New Fuzzy Normed Spaces

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Abstract:
In this paper the research introduces a new definition of a fuzzy normed space then the related concepts such as fuzzy continuous, convergence of sequence of fuzzy points and Cauchy sequence of fuzzy points are discussed in details.

Key words: fuzzy set, fuzzy normed space, sequence of fuzzy points, fuzzy continuous.

S1:Basic Concepts About Fuzzy Sets:

Definition 1.1: [1]
Let X be a nonempty set of elements. A fuzzy set Ã in X is characterized by a membership function, $\mu_{\tilde{A}}: X \rightarrow [0.1]$. Then Ã can be written by $\tilde{A}=\{(x, \mu_{\tilde{A}}(x)) | x \in X, 0 \leq \mu_{\tilde{A}}(x) \leq 1\}$

Definition 1.2: [1]
Let Ã and Ũ be two fuzzy sets in X then

1. $\tilde{A} \subseteq \tilde{U} \Leftrightarrow \mu_{\tilde{A}}(x) \leq \mu_{\tilde{U}}(x)$ for all $x \in X$.
2. $\tilde{A} = \tilde{U} \Leftrightarrow \mu_{\tilde{A}}(x) = \mu_{\tilde{U}}(x)$ for all $x \in X$.
3. Then complement of Ã (denoted by $\tilde{A}^c$) is also a fuzzy set with membership function $\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(X)$ for all $x \in X$.
4. $\tilde{A} = \emptyset \Leftrightarrow \mu_{\tilde{A}}(x) = 0$ for all $x \in X$, where $\emptyset$ is the empty fuzzy set.

Definition 1.3: [2],[3]
A fuzzy point P_x in X is a fuzzy set with membership function

$$\mu_{P_x}(y) = \begin{cases} \alpha & if\ y = x \\ 0 & \text{otherwise} \end{cases}$$

For all $y \in X$ where $0 < \alpha < 1$. We denote this fuzzy point by x_α or (x, α).

Definition 1.4: [4],[5]
Two fuzzy points x_α and y_β are said to be equal if x= y and $\alpha = \beta$ where $\alpha, \beta \in (0,1]$.

Definition 1.5: [5],[6]
Let x_α be a fuzzy point and Ã a fuzzy set in X. then x_α is said to be in Ã or belongs to Ã denoted by $x_\alpha \in \tilde{A}$ if $\alpha \leq \mu_{\tilde{A}}(x)$.

Definition 1.6: [6],[7]
Let f be a function from a nonempty set X into a nonempty set Y. If Ũ is a fuzzy set in Y then $f^{-1}(\tilde{U})$ is a fuzzy set in X with membership function $\mu_{f^{-1}(\tilde{U})} = \mu_{\tilde{U}}$ of.

If Ã is a fuzzy set in X then f (Ã) is a fuzzy set in Y with membership

$$\mu_{f(\tilde{A})}(y) = \begin{cases} \sup \{\mu_{\tilde{A}}(x) \mid x \in f^{-1}(y)\} \\ 0 \quad \text{otherwise} \end{cases}$$

if $f^{-1}(y) \neq \emptyset$

For all $y \in Y$ where $f^{-1}(y) = \{x \in X \mid f(x) = y\}$

Proposition 1.7: [7],[5]
Let $f : X \rightarrow Y$ be a function then for a fuzzy point x_α in X , $f(x_\alpha)$ is a fuzzy point in Y and $f(x_\alpha) = f(x)_\alpha$.

Definition 1.8: [8],[3]
Let X be a vector space over field **K** and let Ã be a fuzzy set in X. then Ã is called a fuzzy subspace of X if for all $x, y \in X$ and $\lambda \in$ **K**.

