

Deriving the Composite Simpson Rule by Using Bernstein Polynomials for Solving Volterra Integral Equations

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Received 12, December, 2012

Accepted 8, May, 2013

Abstract:

In this paper we use Bernstein polynomials for deriving the modified Simpson's 3/8 , and the composite modified Simpson's 3/8 to solve one dimensional linear Volterra integral equations of the second kind , and we find that the solution computed by this procedure is very close to exact solution.

Key words: Integral equation, Bernstein polynomial, Simpson rule.

Introduction:

Integral equations are equations in which the unknown function appears under the sign of integral [1]. It is well known that integral equations arise in many branches of science, for example biological species [2],[3], sliding a bead along a wire [4], human population [4]. Also integral equations have a relation with initial and boundary value problems[1],[3].

The theoretical methods for solving Volterra integral equations are successive approximation, successive substitution, Laplace transformation, Adomian decomposition and series solution methods. Many researchers study the numerical solution[4],[5],[6],[7],[8],[9].

Block-by-block method is used for solving linear Volterra integral equations [10]. Quadrature method is used for solving linear Volterra integral equations of the second kind [11],[12].

Volterra Integral Equations: [1],[3]

The general form of Volterra integral equation is

$$h(x)u(x) = f(x) + \lambda \int_a^x R(x, y, u(y))dy \dots (1)$$

and this equation is said to be :

- Volterra integral equation of the first kind if $h(x) = 0$.
- Volterra integral equation of the second kind if $h(x) = 1$.
- Linear if $R(x, y, u(y)) = k(x, y)u(y)$, otherwise it is nonlinear.
- Homogeneous if $f(x) = 0$, otherwise it is nonhomogeneous.

And,for more details see[1].

Bernstein Polynomials [13]:

Bernstein polynomials are defined by

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i} \dots (2)$$

Where $\binom{n}{i} = \frac{n!}{i!(n-i)!}$

They are $n+1$ polynomials of degree n . For mathematical Convenience, we usually set $B_{i,n} = 0$ if $i < 0$ or $i > n$.

For $n=1$

$$B_{0,1}(t) = 1 - t \quad \text{and} \quad B_{1,1}(t) = t$$

For $n=2$

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$$B_{0,2}(t) = (1 - t)^2, \quad B_{1,2}(t) = 2t(1 - t) \text{ and } B_{2,2}(t) = t^2$$

A recursive definition of Bernstein polynomials is given by

$$B_{i,n}(t) = (1 - t)B_{i,n-1}(t) + tB_{i-1,n-1}(t)$$

These polynomials are non-negative over the interval [0,1] and form a partition of unity

$$\begin{aligned} [B_{0,1}(t) + B_{1,1}(t) &= 1, B_{0,2}(t) + B_{1,2}(t) \\ &+ B_{2,2}(t) \\ &= 1 \text{ and so on}]. \end{aligned}$$

The Modified Simpson's 3/8 Rule:

By the Bernstein polynomials

$$\sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1 - x)^{n-k}$$

Where f is a function, $k = 0, 1, \dots, n$.

Then

$$\begin{aligned} P(x) &= f\left(\frac{0}{n}\right) \binom{n}{0} x^0 (1 - x)^{n-0} \\ &+ f\left(\frac{1}{n}\right) \binom{n}{1} x(1 - x)^{n-1} \\ &+ f\left(\frac{2}{n}\right) \binom{n}{2} x^2 (1 - x)^{n-2} \\ &+ f\left(\frac{3}{n}\right) \binom{n}{3} x^3 (1 - x)^{n-3} \\ &+ \dots + f\left(\frac{n}{n}\right) \binom{n}{n} x^n (1 - x)^{n-n} \\ &= f(0)(1 - x)^n \\ &+ f\left(\frac{1}{n}\right) \left(\frac{n!}{1!(n-1)!}\right) x(1 - x)^{n-1} + \\ &f\left(\frac{2}{n}\right) \left(\frac{n!}{2!(n-2)!}\right) x^2 (1 - x)^{n-2} + \\ &f\left(\frac{3}{n}\right) \left(\frac{n!}{3!(n-3)!}\right) x^3 (1 - x)^{n-3} + \dots \\ &+ f(1)x^n \\ &= f(0)(1 - x)^n \\ &+ nf\left(\frac{1}{n}\right) x(1 - x)^{n-1} \\ &+ \frac{n(n-1)}{2!} f\left(\frac{2}{n}\right) x^2 (1 - x)^{n-2} + \end{aligned}$$

$$\frac{n(n-1)(n-2)}{3!} f\left(\frac{3}{n}\right) x^3 (1 - x)^{n-3} + \dots + f(1)x^n$$

