Open Access **Baghdad Science Journal** **P-ISSN: 2078-8665**
Published Online First: January 2021 **E-ISSN: 2411-7986**

DOI: http://dx.doi.org/10.21123/bsj.2021.18.2.0330

Numerical Solution for Linear Fredholm Integro-Differential Equation Using Touchard Polynomials

Jalil Talab Abdullah

Department of Statistics, Faculty of Administration and Economics, Wasit University, Iraq
E-mail: jalil.talab@gmail.com
ORCID ID: https://orcid.org/0000-0002-6847-3635

Received 25/1/2020, Accepted 4/6/2020, Published Online First 11/1/2021, Published 1/6/2021

Abstract:

A new method based on the Touchard polynomials (TPs) was presented for the numerical solution of the linear Fredholm integro-differential equation (FIDE) of the first order and second kind with condition. The derivative and integration of the (TPs) were simply obtained. The convergence analysis of the presented method was given and the applicability was proved by some numerical examples. The results obtained in this method are compared with other known results.

Key words: Fredholm equation, Integro-differential equation, Integral equation, Touchard polynomial, Numerical solution

Introduction:

Integral and integro-differential equations are originated in many scientific and engineering applications. In particular Fredholm integral equation and (FIDE) can be derived from boundary value problem. The (IDE) contains both differential and integral signs and the derivative of the unknown variable may appear to any order. (FIDE) is an equation derived from the boundary value problem with given initial boundary condition, where both the differential and integral signs appeared together in the same equation. In addition, limits of the integration are constants. The (FIDE) of the first order and second kind contains the unknown variable and its derivative inside and outside the integral sign respectively. It is noted that initial condition should be given for (FIDE) to find the particular solution (1). (FIDEs) often come in applications being the mathematical models of processes in biological problems, physics, economy and chemistry, etc. (2). (FIDEs) are difficult to be solved analytically, so it requires effective numerical methods (3). For these reasons, many scientists have been encouraged to study many numerical methods to solve (FIDEs). All methods have pros or cons but that hasn't stopped scientists from developing various methods such as Bernstein collocation matrix method (4), the well-posedness method (5), reproducing kernel method (6), exponential spline method (7), improved reproducing kernel method (8), priori Nystrom method (9), and Fibonacci polynomials method (10).

The general form of the linear (FIDE) of 1st order and 2nd kind is given by (1, 11):

$$X'(\gamma) = h(\gamma) + \eta \int_{a_1}^{a_2} T(\gamma, u)\, X(u)du \quad , \quad \gamma \in [a_1, a_2\,], \qquad \dots (1)$$

with initial boundary condition $X(a_1) = X_0$, $\dots (1a)$

where a_1, a_2 and η are constants, $T(\gamma, u)$ is a known function of the variables γ and u, called the nucleus (kernel) of the Integral equation. The unknown function $X(\gamma)$ will be calculated, which exists inside and outside the integral sign. $h(\gamma)$ is a given function, $X'(\gamma) = \frac{d}{d\gamma}X(\gamma)$ and $X(a_1) = X_0$ is a constant initial boundary condition.

This paper is ordered as follows: Touchard polynomials, approximation function, solution the (FIDE), convergence analysis, test examples with tables and graphs are presented, brief of conclusions and recommendations, and finally, the references are listed.

Touchard Polynomials:

Let's begin with the definition of the (TPs) that was studied by the French mathematician Jacques Touchard. The (TPs) consist of a polynomial sequence of binomial type, it's defined on [0, 1] as follows (12, 13, 14, and 15):

$$I_n(\gamma) = \sum_{m=0}^{n} E(n,m)\gamma^m = \sum_{m=0}^{n} \binom{n}{m}\gamma^m \quad , \quad \binom{n}{m} = \frac{n!}{m!\,(n-m)!} \qquad \cdots (2)$$

where n and m are the degree and index of the (TPs) respectively.

The 1st five polynomials of the (TPs) are given below:

$I_0(\gamma) = 1$
$I_1(\gamma) = 1 + \gamma$
$I_2(\gamma) = 1 + 2\gamma + \gamma^2$
$I_3(\gamma) = 1 + 3\gamma + 3\gamma^2 + \gamma^3$
$I_4(\gamma) = 1 + 4\gamma + 6\gamma^2 + 4\gamma^3 + \gamma^4$

Approximation Function:

Suppose that the function $X_n(\gamma)$ is approximated using the (TPs) as in the following:

$$X_n(\gamma) = \alpha_0 I_0(\gamma) + \alpha_1 I_1(\gamma) + \cdots + \alpha_n I_n(\gamma) = \sum_{m=0}^{n} \alpha_m I_m(\gamma) \quad 0 \le \gamma \le 1 , \qquad \cdots (3)$$

for m≥0, the function $\{I_m(\gamma)\}_{m=0}^{n}$ denotes the Touchard basis polynomials of nth degree, as defined in Eq. (2), α_m $(m = 0,1, \dots, n)$ are the unknown Touchard coefficients that will be calculated later.

