal I in a hyperring $\mathcal{R}$ is "essential" if for each right hyperideal $0 \neq K$ of $\mathcal{R}$, have $I \cap K \neq 0$. In fact this definition is the same in algebra system. [11,12,13]

Definition 19 [10]**.** A right hyperideal I in $\mathcal{R}$ is called "denes" if given any $0 \neq r_1 \in \mathcal{R}$, $r_2 \in \mathcal{R}$, there is $r \in \mathcal{R}$ such that. $r_1.r \neq 0$, and $r_2.r \in I$.

Definition 2 0[10] **.** The set $\{r \in \mathcal{R}.\ ar = 0\ ,\ \forall a \in I\}$.is called left annihilator of a hyperideal I in $\mathcal{R}$ and denoted by Ann(I) in the same way one can define the right annihilator.

Remark 2[10]**.** Let I be a hyperideal, then Ann(I) = 0 if and only if I is a dense right hyperideal.

Proposition 4. Let I be a hyperideal in a hyperring $\mathcal{R}$, I^c is the complement hyperideal of I in $\mathcal{R}$. Then I+I^c is an essential hyperideal in $\mathcal{R}$.

Proof. Assume that $(I + I^c) \cap H = \{0\}$ for some hyperideal H in $\mathcal{R}$, to prove that H=$\{0\}$. Let $i \in I \cap (I^c + H)$ then i=j+h where $j \in I^c$ and $h \in H$, hence h=i-j and $h \in (I+I^c) \cap H = \{0\}$. Therefore i=j. Since $I \cap I^c = \{0\}$. Therefore i=j=0. Then $I \cap (I^c + H) = \{0\}$. By maximality of I^c, have $H \subseteq I^c$. Hence $H = (I+I^c) \cap H = \{0\}$. ■

Corollary 2. If $\mathcal{R}$ is a hyperring and I is a hyperideal in $\mathcal{R}$, then I is an essential hyperideal if and only if $I^c = \{0\}$

Proof. Suppose that I is an essential hyperideal, if $I \cap I^c = \{0\}$, then $I^c = \{0\}$. Conversely, if $I^c = \{0\}$, then by Proposition 4, I+I^c is an essential hyperideal of $\mathcal{R}$. Therefore $I = I$+I^c is an essential hyperideal in $\mathcal{R}$. ■

The following proposition proved by M. Anbarloei in his paper "The Maximal hyperring of quotient" [10] here it will be proven in another way.

Proposition 5[10]. Every dense hyperideal in $\mathcal{R}$ is essential.

Proof. Let I be a dense hyperideal, and I^c be a complement hyperideal of I in $\mathcal{R}$. Then $I^c I \subseteq I \cap I^c = \{0\}$, thus $I^c I = \{0\}$. Since I is dense, therefore $I^c = \{0\}$. Hence I is essential in $\mathcal{R}$ by Corollary 2. ■

Proposition 6. Let $\mathcal{R}$ be a hyperring and I be a hyperideal in $\mathcal{R}$. If I is an essential hyperideal in $\mathcal{R}$, then it is dense.

Proof. Let I be an essential hyperideal in $\mathcal{R}$. $I \cap$ Ann(I) = $\{0\}$[10] (10, Remark 3.4), whence Ann(I)=$\{r \in \mathcal{R}$: Ir $=\{0\}\}$. But I is essential hyperideal, then Ann(I) = $\{0\}$, and hence I is dense (by Remark 2). ■

Proposition 7. Let f be an epimorphism of a hyperring $\mathcal{R}$ onto $\mathcal{R}'$, and I be essential in $\mathcal{R}'$. Then $f^{-1}(I)$ is an essential hyperideal in $\mathcal{R}$.

Proof. Assuming that $f^{-1}(I)$ is not essential, there is a hyperideal $0 \neq H \in \mathcal{R}$ such that $H \cap f^{-1}(I) = \{0\}$. Then $f(H) \neq 0$ a hyperideal in $\mathcal{R}'$, and $f(H) \cap I = \{0\}$. Hence I is not essential hyperideal that is a contradiction. Therefore $f^{-1}(I)$ is an essential hyperideal in $\mathcal{R}$. ■

Proposition 8. If $\mathcal{R}$ is a hyperring, and I is a hyperideal in $\mathcal{R}$. Then I is essential in $\mathcal{R}$ if and only if for each $c \in \mathcal{R} - I$, there is $r \in \mathcal{R} - I$ such that $cr \in I$.

Proof. Let I be an essential hyperideal in $\mathcal{R}$, $c \in \mathcal{R}$ such that $c \notin I$. Therefore $I \cap c\mathcal{R} \neq \{0\}$. Then there is $r \in \mathcal{R}$ such that $cr \in I \cap c\mathcal{R}$ Hence $cr \in I$.

