On Characterization of Some Extreme Value Distributions Through the Conditional Expectations of Generalized Order Statistics

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Abstract:

Let $X_1, X_2, \ldots\ldots, X_n$ be continuous independent and identically distributed (i.i.d) random variable with d.f. $F(x)$ and p.d.f. $f(x)$.Characterization theorems for a general class of distributions are presented in terms of the function $E\left[g(X_{j:n:m:m+1} \middle| X_{j-p:n:m:m+1} = x, X_{j+q:n:m:m+1} = y\right] = A(x,y)$ where $k = m+1$.In this article We give characterization conditions for the frechet distribution such that $F(x) = 1 - e^{-x^{-\alpha}}, \quad x > 0\ , \alpha > 0$

and generalized extreme value distribution such that $F(x) = e^{-(1-\varepsilon x)^{1/\varepsilon}} if \varepsilon \neq 0$
by conditional expectation of generalized order statistics .

الخلاصة : لتكن $X_1, X_2, \ldots\ldots, X_n$ متغيرات عشوائية مستمرة ومستقلة ومتماثلة التوزيع بدالة الكثافة $F(x)$
ودالة التوزيع$f(x)$. مميزات المبرهنات طبقت على توزيعات عامة بواسطة الدالة
. $E\left[g(X_{j:n:m:m+1} \middle| X_{j-p:n:m:m+1} = x, X_{j+q:n:m:m+1} = y\right] = A(x,y)$ where $k = m+1$
في هذا البحث سوف نعطي مميزات وشروط لتوزيع فرجيت الذي دالته$F(x) = 1 - e^{-x^{-\alpha}}$
$x > 0\ , \alpha > 0$,وكذلك لتوزيع تعميم القيم المتطرفة الذي دالته$F(x) = e^{-(1-\varepsilon x)^{1/\varepsilon}}$
$if \varepsilon \neq 0$ بواسطة التوقع الشرطي لتعميم الإحصاءات المرتبة .

Introduction.

Mathematical foundation of extreme value limit laws, which were first derived heuristically by Fisher and Tippett in 1928. Many scientists were interested in generalized order statistics, beginning with Kamps (1995), who introduced this topic. He considered the ordinary order statistics, record values and sequential order statistics are special cases of this generalized order statistics.
Aleem generalized the results of Kamps and developed some generalized properties of generalized order statistics in 1998. Also, Gajek and Okolewski (2000) gave some evolutions for generalized order statistics. The marginal p.d.f and density function of generalized order statistics are given in Kamps and Cramer (2001).
Let $X_{1:n}, X_{2:n}, \ldots\ldots, X_{n:n}$ be the corresponding order statistics . $X_{1:n:m:k}, X_{2:n:m:k}, \ldots\ldots, X_{n:n:m:k}$ is denoted by generalized order statistics, where $n \in N, k > 0, m_1, \ldots, m_{n-1} \in R, M_r = \sum_{j=r}^{n-1} m_j$, $1 \le r \le n-1$
be parameters such that $\gamma_r = k + (n-r) + M_r > 0$, for all $r \in \{1, \ldots, n-1\}$, and let $m = (m_1, \ldots, m_{n-1})$,if $n \ge 2, m \in R$ arbitrary.
Assume that the random variable X has absolutely continuous and strictly increasing d.f F with left and right extremities a_F and b_F, respectively . Let $h(x)$ be a differentiable real valued function on [0,1] and the condition $h'(x) \neq \frac{h(y)-h(x)}{y-x}$ is valid for all 0< $x < y < 1$, *l*et G be also an absolutely continuous and strictly increasing d.f with left and right extremities $a_G = a_F$ and $b_G = b_F$. Then $F(x) = 1 - \left(1 - G(x)\right)^{1/(m+1)}$ if and only if the representation

$$E\left\{\frac{1}{s}\sum_{p=1}^{s} h'\left(G\left(X_{j:n:m:m+1}\right)\right) \middle| X_{j-p:n:m:m+1} = x, X_{j+s+1-p:n:m:m+1} = y\right\}$$
$$= \frac{h\left(G(y)\right) - h\left(G(x)\right)}{G(y) - G(x)},$$

holds for all $a_X < x < y < b_X$. The number j, n, s are fixed and satisfies the condition $s+1 \le j \le n-s$. In this search by this condition we applied a characterization theorem to frechet and Generalized Extreme Value distributions by using the properties of conditional expectations of generalized order statistics.