Now Eq. (3) can be written as dot product:

$$X_n(\gamma) = [I_0(\gamma) \;\; I_1(\gamma) \dots I_n(\gamma)] \cdot \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n \end{bmatrix}, \qquad \cdots (4)$$

Eq. (4) can be converted into:

$$X_n(\gamma) = [1 \;\; \gamma \;\; \gamma^2 \dots \gamma^n] \cdot \begin{bmatrix} \beta_{00} & \beta_{01} & \beta_{02} & \cdots & \beta_{0n} \\ 0 & \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ 0 & 0 & \beta_{22} & \cdots & \beta_{2n} \\ \vdots & \vdots & \vdots \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & \beta_{nn} \end{bmatrix} \cdot \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n \end{bmatrix}, \dots (5)$$

where β_{rr} $(r = 0,1,2, \dots, n)$ are the coefficients of the power basis that are used to obtain the (TPs) coefficients and the matrix is invertible. For example n=1, 2, and 3, the operational matrices are shown in Eqs. (6), (7), and (8) respectively:

$$X_1(\gamma) = [1 \quad \gamma \;] \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}, \qquad \dots (6)$$

$$X_2(\gamma) = [1 \;\; \gamma \;\; \gamma^2] \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix}, \qquad \cdots (7)$$

$$X_3(\gamma) = [1 \;\; \gamma \;\; \gamma^2 \; \gamma^3 \;] \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}. \qquad \cdots (8)$$

Since the derivative of Eq. (2) is:

$$I_n'(\gamma) = \frac{d}{d\gamma} \sum_{m=0}^{n} \binom{n}{m}\gamma^m = \sum_{m=1}^{n} \binom{n}{m} m\,\gamma^{m-1} \quad . \qquad \cdots (9)$$

Then, the derivative of Eqs. (5), (6), (7) and (8) respectively is:

$$X_n'(\gamma) = [0 \;\; 1 \;\; 2\gamma \dots n\gamma^{n-1}] \cdot \begin{bmatrix} \beta_{00} & \beta_{01} & \beta_{02} & \dots & \beta_{0n} \\ 0 & \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ 0 & 0 & \beta_{22} & \cdots & \beta_{2n} \\ \vdots & \vdots & \vdots \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & \beta_{nn} \end{bmatrix} \cdot \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n \end{bmatrix}, \cdots (10)$$

$$X_1'(\gamma) = [0 \quad 1 \;] \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}, \qquad \dots (11)$$

$$X_2'(\gamma) = [0 \;\; 1 \;\; 2\gamma] \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix}, \qquad \cdots (12)$$

$$X_3'(\gamma) = [0 \;\; 1 \;\; 2\gamma \;\; 3\gamma^2 \;] \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \cdot \cdots (13)$$

Solutions the (FIDE) of 1st order and 2nd Kind:

Since Eq. (1) has the following form:

$$X'(\gamma) = h(\gamma) + \eta \int_{a_1}^{a_2} T(\gamma,u)\,X(u)du \quad , \quad \gamma \in [a_1, a_2\,] \text{ and } X(a_1) = X_0 \dots (14)$$

By using Eqs. (5) and (10), suppose that:

$$X(\gamma) \cong X_n(\gamma) = [1 \;\; \gamma \;\; \gamma^2 \dots \gamma^n] \cdot \begin{bmatrix} \beta_{00} & \beta_{01} & \beta_{02} & \dots & \beta_{0n} \\ 0 & \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ 0 & 0 & \beta_{22} & \cdots & \beta_{2n} \\ \vdots & \vdots & \vdots \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & \beta_{nn} \end{bmatrix} \cdot \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n \end{bmatrix}, \dots (15)$$