By substituting $n=3$. Then

$$\begin{aligned} P(x) &= f(0)(1 - x)^3 \\ &+ 3f\left(\frac{1}{3}\right) x(1 - x)^2 \\ &+ 3f\left(\frac{2}{3}\right) x^2 (1 - x) \\ &+ 3f\left(\frac{3}{3}\right) x^3 (1 - x)^0 \end{aligned}$$

Let

$$\begin{aligned} f(0) &= y_0, f\left(\frac{1}{3}\right) = y_1, f\left(\frac{2}{3}\right) \\ &= y_2, f(1) = y_3 \\ P(x) &= y_0(1 - x)^3 + 3y_1x(1 - x)^2 \\ &+ 3y_2x^2(1 - x) \\ &+ y_3x^3 \dots (3) \end{aligned}$$

By integrating both sides of equation (3) From 0 to 1, one can have:-

$$\begin{aligned} \int_0^1 f(x)dx &\simeq \int_0^1 P(x)dx \\ &= \int_0^1 [y_0(1 - x)^3 + 3y_1x(1 - x)^2 \\ &+ 3y_2x^2(1 - x) \\ &+ y_3x^3] dx \\ &= \int_0^1 [y_0(1 - 3x + 3x^2 - x^3) \\ &+ 3y_1(x - 2x^2 + x^3) \\ &+ 3y_2(x^2 - x^3) \\ &+ y_3x^3] dx \\ &= y_0 \left(x - \frac{3}{2}x^2 + x^3 - \frac{1}{4}x^4\right) \\ &+ 3y_1 \left(\frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4\right) + 3y_2 \left(\frac{1}{3}x^3 - \frac{1}{4}x^4\right) + \frac{1}{4}y_3x^4 \Big|_0^1 \\ &= y_0 \left(1 - \frac{3}{2} + 1 - \frac{1}{4}\right) \\ &+ 3y_1 \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) \\ &+ 3y_2 \left(\frac{1}{3} - \frac{1}{4}\right) + \frac{1}{4}y_3 \\ &= \frac{1}{4}y_0 + \frac{3}{12}y_1 + \frac{3}{12}y_2 + \frac{1}{4}y_3 \\ &= \frac{1}{4}y_0 + \frac{1}{4}y_1 + \frac{1}{4}y_2 + \frac{1}{4}y_3 \end{aligned}$$

$$= \frac{1}{4}(y_0 + y_1 + y_2 + y_3)$$

$$= \frac{1}{4}(f_0 + f_1 + f_2 + f_3)$$

Now, by using the transformation $x = a + t(b - a)$, $h = \frac{b-a}{3}$ and above equation, we get:-

$$\int_a^b f(x)dx = \frac{3h}{4} [f_0 + f_1 + f_2 + f_3] \dots (4)$$

This formula is said to be modified Simpson's 3/8 rule.

The Composite Modified Simpson's 3/8 Rule:

The Composite modified Simpson's 3/8 rule can be derived by extending the modified Simpson's 3/8 rule.

This procedure is begin by dividing [a,b] into n subintervals (n is multiple of three), and applying the modified Simpson's 3/8 rule over each interval, then the sum of the results obtained for each interval is the approximate value of integral, that is