(i) $\mu_{\tilde{A}}(x+y) \geq \min \{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$

(ii) $\mu_{\tilde{A}}(\lambda x) \geq \mu_{\tilde{A}}(x)$

S2: Fuzzy Normed Spaces

Definition 2.1: let X be a vector space over field

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K(**K**= R or **K**= C).Put I=[o,1] then Ñ:X× I → I is said to be a fuzzy norm on X if for each x , y ∈ X and λ ∈ K

(N_1)if $\alpha = 0$ then Ñ(x,α)=0 .

(N_2)if $\alpha \neq 0$ then Ñ(x,α)=0 if and only if x= 0.

(N_3)Ñ(λx,α) = |λ| Ñ(x,α)

(N_4)Ñ(x + y,α) ≤ Ñ(x,α) + Ñ(y,α)

(N5) if $0 < \sigma \leq \alpha < 1$ then Ñ(x,α) ≤ Ñ(x,σ) and there exists $0 < \alpha_n < \alpha$ such that $\lim_{n\to\infty}$Ñ(x,α_n) = Ñ(x,α) .

Then Ñ is called fuzzy norm and (X,Ñ) is called fuzzy normed space.

Now we introduce some propositions to explain the relation between ordinary normed space and fuzzy normed space .

Proposition 2.2:

Let (X, ||.||) be an ordinary normed space, define Ñ(x,α)=$\frac{1}{\alpha}\|x\|$ for $\alpha > 0$ and Ñ(x,α)=0 for α= 0. Then (X,Ñ) is a fuzzy normed space

Proof: let x, y ∈ X and γ ∈ **K**. Then

(N_1) if α = 0 then Ñ(x,α) = 0 .

(N_2)if $\alpha \neq 0$ then Ñ(x,α)=0⇔ $\frac{1}{\alpha}\|x\|=0$ ⇔ $\|x\| = 0$ ⇔ x=0 .

(N_3)Ñ(γx ,α) = $\frac{1}{\alpha} \| \gamma x \| = \frac{|\gamma|}{\alpha} \| x \| = |\gamma|$ Ñ(x,α) .

(N_4)Ñ(x +y,α) = $\frac{1}{\alpha}\| x + y \| \leq \frac{1}{\alpha}\|x\| + \frac{1}{\alpha}\|y\|$ = Ñ(x,α)+Ñ(y,α) .

(N_5) if $0 < \sigma \leq \alpha < 1$ then $\frac{||x||}{\alpha} \leq \frac{||x||}{\sigma}$ that is Ñ(x,α) ≤ Ñ(x,σ) then there exists $0 < \alpha_n < \alpha$ [This is possible by taking $\alpha_n = (1-\frac{1}{n})\alpha$] such that $\lim_{n\to\infty}$Ñ(x,α_n) = Ñ(x,α) **.**

The proof of the following result is clear. Hence is omitted

Proposition 2.3:

Let (X,Ñ) be a fuzzy normed space if for each x ∈ X define

|| x || = Ñ(x, α) ,for some α ∈ (0,1] .Then (X,‖.‖) is an ordinary normed space.

Example 2.4:

Let X=**R**, then Ñ(x,α) = $\frac{1}{\alpha}|x|$ is a fuzzy norm on **R** by proposition 2.2 called the usual fuzzy norm.

Remark 2.5:

From the definition 2.1 we obtain by induction the generalized of (N_4)

Ñ$(x_1 - x_n,\alpha)$ ≤ Ñ(x_1-x_2, α) + Ñ$(x_2 - x_3, \alpha)$ +…+ Ñ$(x_{n-1} - x_n, \alpha)$

Where $(x_2, \alpha), (x_3, \alpha), \ldots, (x_{n-1}, \alpha) \in X$.

Definition 2.6:

A fuzzy subspace Ŷ of a fuzzy normed space (X, Ñ) is a fuzzy subspace of X considered as a vector space with the fuzzy norm obtained by restricting the fuzzy norm on X to Ŷ.