and

$$X'(\gamma) \cong X'_n(\gamma) = [0\ \ 1\ \ 2\gamma \dots n\gamma^{n-1}].\begin{bmatrix} \beta_{00} & \beta_{01} & \beta_{02} & \dots & \beta_{0n} \\ 0 & \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ 0 & 0 & \beta_{22} & \cdots & \beta_{2n} \\ \vdots & \vdots & \vdots \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & \beta_{nn} \end{bmatrix}.\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ . \\ . \\ \alpha_n \end{bmatrix}, \quad \dots (16)$$

now, by substituting Eqs. (15) and (16) into Eq. (14) yields:

$$[0\ \ 1\ \ 2\gamma \dots n\gamma^{n-1}].\begin{bmatrix} \beta_{00} & \beta_{01} & \beta_{02} & \dots & \beta_{0n} \\ 0 & \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ 0 & 0 & \beta_{22} & \cdots & \beta_{2n} \\ \vdots & \vdots & \vdots \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & \beta_{nn} \end{bmatrix}.\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ . \\ . \\ \alpha_n \end{bmatrix}$$

$$= h(\gamma) + \eta \int_{a_1}^{a_2} T(\gamma, u)\,[1\ u\ u^2 \dots u^n].$$

$$\begin{bmatrix} \beta_{00} & \beta_{01} & \beta_{02} & \dots & \beta_{0n} \\ 0 & \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ 0 & 0 & \beta_{22} & \cdots & \beta_{2n} \\ \vdots & \vdots & \vdots \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & \beta_{nn} \end{bmatrix}.\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ . \\ . \\ \alpha_n \end{bmatrix} du. \quad \cdots (17)$$

Now, in order to determine $(\alpha_0, \alpha_1, \dots, \alpha_n)$ the integrations in Eq. (17) must be computed by selecting $\gamma_r \in [0, 1], (r = 0,1, \dots, n)$, and applying the given initial condition, to get a system of (n+1) linear algebraic equations with the (n+1) of the unknown values. Solving this system by using Gauss elimination method, the unknown coefficients $(\alpha_0, \alpha_1, \dots, \alpha_n)$ are obtained and have a unique solution, hence, by substituting Touchard coefficients into Eq. (3) to get the approximate numerical solution for Eq. (1)

Convergence Analysis:

In this section, the convergence for the suggested method is proved.

The unknown Touchard coefficients $(\alpha_0, \alpha_1, \dots, \alpha_n)$ are uniquely determined by Eq. (17). Therefore Eq. (1) with the boundary condition has a unique solution and this solution is given by truncated Touchard series in Eq. (3). Now, when the approximate numerical solution $X_n(\gamma)$ and its derivatives are substituted in Eq. (1), the following equation should be satisfied approximately, then for $\gamma = \gamma_\lambda \in [0, 1], \lambda = 0, 1, 2, \dots, n$,

$$ER_n(\gamma_\lambda) = \left| \left(\sum_{m=0}^{n} \alpha_m I_m(\gamma_\lambda)\right)' - h(\gamma_\lambda) - \eta \int_{a_1}^{a_2} T(\gamma_\lambda, u) \sum_{m=0}^{n} \alpha_m I_m(u)\, du \right| \cong 0,$$

and $ER_n(\gamma_\lambda) \leq 10^{-\gamma_\lambda}$.

If $\max(10^{-\gamma_\lambda}) = 10^{-\gamma}$ is specified,

then the truncation limit n is increased until the difference $ER(\gamma_\lambda)$ between each of the points γ_λ becomes less than or equal $10^{-\gamma}$. In other words, the error function $ER_n(\gamma_\lambda)$ can be estimated by the relation:

$$ER_n(\gamma) = \left(\sum_{m=0}^{n} \alpha_m I_m(\gamma)\right)' - h(\gamma) - \eta \int_{a_1}^{a_2} T(\gamma, u) \sum_{m=0}^{n} \alpha_m I_m(u)\, du,$$

then, $ER_n(\gamma) \to 0$ when n is a very large, then the error function decreases (3 and 10)

Numerical Examples:

This section checks the computational accuracy of the (TPs) method, by testing three examples of linear (FIDE) and one example of a nonlinear (FIDE). The accuracy of the solution method was measured by the absolute error for the first three examples, while the fourth example was measured by the maximum absolute error, convergence rate and time of CPU, also all calculations and charts were accomplished on my PC using the matlab2018 program.