$$\int_b^a f(x)dx$$

$$= \int_a^{a+3h} f(x)dx + \int_{a+3h}^{a+6h} f(x)dx$$

$$+ \dots + \int_{a+(n-6)h}^{a+(n-3)h} f(x)dx$$

$$+ \int_{a+(n-3)h}^b f(x)dx, \text{ where } h = \frac{b-a}{n}$$

$$\int_a^b f(x)dx = \frac{3h}{4} [f(a) + f(a+h) + f(a+2h) + f(a+3h)]$$

$$+ \frac{3h}{4} [f(a+3h) + f(a+4h) + f(a+5h) + f(a+6h)] + \dots$$

$$+ \frac{3h}{4} [f(a+(n-6)h) + f(a+(n-5)h) + f(a+(n-4)h) + f(a+(n-3)h)]$$

$$+ \frac{3h}{4} [f(a+(n-3)h) + f(a+(n-2)h) + f(a+(n-1)h) + f(b)]$$

$$= \frac{3h}{4} [f(a) + f(a+h) + f(a+2h) + 2f(a+3h) + f(a+4h) + f(a+5h) + \dots + 2f(a+(n-3)h) + f(a+(n-2)h) + f(a+(n-1)h) + f(b)]$$

$$= \frac{3h}{4} \left[f(a) + \sum_{j=1,4,7,\dots}^{n-2} [f(x_j) + f(x_{j+1})] + 2 \sum_{j=3,6,9,\dots}^{n-3} f(x_j) + f(b) \right] \dots (5)$$

This formula is said to be the composite modified Simpson's 3/8 rule.

Numerical Solution for Solving The One-dimensional Volterra Linear Integral Equation Using The Composite Modified Simpson's 3/8 Rule:

In this section, we use the composite modified Simpson's 3/8 rule for solving the one-dimensional Volterra linear integral equations of the second kind given by

$$u(x) = f(x) + \lambda \int_a^x K(x,y)u(y)dy \quad , \quad x \geq a \dots (6)$$

First, we divide the interval [a, b] into n subintervals $[x_i, x_{i+1}]$, $i = 0, 1, 2, \dots, n - 1$, such that $x_i = a + ih$, $i = 0, 1, \dots, n$ where n is multiple of three and $h = \frac{b-a}{h}$. So, the problem here is to find the numerical solution of equation (6) at each x_i , $i = 0, 1, \dots, n$. Then by setting $x = x_i$ in equation (6), we get

$$\begin{aligned} u(x_i) &= f(x_i) \\ &+ \lambda \int_a^{x_i} k(x_i, y)u(y)dy, \quad i \\ &= 0, 1, \dots, n \dots (7) \end{aligned}$$

For $i = 3, 6, 9, \dots, n$. We approximate the integral that appeared in the right hand side of equation (7) by the composite modified Simpson's 3/8 rule to obtain:-

$$\begin{aligned} u_0 &= f_0 \\ u_i &= \frac{3h}{4} [k(x_i, x_0)u_0 \\ &+ \sum_{j=1,4,7,\dots}^{i-2} [k(x_i, x_j)u_j \\ &+ k(x_i, x_{j+1})u_{j+1}] \\ &+ 2 \sum_{j=3,6,9,\dots}^{i-3} k(x_i, x_j)u_j + k(x_i, x_i)u_i], \quad i = 3, 6, 9, \dots, n \dots (8) \end{aligned}$$

And, for $i \neq 3, 6, 9, \dots, n$, we approximate the integral that appeared in the right hand side of equation (7) by the composite modified Trapezoidal rule [13] to get

$$\begin{aligned} u_i &= f_i + \frac{\lambda h}{2} \left[k(x_i, x_0)u_0 \right. \\ &+ 2 \sum_{j=1}^{i-2} [k(x_i, x_j)u_j \\ &+ k(x_i, x_{j+1})u_{j+1}] \left. \right], \quad i \\ &\neq 3, 6, 9, \dots, n \dots (9) \end{aligned}$$

To illustrate this method, we consider the following examples:

Example (1):

Consider the one-dimensional Volterra linear integral equation of the second kind:-

$$u(x) = x + \frac{2}{5} \int_0^x xy u(y)dy \quad 0 \leq x \leq 2$$

whose exact solution is $u(x) = xe^{\frac{x^2}{5}}$, this equation can be solved numerically with the composite modified Simpson's 3/8 rule. First, we divide the interval [0, 2] into 9 subintervals, such that