S3: Open Fuzzy Sets, Closed Fuzzy Sets, Fuzzy Continuity of Functions

In this section we introduce some new concepts

Definition 3.1:

Let (X, Ñ) be a fuzzy normed space. Given $x_\alpha \in X$, where $\alpha \in (0,1]$ and a real number $r > 0$

(i) Õ(x_α,r) $=\{y_\beta \in X$: Ñ$(x- y,\lambda) < r\}$ is open fuzzy ball, where $\lambda=\min\{\alpha,\beta\}$.

(ii) B̃(x_α,r) $=\{y_\beta \in X$: Ñ$(x- y,\lambda) \leq r\}$ is closed fuzzy ball, where $\lambda=\min\{\alpha, \beta\}$.

(iii) $S(x_\alpha,r)$ $=\{y_\beta \in X$: Ñ$(x- y,\lambda) = r\}$is fuzzy sphere, where $\lambda=\min\{\alpha, \beta\}$.

In all three cases, x_α is called the center and r is radius.

Definition 3.2:

A fuzzy set Ã in fuzzy normed space (X, Ñ) is said to be open if it contains a fuzzy ball about each of its, fuzzy element.

A fuzzy set D̃ is said to be closed fuzzy set if it's complement is open fuzzy set.

Definition 3.3:

Let (X, **Ñ**) be a fuzzy normed space, an open fuzzy ball Õ(x_α, ε) of radius ε is often called an ε -neighborhood of x_α (here ε >0).

By a neighborhood of x_α we mean a fuzzy set of X which contains an ε -neighborhood of x_α.

Definition 3.4:

The fuzzy point x_α is called an interior point of the fuzzy set Ã if Ã is a neighborhood of x_α .The interior of Ã is the set of all interior fuzzy points of Ã and is denoted by int(Ã).

Int(Ã) is open fuzzy set and is the largest open fuzzy set contained in Ã.

Definition 3.5:

Let (X, $\tilde{N}_1$) and (Y,$\tilde{N}_2$) be a fuzzy normed spaces. A mapping T : X → Y is said to be fuzzy continuous at the fuzzy point x_α ∈ X where α ∈ (0,1] if for every ε >0 there is δ >0 such that $\tilde{N}_2(T(y) - T(x), \lambda) < \varepsilon$, for all y_β ∈ X satisfying $\tilde{N}_1(y - x, \lambda) < \delta$, where λ = min{α, β}. T is said to be fuzzy continuous if it is fuzzy continuous at every fuzzy point x_α ∈ X.

Theorem 3.6:

A mapping T of a fuzzy normed space (X, $\tilde{N}_1$) into a fuzzy normed space (Y,$\tilde{N}_2$) is fuzzy continuous if and only if the inverse image of any open fuzzy set in Y is open fuzzy set in X.

Proof:

Suppose T is fuzzy continuous. Let Õ be open fuzzy set in Y and Ũ is the inverse image of Õ i.e T^{-1} (Õ) = Ũ. If Ũ=∅ it is open fuzzy set. Let Ũ≠ ∅, for any x_α ∈ Ũ where α ∈ (0,1]. Let y_α =T(x_α)= $T(x)_\alpha$ [By proposition 1.7] since Õ is open, it contains as ε-neighborhood N_2 of y_α. Since T is fuzzy continuous, x_α has an δ-neighborhood N_1 which is mapped into N_2. Since N_2 ⊂ Õ we have N_1 ⊂ Ũ so that Ũ is open fuzzy set because x_α ∈ Ũ was arbitrary.

Conversely, assume that the inverse image of every open fuzzy set in Y is open fuzzy set in X. Then for each x_α∈ X where α ∈(0,1] and any ε-neighborhood N_2 of $T(x)_\alpha$ the inverse image of N_1 is open since N_2 is open and N_1 contains x_α. Hence N_1 is also contain a δ-neighborhood of x_α which is mapped into N_2 because N_1 is mapped into N_2. Consequently T is fuzzy continuous at x_α. Since x_α ∈ X was arbitrary T is fuzzy continuous.