The general formulas of the testes were defined as follows:

Absolute error:

$|ER| = |X(\gamma_\lambda) - X_n(\gamma_\lambda)|$, $\gamma_{\lambda \in [0,1]}$ and $\lambda = 0, 1, \dots, n$

Maximum error: $\|ER\|_\infty = \max_{\gamma_\lambda \in [0,1]} |X(\gamma_\lambda) - X_n(\gamma_\lambda)|$, where $X(\gamma_\lambda)$, and $X_n(\gamma_\lambda)$ are the exact and approximate numerical solutions with the Touchard's approximation of the (FIDEs), respectively.

Convergence rate: Ratio = $\frac{\|ER^{n-1}\|_\infty}{\|ER^n\|_\infty}$, where $\|ER^n\|_\infty$ and $\|ER^{n-1}\|_\infty$ are the maximum absolute errors of degree n and $n-1$, respectively.

Example1: Solve the linear (FIDE) given in (1):

$$X'(\gamma) = h(\gamma) + \int_0^1 xu\, X(u)\, du, \qquad 0 \leq \gamma \leq 1$$

where $h(\gamma) = 3 + 6\gamma$, $\eta = 1, T(\gamma, u) = xu$, $X(0) = 0$, and the exact solution $X(\gamma) = 3\gamma + 4\gamma^2$

Now, by applying the (TPs), the approximate solutions of this example for n = 1, 2 and 3 are respectively:

$$X_1(\gamma) = \sum_{m=0}^{1} \alpha_m I_m(\gamma) = \alpha_0 I_0(\gamma) + \alpha_1 I_1(\gamma) = (-3.01) I_0(\gamma) + (3.0) I_1(\gamma).$$

$$X_2(\gamma) = \sum_{m=0}^{2} \alpha_m I_m(\gamma) = \alpha_0 I_0(\gamma) + \alpha_1 I_1(\gamma) + \alpha_2 I_2(\gamma) = (0.76)\, I_0(\gamma) + (-4.5556) I_1(\gamma) + (3.7778)\, I_2(\gamma)$$

Baghdad Science Journal
P-ISSN: 2078-8665
E-ISSN: 2411-7986

$$X_3(\gamma) = \sum_{m=0}^{3} \alpha_m I_m(\gamma) = \alpha_0 I_0(\gamma) + \alpha_1 I_1(\gamma) + \alpha_2 I_2(\gamma) + \alpha_3 I_3(\gamma) = (0.8)I_0(\gamma) + (-4.5556)I_1(\gamma) + (3.7778)I_2(\gamma) + (0)I_3(\gamma).$$

The approximate solutions and absolute error were compared in Tables 1 and 2, respectively, showing that the accuracy of the results increases as n increases. In Fig.1, the exact solution was compared with Touchard solution for n = 3.

Table 1.Approximate Numerical and Exact Solutions of Example 1.

γ	Exact Solution	Approximate Solutions		
		n = 1	n = 2	n = 3
0.0	0.0	−1.0000e−02	−1.7778e−02	2.2222e−02
0.1	3.4000e−01	2.9000e−01	3.2000e−01	3.6000e−01
0.2	7.6000e−01	5.9000e−01	7.3333e−01	7.7333e−01
0.3	1.2600e+00	8.9000e−01	1.2222e+00	1.2622e+00
0.4	1.8400e+00	1.1900e+00	1.7867e+00	1.8267e+00
0.5	2.5000e+00	1.4900e+00	2.4267e+00	2.4667e+00
0.6	3.2400e+00	1.7900e+00	3.1422e+00	3.1822e+00
0.7	4.0600e+00	2.0900e+00	3.9333e+00	3.9733e+00
0.8	4.9600e+00	2.3900e+00	4.8000e+00	4.8400e+00
0.9	5.9400e+00	2.6900e+00	5.7422e+00	5.7822e+00
1.0	7.0000e+00	2.9900e+00	6.7600e+00	6.8000e+00

Table 2. Comparison of the Absolute Error of Example 1.

γ	Absolute Errors		
	n = 1	n = 2	n = 3
0.0	1.0000e − 02	1.7778e − 02	2.2222e − 02
0.1	5.0000e − 02	1.9999e − 02	2.0001e − 02
0.2	1.7000e − 01	2.6666e − 02	1.3334e − 02
0.3	3.7000e − 01	3.7778e − 02	2.2223e − 03
0.4	6.5000e − 01	5.3333e − 02	1.3333e − 02
0.5	1.0100e + 00	7.3333e − 02	3.3333e − 02
0.6	1.4500e + 00	9.7777e − 02	5.7777e − 02
0.7	1.9700e + 00	1.2667e − 01	8.6667e − 02
0.8	2.5700e + 00	1.6000e − 01	1.2000e − 01
0.9	3.2500e + 00	1.9778e − 01	1.5778e − 01
1.0	4.0100e + 00	2.4000e − 01	2.0000e − 01