$x_i = \frac{2i}{9}$, $i = 0, 1, \dots, 9$. Then $u_0 = f(0) = 0$, and the equation (8) becomes:-

$$\begin{aligned} u_i &= x_i + \frac{1}{30} \sum_{j=1,4,7,\dots}^{i-2} (x_i x_j u_j \\ &+ x_i x_j u_{j+1}) \\ &+ \frac{1}{15} \sum_{j=3,6,9,\dots}^{i-3} x_i x_j u_j \\ &+ \frac{1}{30} x_i^2 u_i, \quad i = 3, 6, 9, \dots (10) \end{aligned}$$

and the equation (9) becomes:-

$$\begin{aligned} u_i &= x_i + \frac{2}{45} \sum_{j=1}^{i-1} x_i x_j u_j \\ &+ \frac{1}{45} x_i^2 u_i, \quad i \\ &\neq 3, 6, 9, \dots (11) \end{aligned}$$

By setting $i = 1$ in the equation (11) one can get $u_1 = 0.2224663554$

By setting $i = 2$ in the equation (11) one can get $u_2 = 0.4473848062$

By setting $i = 3$ in the equation (10) one can get $u_3 = 0.6822919096$.

By continuing in this manner one can get the following values:

U ₀ =0	U ₁ =0.2224663554	U ₂ =0.4473848062
U ₃ =0.6822919096	U ₄ =0.9330498672	U ₅ =1.2202686732
U ₆ =1.5752706589	U ₇ =2.0084545071	U ₈ =2.6015170971
U ₉ =3.4840871196		

Second, we divide the interval [0, 2] into 18 subintervals, such that $x_i = \frac{i}{9}$, $i = 0, 1, \dots, 18$. Then $u_0 = f(0) = 0$, and the equations (8), (9) become:-

$$u_i = x_i + \frac{1}{60} \left[\sum_{j=1,4,7,\dots}^{i-2} (x_i x_j u_j + x_i x_{j+1} u_{j+1}) + 2 \sum_{j=3,6,9,\dots}^{i-3} x_i x_j u_j + x_i^2 u_i \right]$$

U ₀ =0	U ₁ =0.1111263548	U ₂ =0.2224052300
U ₃ =0.3342955701	U ₄ =0.4471364570	U ₅ =0.5620748374
U ₆ =0.6805480476	U ₇ =0.8028413818	U ₈ =0.9318813651
U ₉ =1.0704371891	U ₁₀ =1.2182194913	U ₁₁ =1.3813511291
U ₁₂ =1.5650052777	U ₁₃ =1.7675583096	U ₁₄ =2.0014643364
U ₁₅ =2.2770419403	U ₁₆ =2.5897590702	U ₁₇ =2.9658216063
U ₁₈ =3.4276679769		

Third, we divide the interval [0, 2] into 36 and 72 subintervals such that $x_i = \frac{i}{18}$, $i = 0, 1, 2, \dots, 36$, $x_i = \frac{i}{36}$, $i = 0, 1, 2, \dots, 72$

$$, i = 3, 6, 9, \dots, 18. \dots (12)$$

and

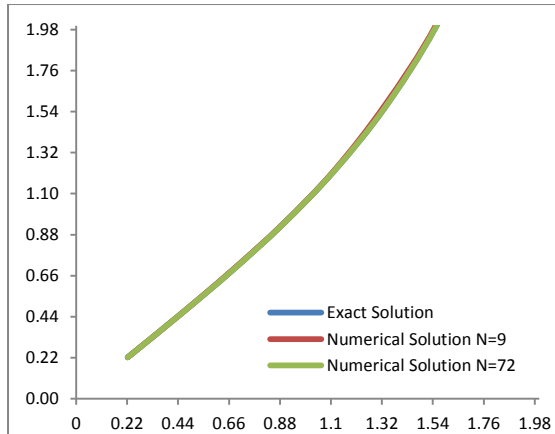
$$u_i = x_i + \frac{1}{45} \sum_{j=1}^{i-1} x_i x_j u_j + \frac{1}{90} x_i^2 u_i, \quad i \neq 3, 6, \dots, 18. \dots (13)$$

By setting $i = 1$ in the equation (13) one can get $u_1 = 0.1111263548$
 By setting $i = 2$ in the equation (13) one can get $u_2 = 0.2224052300$
 and by setting $i = 3$ in the equation (12) one can get $u_3 = 0.3342955701$
 And, By continuing in this manner one can get the following values:-

Respectively and by following the same previous steps, one can get the results that can be found in the appendix of example (1). Some of these results are tabulated down with the comparison with the exact solution.