Definition 3.7:

Let Ã be a fuzzy set in a fuzzy normed space (X,Ñ). Then a fuzzy point x_α ∈ X where α ∈(0,1] (which may or not be a fuzzy element of Ã) is called a limit of Ã if every neighborhood of x_α contains at least one fuzzy element y_β ∈ Ã distinct from x_α. The fuzzy set consisting of Ã and its limit fuzzy points is called closure of Ã and is denoted by cl(Ã). It is the smallest closed fuzzy set containing Ã.

Definition 3.8:

A fuzzy set Ã of a fuzzy normed space (X,Ñ) is said to be dense in X if cl(Ã)=X.

S4: Convergence, Cauchy Fuzzy Sequences

In this section we will introduce some new concepts and results

Definition4.1:

A sequence of fuzzy points {(x_n, α_n)} in a fuzzy normed space (X, Ñ) is said to be convergent to x_α in X where α, α_n ∈ (0,1] for i=1, 2, … if $\lim_{n\to\infty} \tilde{N}(x_n - x, \lambda) = 0$, x_α is called the limit if {(x_n, α_n)}and we write $\lim_{n\to\infty}$ (x_n, α_n)= x_α or simply (x_n, α_n)→ x_α ,if {(x_n, α_n)} is not convergent then it is called divergent.

Remark 4.2:

If (x_n, α_n) → x_α, an ε > 0 being given, there is a positive integer N such that (x_n, α_n) with n > N lie in ε-neighborhood Õ(x_α, ε) of x_α that is: $\tilde{N}(x_n - x, \lambda) < \varepsilon$, for all n > N.

Definition4.3:

We call a nonempty fuzzy set Ã in (X,Ñ) bounded if its fuzzy diameter

$\delta(\tilde{A}) = \sup \{\tilde{N}(x - y, \lambda): x_\alpha, y_\beta \in \tilde{A}, \lambda = \min\{\alpha, \beta\}\}$ is finite.

Definition4.4:

In a fuzzy normed space $(X,\tilde{N})$ we call a sequence $\{(x_n, \alpha_n)\}$ is bounded if the corresponding fuzzy set is bounded.

Remark 4.5:

If $\tilde{A}$ is a bounded fuzzy set then $\tilde{A} \subset \tilde{O}(x_\alpha, r)$ where $x_\alpha \in \tilde{A}$ is any fuzzy element and $r>0$ is a (sufficiently large) real number.

Theorem 4.6:

Let $(X,\tilde{N})$ be a fuzzy normed space. Then (i) a convergent fuzzy sequence in X is bounded and its limit is unique.

(ii) if $(x_n, \alpha_n) \to x_\alpha$ and $(y_m, \beta_m) \to y_\beta$ in X ,where $\alpha, \alpha_i, \beta, \beta_i \in (0.1]$ i= 1, 2, …

Then $\tilde{N}(x_n - y_m, \lambda) \to \tilde{N}(x - y, \lambda)$,where $\lambda = \min\{\alpha, \alpha_i, \beta\, \beta_i\}$.

Proof:

(i)Suppose that $(x_n, \alpha_n) \to x_\alpha$ then taking $\varepsilon = 1$ we can find $N > 0$ such that $\tilde{N}(x_n - x_\alpha) < 1$ for all $n > N$. Hence by remark 2.5 for all n we have $\tilde{N}(x_n - x_\alpha, \lambda) < 1 + a$, where $a = \max\{\tilde{N}(x_1 - x_2, \lambda), \tilde{N}(x_2 - x_3, \lambda), \dots, \tilde{N}(x_n - x_\alpha, \lambda)\}$, where $\lambda = \min\{\alpha_1, \alpha_2, \dots, \alpha_n, \dots\}$ [Here λ exists since this set is bounded].This shows that $\{(x_n, \alpha_n)\}$ is bounded. Now, assuming that $(x_n, \alpha_n) \to x_\alpha$ and $(x_n, \alpha_n) \to z_\beta$, we obtain from (N_4) $0 \le \tilde{N}(x_\alpha - z_\beta, \lambda) \le \tilde{N}(x_\alpha - x_n, \lambda) + \tilde{N}(x_n - z_\beta, \lambda) \to 0+0$ .where $\lambda = \min\{\beta, \alpha_1, \alpha_2, \dots, \alpha_n, \dots\}$[Here λ exists since this set is bounded] .