1. Characterization for Frechet Distribution through the Conditional Expectations of Generalized Order Statistics

The absolutely continuous random variable X strictly increasing d.f having support $[0,\infty)$ that has a Frechet distribution

$$F(x) = 1 - e^{-x^{-\alpha}}, \qquad x > 0 \quad , \qquad \alpha > 0$$

if and only if the representation

$$\frac{1}{s}\sum_{p=1}^{s} E\left[X_{j:n:m:m+1} \middle| X_{j-p:n:m:m+1} = x, X_{j+s+1-p:n:m:m+1} = y\right] = \frac{h\big(G(y)\big) - h\big(G(x)\big)}{G(y) - G(x)}$$

$$= \frac{h\left(1 - \left(e^{-y^{-\alpha}}\right)^{m+1}\right) - h\left(1 - \left(e^{-x^{-\alpha}}\right)^{m+1}\right)}{(e^{-x^{-\alpha}})^{m+1} - (e^{-y^{-\alpha}})^{m+1}}$$

$$= \frac{-ln\left(e^{-y^{-\alpha}}\right)\left(1 - \left(e^{-y^{-\alpha}}\right)^{m+1}\right) + \frac{1 - \left(e^{-y^{-\alpha}}\right)^{m+1}}{m+1} + \frac{ln\left(e^{-y^{-\alpha}}\right)^{m+1}}{m+1}}{(e^{-x^{-\alpha}})^{m+1} - (e^{-y^{-\alpha}})^{m+1}}$$

$$\frac{+ln\left(e^{-x^{-\alpha}}\right)\left(1 - \left(e^{-x^{-\alpha}}\right)^{m+1}\right) - \frac{1 - \left(e^{-x^{-\alpha}}\right)^{m+1}}{m+1} - \frac{ln\left(e^{-x^{-\alpha}}\right)^{m+1}}{m+1}}{(e^{-x^{-\alpha}})^{m+1} - (e^{-y^{-\alpha}})^{m+1}}$$

$$= \frac{y^{-\alpha}\left(1 - \left(e^{-y^{-\alpha}}\right)^{m+1}\right) - \frac{\left(e^{-y^{-\alpha}}\right)^{m+1}}{m+1} - y^{-\alpha}}{(e^{-x^{-\alpha}})^{m+1} - (e^{-y^{-\alpha}})^{m+1}}$$

$$+ \frac{-x^{-\alpha}\left(1 - \left(e^{-x^{-\alpha}}\right)^{m+1}\right) + \frac{\left(e^{-x^{-\alpha}}\right)^{m+1}}{m+1} + x^{-\alpha}}{(e^{-x^{-\alpha}})^{m+1} - (e^{-y^{-\alpha}})^{m+1}}$$

$$= \frac{1}{m+1} + \frac{x^{-\alpha}\left(e^{-x^{-\alpha}}\right)^{m+1} - y^{-\alpha}\left(e^{-y^{-\alpha}}\right)^{m+1}}{(e^{-x^{-\alpha}})^{m+1} - (e^{-y^{-\alpha}})^{m+1}},$$

holds for all $0 \le x < y < \infty$. The result follows from Theorem by a choice of

$$h(x) = -ln(1-x)^{1/(m+1)}x + \frac{x}{m+1} + \frac{ln(1-x)}{m+1} - \frac{1}{m+1}$$

$$h'(x) = -ln(1-x)^{1/(m+1)},$$

and

$$G(x) = 1 - \left(e^{-x^{-\alpha}}\right)^{m+1}. \blacksquare$$

2. Characterization for Generalized Extreme Value Distribution through the Conditional Expectations of Generalized Order Statistics

The absolutely continuous random variable X strictly increasing d.f has a distribution Generalized Extreme Value Distribution(GEV)

$F(x) = e^{-(1-\varepsilon x)^{1/\varepsilon}} if \varepsilon \neq 0$

Let $\qquad G(x) = 1 - \left(1 - e^{-(1-\varepsilon x)^{1/\varepsilon}}\right)^{m+1}$