Table (1) represents the exact and the numerical solution of example (1) at specific points for different values of n

X	Exact Solution	Numerical Solution			
		N=9	N=18	N=36	N=72
0.222222222	0.2223848585	0.2224663554	0.2224052300	0.2223899536	0.2223861320
0.444444444	0.4470533010	0.4473848062	0.4471364570	0.4470740553	0.4470584955
0.666666667	0.6799663130	0.6822919096	0.6805480476	0.6801117669	0.6800026778
0.888888889	0.9314983085	0.9330498672	0.9318813651	0.9315949132	0.9315223638
1.111111111	1.2175126789	1.2202686732	1.2182194913	1.2176873746	1.2175566340
1.333333333	1.5615934837	1.5752706589	1.5650052777	1.5624459538	1.5618065712
1.555555556	1.9992459998	2.0084545071	2.0014643364	1.9998129746	1.9993862624
1.777777778	2.5855576010	2.6015170971	2.5897590702	2.5865822277	2.5858171455
2	3.4092097306	3.4840871196	3.4276679769	3.4134233850	3.4103583731



Fig(1): Graph of exact and numerical solution of example 1.

Example(2):

Consider the one-dimensional Volterra linear integral equation of the second kind:-

$$u(x) = x - \frac{4}{35}x^{\frac{7}{2}} + \int_0^x (x - y)^{\frac{3}{2}} u(y) dy \quad 0 \leq x \leq 2$$

Whose exact solution is $u(x) = x$. We solve this equation numerically with the composite modified Simpson's 3/8 rule. First, we divide the interval $[0, 2]$ into 9 subintervals such that

$U_0=0$	$U_1=0.2216310035$	$U_2=0.4429149690$
$U_3=0.6576958875$	$U_4=0.8844957113$	$U_5=1.1046072143$
$U_6=1.3079521615$	$U_7=1.5427891825$	$U_8=1.7602438383$
$U_9=1.9464920052$		

Second, if we divide the interval $[0, 2]$ into 18 subintervals, such that

$x_i = \frac{i}{9}, i = 0, 1, \dots, 18$. Then the equations (8), (9) become:-

$$u_i = x_i - \frac{4}{35}x_i^{\frac{7}{2}} + \frac{1}{12} \sum_{j=1,4,7,\dots}^{i-2} \left[(x_i - x_j)^{\frac{3}{2}} u_j + (x_i - x_{j+1})^{\frac{3}{2}} u_{j+1} \right]$$

$x_i = \frac{2i}{9}, i = 0, 1, \dots, 9$. Then $u_0 = f(0) = 0$, and the equations (8),(9) become:-

$$u_i = x_i - \frac{4}{35}x_i^{\frac{7}{2}} + \frac{1}{6} \left\{ \sum_{j=1,4,7,\dots}^{i-2} \left[(x_i - x_j)^{\frac{3}{2}} u_j + (x_i - x_{j+1})^{\frac{3}{2}} u_{j+1} \right] \right\} + \frac{1}{3} \sum_{j=3,6,9,\dots}^{i-3} (x_i - x_j)^{\frac{3}{2}} u_j, \quad i = 3, 6, 9, \dots (14)$$

and

$$u_i = x_i - \frac{4}{35}x_i^{\frac{7}{2}} + \frac{2}{9} \sum_{j=1}^{i-1} (x_i - x_j)^{\frac{3}{2}} u_j, \quad i \neq 3, 6, 9, \dots (15)$$

By setting $i = 1$ in equation (15) one can get $u_1 = 0.2216310035$.

By setting $i = 2$ in equation (15) one can get $u_2 = 0.4429149690$.

By setting $i = 3$ in equation (14) one can get $u_3 = 0.6576958875$.

And, by continuing in this manner one can get the following values:-

$$+ \frac{1}{6} \sum_{j=3,6,9,\dots}^{i-3} (x_i - x_j)^{\frac{3}{2}} u_j, \quad i = 3, 6, 9, \dots, 18. \quad \dots (16)$$

and

$$u_i = x_i - \frac{4}{35}x_i^{\frac{7}{2}} + \frac{1}{9} \sum_{j=1}^{i-1} (x_i - x_j)^{\frac{3}{2}} u_j, \quad i \neq 3, 6, 9, \dots, 18. \quad \dots (17)$$

By setting $i = 1$ in equation (17) one can get $u_1 = 0.1110588543$.