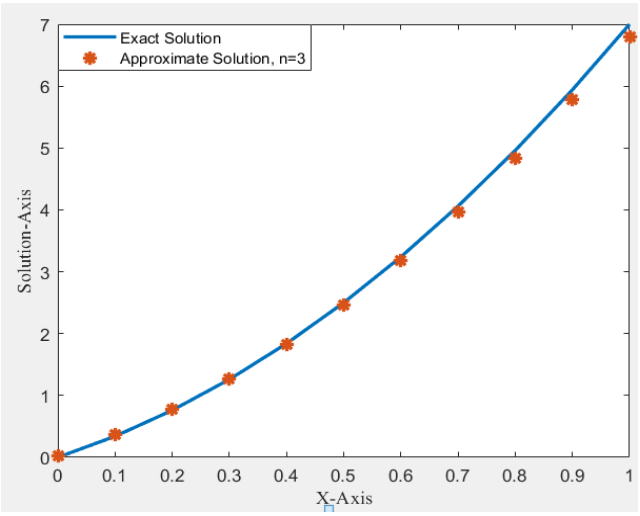

Figure 1. Numerical Result and Exact Solution of Example 1 for n = 3.

Example 2: Solve the linear (FIDE) given in (16, and 17)

$$X'(\gamma) = h(\gamma) + \int_0^1 \gamma\, X(u)\, du\,, \qquad 0 \le \gamma \le 1$$

Baghdad Science Journal
P-ISSN: 2078-8665
E-ISSN: 2411-7986

where $h(\gamma) = \gamma\, e^{\gamma} + e^{\gamma} - \gamma\,, \eta = 1,\ T(\gamma, u) = \gamma\,, X(0) = 0$ and the exact solution is $X(\gamma) = \gamma\, e^{\gamma}$. The approximate numerical results are obtained for n = 1, 3 and 4, respectively:

$$X_1(\gamma) = (-0.97)I_0(\gamma) + (0.9657)I_1(\gamma).$$

$$X_3(\gamma) = (-0.45)I_0(\gamma) + (0.5351)I_1(\gamma) + (-0.7436)I_2(\gamma) + (0.6523)I_3(\gamma)$$

$$X_4(\gamma) = (-0.15)I_0(\gamma) + (-0.6898)I_1(\gamma) + (1.0507)I_2(\gamma) + (-0.4411)I_3(\gamma) + (0.2278)\, I_4(\gamma).$$

Table 3 shows the absolute errors for n=4, and compares with methods included in (16 and 17). In Fig. 2, the exact solution is compared with Touchard solution for n = 4.

Table 3. Comparison of the Absolute Error of Example 2.

γ	Absolute Errors, n=4		
	Current Method	Method in (16)	Method in (17)
0.1	1.5839709e − 03	1.34917637e − 03	1.00118319e − 02
0.2	1.0428977e − 03	1.15960044e − 03	2.78651355e − 02
0.3	5.4169991e − 03	5.67152531e − 03	5.08730892e − 02
0.4	1.1544044e − 02	5.93105650e − 02	7.55356316e − 02
0.5	1.9394491e − 02	1.32330751e − 02	9.71888592e − 02
0.6	2.8827524e − 02	4.39287720e − 02	1.09551714e − 01
0.7	3.9505430e − 02	1.41201624e − 02	1.04133232e − 01
0.8	5.0796945e − 02	1.34514117e − 02	6.94512700e − 02
0.9	6.1668672e − 02	1.32045209e − 02	1.00034260e − 02

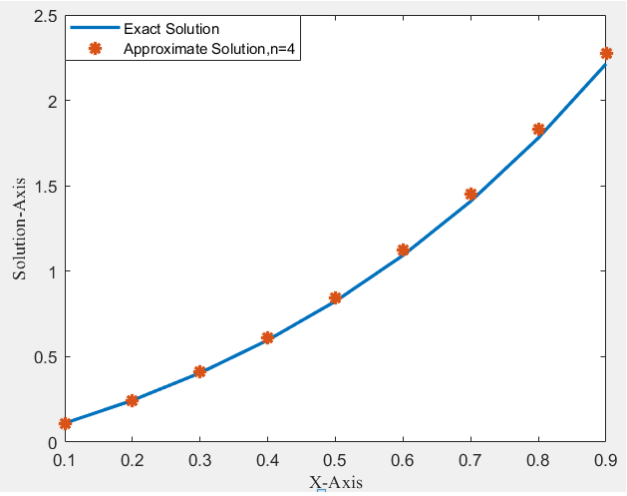

Figure 2. Numerical Result and Exact Solution of Example 2 for n = 4.