And $\qquad h\,(x) = -ln(1-x)^{\frac{1}{m+1}}x + \frac{x}{m+1} + \frac{ln(1-x)}{m+1} - \frac{1}{m+1}$

$$\frac{1}{s}\sum_{p=1}^{s} E\left[X_{j:n:m:m+1}\middle|X_{j-p:n:m:m+1} = x, X_{j+s+1-p:n:m:m+1} = y\right]$$

$$= \frac{h\big(G(y)\big) - h\big(G(x)\big)}{G(y) - G(x)}$$

$$= \frac{h\left(1-\left(1-e^{-(1-\varepsilon y)^{1/\varepsilon}}\right)^{m+1}\right) - h\left(1-\left(1-e^{-(1-\varepsilon x)^{1/\varepsilon}}\right)^{m+1}\right)}{\left(1-e^{-(1-\varepsilon x)^{1/\varepsilon}}\right)^{m+1} - \left(1-e^{-(1-\varepsilon y)^{1/\varepsilon}}\right)^{m+1}}$$

$$= \frac{-ln\left(1-e^{-(1-\varepsilon y)^{1/\varepsilon}}\right)\left(1-\left(1-e^{-(1-\varepsilon y)^{1/\varepsilon}}\right)^{m+1}\right) + \frac{\left(1-\left(1-e^{-(1-\varepsilon y)^{1/\varepsilon}}\right)^{m+1}\right)}{m+1}}{\left(1-e^{-(1-\varepsilon x)^{1/\varepsilon}}\right)^{m+1} - \left(1-e^{-(1-\varepsilon y)^{1/\varepsilon}}\right)^{m+1}}$$

$$\frac{+\frac{ln\left(1-e^{-(1-\varepsilon y)^{1/\varepsilon}}\right)^{m+1}}{m+1} + ln\left(1-e^{-(1-\varepsilon x)^{1/\varepsilon}}\right)\left(1-\left(1-e^{-(1-\varepsilon x)^{1/\varepsilon}}\right)^{m+1}\right)}{\left(1-e^{-(1-\varepsilon x)^{1/\varepsilon}}\right)^{m+1} - \left(1-e^{-(1-\varepsilon y)^{1/\varepsilon}}\right)^{m+1}}$$

$$\frac{-\frac{\left(1-\left(1-e^{-(1-\varepsilon x)^{1/\varepsilon}}\right)^{m+1}\right)}{m+1} - \frac{ln\left(1-e^{-(1-\varepsilon x)^{1/\varepsilon}}\right)^{m+1}}{m+1}}{\left(1-e^{-(1-\varepsilon x)^{1/\varepsilon}}\right)^{m+1} - \left(1-e^{-(1-\varepsilon y)^{1/\varepsilon}}\right)^{m+1}}$$

$$= \frac{-(1-\varepsilon y)^{1/\varepsilon}\left(1-\left(1-e^{-(1-\varepsilon y)^{1/\varepsilon}}\right)^{m+1}\right) - \frac{\left(1-e^{-(1-\varepsilon y)^{1/\varepsilon}}\right)^{m+1}}{m+1}}{\left(1-e^{-(1-\varepsilon x)^{1/\varepsilon}}\right)^{m+1} - \left(1-e^{-(1-\varepsilon y)^{1/\varepsilon}}\right)^{m+1}}$$

$$\frac{+\frac{ln\left(1-e^{-(1-\varepsilon y)^{1/\varepsilon}}\right)^{m+1}}{m+1} + (1-\varepsilon x)^{1/\varepsilon}\left(1-\left(1-e^{-(1-\varepsilon x)^{1/\varepsilon}}\right)^{m+1}\right)}{\left(1-e^{-(1-\varepsilon x)^{1/\varepsilon}}\right)^{m+1} - \left(1-e^{-(1-\varepsilon y)^{1/\varepsilon}}\right)^{m+1}}$$

$$\frac{+\frac{\left(1-e^{-(1-\varepsilon x)^{1/\varepsilon}}\right)^{m+1}}{m+1} - \frac{ln\left(1-e^{-(1-\varepsilon x)^{1/\varepsilon}}\right)^{m+1}}{m+1}}{\left(1-e^{-(1-\varepsilon x)^{1/\varepsilon}}\right)^{m+1} - \left(1-e^{-(1-\varepsilon y)^{1/\varepsilon}}\right)^{m+1}}$$

$$= \frac{1}{m+1} + \frac{(1-\varepsilon y)^{1/\varepsilon}\left(1-e^{-(1-\varepsilon y)^{1/\varepsilon}}\right)^{m+1} - (1-\varepsilon x)^{1/\varepsilon}\left(1-e^{-(1-\varepsilon x)^{1/\varepsilon}}\right)^{m+1}}{\left(1-e^{-(1-\varepsilon x)^{1/\varepsilon}}\right)^{m+1} - \left(1-e^{-(1-\varepsilon y)^{1/\varepsilon}}\right)^{m+1}}.$$

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