By setting $i = 1$ in equation (17) one can get $u_2 = 0.4429149690$.

By setting $i = 3$ in equation (16) one can get $u_3 = 0.3325444915$

$u_0=0$	$u_1=0.1110588543$	$u_2=0.2220880359$
$u_3=0.3325444915$	$u_4=0.4440838939$	$u_5=0.5550527639$
$u_6=0.6645903134$	$u_7=0.7769297456$	$u_8=0.8878314093$
$u_9=0.9962027164$	$u_{10}=1.1095483963$	$u_{11}=1.2203506870$
$u_{12}=1.3273025035$	$u_{13}=1.4418215959$	$u_{14}=1.5524695371$
$u_{15}=1.6577165689$	$u_{16}=1.7735580777$	$u_{17}=1.8839681446$
$u_{18}=1.9871689996$		

And, by continuing in this manner one can get the following values:-

Third, we divide the interval $[0, 2]$ into 36 and 72 subintervals such that

$$x_i = \frac{i}{18}, \quad i = 0, 1, \dots, 36 \quad \text{and} \quad x_i = \frac{i}{36}, \quad i = 0, 1, \dots, 72$$

Respectively and by following the same previous steps one can get the results that can be found in the appendix of example (2). Some of these results are tabulated down with the comparison with the exact solution.

Table (2) represents the exact and the numerical solution of example (2) at specific points for different values of n

X	Exact Solution	Numerical Solution			
		N=9	N=18	N=36	N=72
0.222222222	0.222222222	0.2216310035	0.2220880359	0.2221907850	0.2222147077
0.444444444	0.444444444	0.4429149690	0.4440838939	0.4443581710	0.4444235479
0.666666667	0.666666667	0.6576958875	0.6645903134	0.6661767275	0.6665490295
0.888888889	0.888888889	0.8844957113	0.8878314093	0.8886332252	0.8888264788
1.111111111	1.111111111	1.1046072143	1.1095483963	1.1107322356	1.1110185287
1.333333333	1.333333333	1.3079521615	1.3273025035	1.3318853591	1.3329816693
1.555555556	1.555555556	1.5427891825	1.5524695371	1.5548073905	1.5553725636
1.777777778	1.777777778	1.7602438383	1.7735580777	1.7767539344	1.7775274066
2	2.000000000	1.9464920052	1.9871689996	1.9969012387	1.9992444590

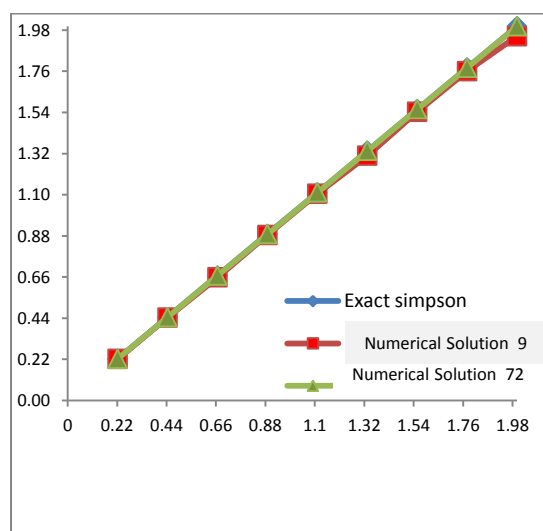


Fig (2): Graph of exact and numerical solution of example 2

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اشتقاق قاعدة سمبسون المعدلة باستخدام متعددات حدود بيرنشتاين لحل معادلات فولتيرا التكاملية

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الخلاصة :

قد تم استخدام متعددات حدود بيرنشتاين لأشتقاق قاعدة سمبسون (3/8) المعدلة وذلك لحل معادلات فولتيرا التكاملية الخطية من النوع الثاني. وتبين ان الحل باستخدام هذه الطريقة قريب جدا من الحل التحليلي (المضبوط).