Conversely, assume that for each $c \in \mathcal{R}$, there is $r \in \mathcal{R}$ such that. $cr \in I$. Let H be any nonzero hyperideal in $\mathcal{R}$, and let $0 \neq h \in H$ be any element. Hence there is element $r \in \mathcal{R}$ such that $rh \in I$. Hence $I \cap \mathcal{R}h \neq \{0\}$. But $\mathcal{R}h \subseteq H$. Therefore $I \cap \mathcal{R}h = I \cap H$ and $I \cap H \neq \{0\}$. Thus I is an essential hyperideal in $\mathcal{R}$. ■

Theorem 4. Let $\mathcal{R}$ be a hyperring. Then $\mathcal{R}$ is (NSI) if and only if for each essential hyperideal I in $\mathcal{R}$, and every an $\mathcal{R}$-homomorphism f: $I \longrightarrow \mathcal{R}$, there is an element $r \in \mathcal{R}$ such that. $f(x) = rx$, $\forall x \in I$.

Proof. Suppose that, $\mathcal{R}$ is (NSI) hyperring, so by Proposition.1 Baer´s condition holds for $\mathcal{R}$, in particular holds for every essential hyperideal in $\mathcal{R}$.

Conversely, to prove $\mathcal{R}$ is (NSI) hyperring. Let g be an $\mathcal{R}$-homomorphism of a hyperideal J in $\mathcal{R}$ into $\mathcal{R}$. Consider I^c a complement hyperideal of J in $\mathcal{R}$. By Proposition 4, I =J+J^c is an essential hyperideal in $\mathcal{R}$. Define f: $I \longrightarrow \mathcal{R}$ as follows f(x)=g(x) if x ∈ J , and f(x) =0 if x ∈ J^c. f is an $\mathcal{R}$-homomorphism of I into $\mathcal{R}$, then there is an element r ∈ $\mathcal{R}$ such that f(x)=rx for all x ∈ I. Hence g(x) = rx for all x ∈ J. ■

Conclusion:

Normal injective are introduced and studied by more than one author on a hypermodules. The goal of this paper is to shed light on the definition of normal injective but on the hyperring named by normal self-injective hyperring. Also, the extension of the Baer's condition on a ring to Baer's condition on a hyperring.

Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Baghdad.

Authors' contributions statement:

T. A. I. contributed to the interpretation and review of the research, checking the results and verifying the validity of what was stated in the research. M. F. A. contributed in designing and implementing the research, analyzing the results and writing this manuscript. The authors discussed the results and contributed to the final manuscript.

References.

1. Ameri R, Shojaei H, Projective and injective Krasner hypermodules, J Algebra Appl. 2020 Jun 13; 2(10): 2150186: https://doi.org/10.1142/S0219498821501863.
2. Bordbar H, Cristea I. Divisible hypermodules. Analele Stiint ale Univ Ovidius Constanta Ser Mat. 2022 Mar 11; 30(1): 57–74. DOI: 10.2478/auom-2022-0004
3. Aghabozorgi H, Jafarpour M, Davvaz B. On some classes of hypergroups. Cogent Math Stat. 2018 Jan 1; 5(1): 1510357.
4. Abumghaiseeb A. On $\boldsymbol{\delta}$-Primary hyperideals and fuzzy hyperideals expansions, Republic of Turkey Yildiz Technical University Graduate School of Natural and Applied Sciences. PhD Thesis. 2018. https://acikbilim.yok.gov.tr/bitstream/handle/20.500.12812/381645/yokAcikBilim_10195937.pdf?sequence=-1&isAllowed=y.
5. Kuhlmann K, Linzi A, Stojałowska H. Orderings and valuations in hyperfields. J Algebra. 2021 Jun 9; 611: 399–421.
6. Bordbar H, Cristea I. Regular parameter elements and regular local hyperrings. Math. 2021; 9(3): 1–13.
7. Bordbar H, Cristea I. About the normal projectivity and injectivity of Krasner hypermodules, Axioms, 2021 May 4; 10(2): 83. https://doi.org/10.3390/axioms10020083.
8. Boriboon S, Pianskool S. Baer hypermodules over Krasner hyperrings. 22ndAnnu. Meet Math. 2017; 1–9
9. Türkmen BN. t-Extending Krasner Hypermodules. Bol. Soc. Paran. Mat.. 2022; 40(40): 1–6.
10. Yazarli H, Yılmaz D, Davvaz B. The maximal hyperrings of quotients. Ital J Pure Appl Math. 2020; 44. 938–951.
11. Ahmed M. Some Results on Weak Essential Submodules. Baghdad Sci J [Internet]. 2016; 13(3): 599–606. Available from: http://csw-journal.org/show-articles.php?id_article=1834
12. Ahmed M. Weak Essential Submodules. Baghdad Sci J [Internet]. 2009; 6(1): 214-221. Available from: https://bsj.uobaghdad.edu.iq/index.php/BSJ/article/view/972
13. Ahmed M, Ibrahiem T. H-essential submodules and Homessential Modules. Iraqi J Sci. 2019; 60(6): 1381-1386.

الحلقات الفوقية السوية ذاتية الاغمار

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الخلاصة:
في هذه الورقة البحثية تم تقديم مفهوم الحلقة الفوقية السوية ذاتية الاغمار، كما تم ايجاد بعض العلاقات بين هذا المفهوم ومفاهيم اخرى مثل المثالي الجوهري والمثالي الكثيف والحلقة الفوقية القابلة للقسم.

الكلمات المفتاحية : شرط بير، مثالي فوقي كثيف، الحلقات الفوقية القابلة للقسمة ،مثالي فوقي جوهري، الحلقات الفوقية السوية ذاتية الاغمارز