Example 3: Solve the linear (FIDE) given in (11, 16, and 17)

$$X'(\gamma) = h(\gamma) + \int_0^1 \gamma u\, X(u)\, du\,, \qquad 0 \le \gamma \le 1$$

where $h(\gamma) = 1 - \frac{1}{3}\gamma\,,\ \eta = 1,\ T(\gamma, u) = \gamma u\,, X(0) = 0$ and the exact solution is $X(\gamma) = \gamma$. By applying suggested method for this example, for n = 5, the Touchard solution is:

$$X_5(\gamma) = (-1)I_0(\gamma) + (1)I_1(\gamma) + 0 = X(\gamma) = \gamma.$$

In Table 4, the absolute error in the current method is compared with those in (11, 16 and 17), and it is found that the absolute error in the current method is the highest accuracy. In Fig. 3, for n = 5, the Touchard solution is compared with the exact solution.

Baghdad Science Journal
P-ISSN: 2078-8665
E-ISSN: 2411-7986

Table 4. Comparison of the Absolute Error of Example 3.

	Absolute Errors			
γ	Current Method n=5	Method in (11)	Method in (16)	Method in (17)
0.1	0.0	3.7900e − 06	2.1794e − 04	1.6667e − 03
0.2	0.0	1.5160e − 05	6.3855e − 04	6.0939e − 03
0.3	0.0	3.4110e − 05	7.9137e − 04	1.3202e − 02
0.4	0.0	6.0640e − 05	2.1559e − 02	2.2914e − 02
0.5	0.0	9.4750e − 05	4.9936e − 03	3.5158e − 02
0.6	0.0	1.3644e − 04	2.2173e − 02	6.6965e − 02
0.7	0.0	1.8571e − 04	1.0565e − 04	7.1243e − 02
0.8	0.0	2.4256e − 04	1.4323e − 03	8.6398e − 02
0.9	0.0	3.0699e − 04	2.0775e − 02	1.0810e − 01

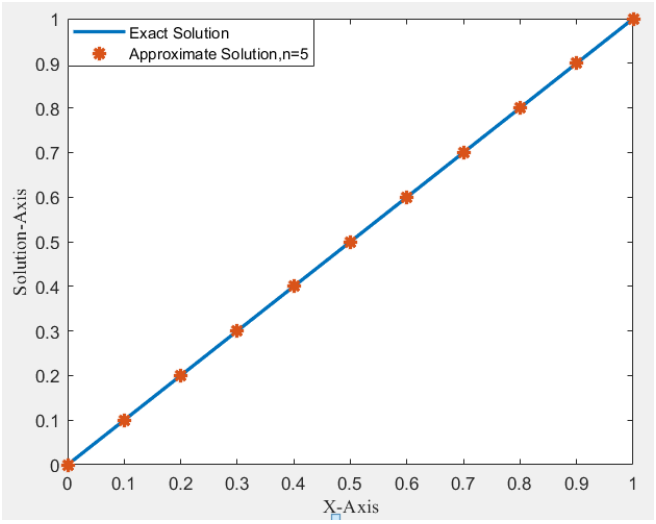

Figure 3. Numerical Result and Exact Solution of Example 3 for n = 5.

Example 4: Finally, solve the nonlinear (FIDE) given in (18)

$$X'(\gamma) = h(\gamma) + \int_0^1 \gamma^3 (X(u))^2 \, du, \qquad 0 \le \gamma \le 1$$

where $h(\gamma) = 1 - \frac{1}{3}\gamma^3$, $\eta = 1$, $T(\gamma, u) = \gamma^3$, $X(0) = 0$ and the exact solution is $X(\gamma) = \gamma$.
By applying suggested method for n= 1, 2 and 3, the Touchard solutions are obtained respectively:

$$X_1(\gamma) = (-1)I_0(\gamma) + (1)I_1(\gamma) = X(\gamma) = \gamma.$$

$$X_2(\gamma) = (-1)I_0(\gamma) + (1)I_1(\gamma) + (4.4522e-16)I_2(\gamma).$$

$$X_3(\gamma) = (-1)I_0(\gamma) + (1)I_1(\gamma) + (-7.1067e-15)I_2(\gamma) + (1.8000e-15)I_3(\gamma).$$

The comparison of the maximum error, ratio of error and CPU times of the current method with those in (18) is shown in Table 5, and shows that the current method for n = 1, 2 and 3 has a much higher accuracy than those in (18) for n = 5, 9 and 17. In Fig. 4, for n = 2 and 3, the Touchard solutions were compared with the exact solution.