Thus $\tilde{N}(x_\alpha - z_\beta, \lambda) = 0$ which implies that $x_\alpha = z_\beta$.

(ii)By remark 2.5 we have $\tilde{N}(x_n - y_m, \lambda) \le \tilde{N}(x_n - x_\alpha, \lambda) + \tilde{N}(x_\alpha - y_\beta, \lambda) + \tilde{N}(y_\beta - y_m, \lambda)$.

Hence we obtain

$\tilde{N}(x_n - y_m, \lambda) - \tilde{N}(x_\alpha - y_\beta, \lambda) \le \tilde{N}(x_n - x_\alpha, \lambda) + \tilde{N}(y_m - y_\beta, \lambda)$.

And a similar inequality by interchanging (x_n, α_n) and x_α as well as (y_m, β_m) and y_β and multiplying by -1. Together $|\tilde{N}(x_n - y_m, \lambda) - \tilde{N}(x_\alpha - y_\beta, \lambda)| \le \tilde{N}(x_n - x_\alpha, \lambda) + \tilde{N}(y_m - y_\beta, \lambda) \to 0$ as $n \to \infty$. Where $\lambda = \min\{\alpha, \beta, \alpha_n, \beta_m\}$.

Definition 4.7:

A sequence $\{(x_n,\alpha_n)\}$ in a fuzzy normed space $(X,\tilde{N})$ is said to be Cauchy if for

every $\varepsilon > 0$ there is integer $N > 0$ such that $\tilde{N}(x_m - x_n, \lambda) < \varepsilon$ for every $m,n > N$,

where $\lambda = \min\{\alpha_1, \alpha_2, \dots, \alpha_n, \dots\}$[Here λ exists since this set is bounded] .

Theorem 4.8:

Every convergent fuzzy sequence in a fuzzy normed space $(X,\tilde{N})$ is Cauchy.

Proof:

Let $\{(x_n, \alpha_n)\}$ be a sequence of fuzzy points in X such that $(x_n, \alpha_n) \to x_\alpha$ then for every $\varepsilon > 0$ there is an integer $N > 0$ such that $\tilde{N}(x_n - x_\alpha, \lambda) < \frac{\varepsilon}{2}$ for all $n > N$.

Hence by (N_4) we obtain for $m,n > N$.

$\tilde{N}(x_m - x_n, \lambda) \le \tilde{N}(x_m - x_\alpha, \lambda) + \tilde{N}(x_\alpha - x_n, \lambda) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

This shows that $\{(x_n, \alpha_n)\}$ is Cauchy. Where $\lambda = \min\{\alpha, \alpha_1, \alpha_2, \dots, \alpha_n, \dots\}$[Here λ exists since this set is bounded] .

Theorem 4.9:

Let $\tilde{A}$ be a nonempty fuzzy set in a fuzzy normed space $(X,\tilde{N})$ and $cl(\tilde{A})$ its closure.

Then

(i) $x_\alpha \in cl(\tilde{A})$ if and only if there is a fuzzy sequence $\{(x_n, \alpha_n)\}$ in $\tilde{A}$ such that $(x_n, \alpha_n) \to x_\alpha$ where $\alpha \in (0,1]$.

(ii) $\tilde{A}$ is closed if and only if the situation $(x_n, \alpha_n) \in \tilde{A}$ and $(x_n, \alpha_n) \to x_\alpha$ implies $x_\alpha \in \tilde{A}$.