Table 5. Comparison of the Maximum Absolute Error, Error Ratio and CPU Time of Example 4.

Current Method				Method in (18)			
n	$\|e\|_\infty$	Ratio	Time	n	$\|e\|_\infty$	Ratio	Time
1	1.3989 e−14	−	0.036	5	3.26 e −03	−	0.37
2	6.2172 e−15	2.25	0.039	9	8.44 e −04	3.87	0.41
3	3.3307 e−15	1.87	0.092	17	2.16 e −04	3.90	0.56

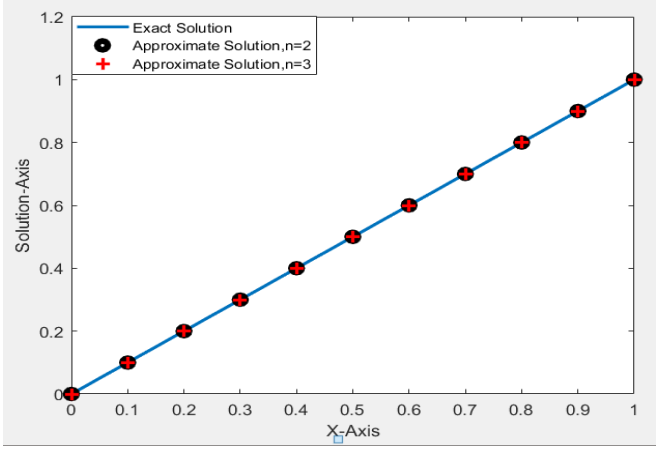

Figure 4. Numerical results and Exact Solution of Example 4 for n = 2 and 3.

Conclusions and Recommendations:

In this study, numerical solutions are obtained for linear (FIDEs) of the first order and second kind under condition, using Touchard polynomials, and different degrees for purpose of comparing. This method reduces the (FIDEs) into a set of algebraic equations. It's worth noting that one of the important features of this method is that the Touchard coefficients of the solutions are found easily by using PC programs. Also, another advantage is the obtaining solution is polynomials of the degree equal or less than selected n. However, the solution converges rapidly to the exact solution when n increases. The comparison between the absolute errors for four test examples and those methods included in (11, 16, 17 and 18), shows that the accuracy of the current method is almost similar or better than those of the existing methods. As a future work, the current method can also applied to the system of linear (FIDEs), because it is effective and applicable for the linear and nonlinear for these kinds of equations and the results obtained support this claim. Because the solutions obtained here are approximate solutions, it is expected in some examples that the absolute error increases when γ approaches 1 in the interval [0, 1] as in examples 1 and 2.

All the methods referred to in the introduction to this study are approximate numerical methods that have been used to solve the Fredholm integro-differential equations that are difficult to solve analytically. These methods have been used to solve them numerically. The pros of these methods are to obtain approximate solutions and the possibility of writing algorithms for solutions in these methods. Programming these algorithms on personal computers by writing computer programs to identify unknown values and then the possibility of comparing the results obtained in these methods by graphs. The cons of these methods are the existence of errors in the accuracy of the results in reaching the approximate solutions.

Author's declaration:

- Conflicts of Interest: None.
- I hereby confirm that all the Figures and Tables in the manuscript are mine. Besides, the Figures and images, which are not mine, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in Wasit University.

References:

1. Wazwaz A. M. Linear and Nonlinear Integral Equations Methods and Applications. Springer Berlin, Heidelberg; 2011. 33-38 p.
2. Dzhumabaev D. S. New general solutions to linear Fredholm integro-differential equations and their applications on solving the boundary value problems. J. Comput Appl. Math. 2018 Jan. 1; 327: 79-108.
3. Biçer G. G, Öztürk Y, Gülsu M. Numerical approach for solving linear Fredholm integro-differential equation with piecewise intervals by Bernoulli polynomials. IJCM. 2018; 95(10): 2100-2111.
4. S. Davaeifar, J. Rashidinia, M. Amirfakhrian. Bernstein Polynomial Approach for Solution of Higher-Order Mixed Linear Fredholm Integro-Differential-Difference Equations with Variable Coefficients. TWMS J. Pure Appl. Math. 2016; 7(1): 46-62
5. Dzhumabaev D. S. On One Approach to Solve the Linear Boundary Value Problems for Fredholm Integro-Differential Equations. J. Comput. Appl. Math. 2016 March. 1; 294: 342-357.
6. Du H, Zhao G, Zhao C. Reproducing Kernel Method for Solving Fredholm Integro-Differential Equations with Weakly singularity. J. Comput. Appl. Math. 2014 Jan. 1; 255 (2014): 122-132.
7. R. Jalilian, T. Tahernezhad. Exponential Spline Method for Approximation Solution of Fredholm Integro-Differential Equation. IJCM. 2019 Feb 18; 97(4): 791-801.

https://doi.org/10.1080/00207160.2019.1586891
8. Xue Q, Niu J, Yu D, Ran C. An Improved Reproducing Kernel Method for Fredholm Integro-Differential Type Two-Point Boundary Value Problems. IJCM. 2018; 95(5): 1015-1023
9. Fairbairn A. I, Kelmanson M. A. A priori Nyström-Method Error Bounds in Approximate Solutions of 1-D Fredholm Integro-Differential Equations. IJMS. 2018 Nov 6; 150 (2019): 755– 766.
10. Kurt A, Yalçınbaş S, Sezer M. Fibonacci Collocation Method for Solving High-Order Linear Fredholm Integro-Differential-Difference Equations. Math. comput. appl. 2013 Jun; 18(3):448-458. Available from: https://doi.org/10.1155/2013/486013
11. Berenguer M. I, Muñoz M. F, Garralda-Guillem A. I, Galán M. R. A sequential Approach for Solving the Fredholm Integro-Differential Equation. Appl. Numer. Math. 2012; 62(2012): 297–304
12. Nazir A, Usman M, Mohyud-Din S. T. Tauseef Mohyud-din.Touchard Polynomials Method for Integral Equations. Int. J. Modern Theo. Physics. 2017 Aug 30; 3(1): 74-89.
13. Paris R. B. The Asymptotes of the Touchard Polynomials: A Uniform Approximation. Mathematica Aeterna. 2016 Jun 28; 6(5): 765-779.
14. Mihoubi M, Maamra M. S.Touchard Polynomials, Partial Bell Polynomials and Polynomials of Binomial Type. J. Integer Seq. 2011 Mar 25; 14(2011): 1-9.
15. Yasmin G. Some Identities of the Apostol Type Polynomials Arising From Umbral Calculus. PJM. 2018; 7(1): 35-52
16. Danfu H, Xufeng S. Numerical Solution of Integro-Differential Equations by Using CAS Wavelet Operational Matrix of Integration. App. Math. Comput. 2007 Dec 15; 194(2): 460-466. Available from:https://doi.org/10.1016/j.amc.2007.04.048
17. Darania P, Ebadian A. A Method for the Numerical Solution of the Integro-Differential Equations. Appl. Math. Comput [Internet]. 2007 May 1; 188(1): 657-668. https://doi.org/10.1016/j.amc.2006.10.046
18. Dehghan M, Salehi R.The Numerical Solution of the Non-Linear Integro-Differential Equations Based on The Mesh Less Method. J. Comput. Appl. Math. 2012; 236(9): 2367-2377. https://doi.org/10.1016/j.cam.2011.11.022

حل عددي لمعادلة فريدهولم التفاضلية التكاملية الخطية باستخدام متعددة حدود تشارد.

جليل طلب عبدالله

قسم الاحصاء، كلية الادارة والاقتصاد، جامعة واسط، واسط، العراق

الخلاصة:
تم تقديم طريقة جديدة تستند الى متعددة حدود تشارد للحل العددي لمعادلات فريدهولم التفاضلية التكاملية من المرتبة الاولى والنوع الثاني مع الشرط. تم الحصول ببساطة على مشتقة متعددة حدود تشارد وتكاملها. واعطيت دقة الطريقة المقدمة وثبتت قابلية تطبيقها ببعض الامثلة العددية. تتم مقارنة النتائج التي تم الحصول عليها مع النتائج المعروفة الاخرى.

الكلمات المفتاحية: معادلة فريدهولم، معادلة تفاضلية تكاملية ، معادلة تكاملية، متعددة حدود تشارد، الحل العددي .