Proof:

(i)Let $x_\alpha \in cl(\tilde{A})$. If $x_\alpha \in \tilde{A}$ a fuzzy sequence of that type is $x_\alpha, x_\alpha, \dots$

If $x_\alpha \notin \tilde{A}$ then it must be a limit of $\tilde{A}$. Hence for each n=1,2,… the fuzzy ball $\tilde{O}(x_\alpha, \frac{1}{n})$ contains an $(x_n, \alpha_n) \in \tilde{A}$ and $(x_n, \alpha_n) \to x_\alpha$ because $\frac{1}{n} \to 0$ as $n \to \infty$. Conversely, if $\{(x_n, \alpha_n)\}$ is in $\tilde{A}$ and $(x_n, \alpha_n) \to x_\alpha$ then $x_\alpha \in \tilde{A}$ or every neighborhood of x_α contains a fuzzy point $(x_n, \alpha_n) \neq x_\alpha$ so that is x_α a limit of $\tilde{A}$. Hence $x_\alpha \in cl(\tilde{A})$. It is clear that $\tilde{A} = cl(\tilde{A})$.

(ii)$\tilde{A}$ is closed if and only if $\tilde{A} = cl(\tilde{A})$ so that (ii) follows readily from (i).

Theorem 4.10:

A mapping $T : X \to Y$ of a fuzzy normed space $(X,\tilde{N}_1)$ into a fuzzy normed space $(Y,\tilde{N}_2)$ is fuzzy continuous at a fuzzy point $x_\alpha, \in X$ if and only if $(x_n, \alpha_n) \to x_\alpha$ implies $(T(x_n), \alpha_n) \to T(x)_\alpha$.

Proof:

Assume that T is fuzzy continuous at x_α. Then given $\varepsilon > 0$ there is $\delta > 0$ such that $\tilde{N}_1(y - x,\lambda) < \delta$ implies $\tilde{N}_2(T(y) - T(x),\lambda) < \varepsilon$. Let $(x_n, \alpha_n) \to x_\alpha$. Then there is $N > 0$ such that $\tilde{N}(x_n - x_\alpha ,\lambda) < \varepsilon$ for all $n > N$.Where $\lambda = \min\{\alpha,\beta\}$.Hence for all $n > N$, $\tilde{N}_2(T(x_n) - T(x),\lambda) < \varepsilon$ this means that $(T(x_n), \alpha_n) \to T(x)_\alpha$.

Conversely, we assume that $(x_n, \alpha_n) \to x_\alpha$ implies $(T(x_n), \alpha_n) \to T(x)_\alpha$, and prove that T is fuzzy continuous at x_α. Suppose this is false. Then there is an $\varepsilon > 0$ such that for every $\delta > 0$ there is $y_\beta \neq x_\alpha$ satisfying $\tilde{N}_1(y - x,\lambda) < \delta$ but $\tilde{N}_2(T(y) - T(x),\lambda) \geq \varepsilon$.

In particular for $\delta = \frac{1}{n}$ there is $\{(x_n, \alpha_n)\}$ satisfying $\tilde{N}_1(x_n - x_\alpha,\lambda) < \frac{1}{n}$ but $\tilde{N}_2(T(x_n) - T(x),\lambda) \geq \varepsilon$. Clearly $(x_n, \alpha_n) \to x_\alpha$ but $\{T(x_n), \alpha_n)\}$ does not converse to $T(x)_\alpha$. This contradicts $(T(x_n), \alpha_n) \to T(x)_\alpha$ and proves the theorem.

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فضاءات قيا سية ضبا بية جديدة

جها د رمضا ن خضر*

*الجا معة التكنولوجية.

الخلاصة:
في هذا البحث قدمنا تعريف جديد للفضاء القياسي الضبابي ثم بعد ذلك قمنا بدراسة المفاهيم المتعلقة بهذا التعريف مثلا الاستمرارية الضبابيه والتقارب للمتتابعات التي عناصرها نقاط ضبابيه ومتتابعات كوشي بتفاصيل اكثر.