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Boubaker Wavelets Functions: Properties and Applications

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Abstract:

This paper is concerned with introducing an explicit expression for orthogonal Boubaker polynomial functions with some important properties. Taking advantage of the interesting properties of Boubaker polynomials, the definition of Boubaker wavelets on interval [0,1) is achieved. These basic functions are orthonormal and have compact support. Wavelets have many advantages and applications in the theoretical and applied fields, and they are applied with the orthogonal polynomials to propose a new method for treating several problems in sciences, and engineering that is wavelet method, which is computationally more attractive in the various fields. A novel property of Boubaker wavelet function derivative in terms of Boubaker wavelet themselves is also obtained. This Boubaker wavelet is utilized along with a collocation method to obtain an approximate numerical solution of singular linear type of Lane-Emden equations. Lane-Emden equations describe several important phenomena in mathematical science and astrophysics such as thermal explosions and stellar structure. It is one of the cases of singular initial value problem in the form of second order nonlinear ordinary differential equation. The suggested method converts Lane-Emden equation into a system of linear differential equations, which can be performed easily on computer. Consequently, the numerical solution concurs with the exact solution even with a small number of Boubaker wavelets used in estimation. An estimation of error bound for the present method is also proved in this work. Three examples of Lane-Emden type equations are included to demonstrate the applicability of the proposed method. The exact known solutions against the obtained approximate results are illustrated in figures for comparison.

Keywords: Boubaker polynomial, Collocation method, Convergence criteria, Error analysis, Wavelet polynomial

Introduction:

Wavelet theory is an emerging area in mathematical research and it has a wide range application in engineering discipline, singular analysis, and time frequency analysis (1-5). It permits the accurate representation of different functions and operators. Furthermore, wavelets functions construct a connection with variety numerical techniques. Many authors have increasingly considered the application of Chebyshev wavelets and shifted Chebyshev wavelets. For example, in (6), the modified Chebyshev wavelets have been applied for solving this film of non-Newtonian fluid problem while Chebyshev wavelets utilized for fractional differential equations by (7) have been shifted, also Chebyshev wavelets have been used for solving problems in mathematics and physics. It is well known that there are other types of wavelet functions and all of them have been applied for solving many practical problems arising in numerous branches of science and engineering, that require solving singular initial value problems and boundary value problems of partial differential equations, linear and nonlinear fractional differential equations. Wavelet functions have been previously applied for obtaining approximate solutions for some of these problems. Authors in (8) have constructed a fast algorithm for some linear and nonlinear wave equations using Taylor wavelets. In the two papers (9,10), the authors treated weakly kernel integral equation of the second kind and fractional delay differential equation respectively using Hermit wavelets. In (11-13), Legendre wavelets method have been introduced for solving respectively, optimal control problem, fractional differential equations and partial differential equations.

In this paper, first a new form of polynomials is introduced, the orthogonal Boubaker polynomials and derive many interest and useful properties, then present the definition of Boubaker wavelet depending on the orthogonal Boubaker polynomials.

The main goals of the present work are

- Introducing Boubaker wavelets functions and deriving explicit forms of their derivatives.
- Presenting the bounded of Boubaker wavelet coefficients.
- Using Boubaker wavelets functions together with their properties to solve Lane-Emden type equation with the aid of collocation method.

Lane Emden equation is singular initial value problem and many authors have studied them. The solution of Lane Emden equation is numerically found, for example (14-17).

The paper is organized as follows: In section two, a new explicit formula for defining orthogonal Boubaker polynomial is introduced with some important properties. Then Boubaker wavelets are constructed in section 3. Boubaker coefficients discussion is given in section 4. In section 5 the derivative of Boubaker wavelets in terms of Boubaker wavelets is given while the introduced Boubaker wavelets with the new property is applied for solving Lane-Emden equation using collocation method is included, in section 6. Some conclusion remarks illustrating the validity of the suggested basis functions are listed in section 7.

New Explicit Definition for Orthogonal Boubaker Polynomials

The Boubaker polynomials were deals with in physics studies that get a thermal model of the spray pyrolysis disposal (18). They are merged from an attempt to obtain a solution to heat equation in a determination step through resolution process (19).

Definition 1 (20)

The expression of Boubaker polynomials $B_m(\tau)$is described as

$$B_m(\tau) = \sum_{p=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^p \frac{(n-4p)}{(n-p)} \binom{n-p}{p} \tau^{n-2p} \qquad (1)$$

and it can be defined with the recurrence relation given below

$$B_m(\tau) = \tau\, B_{m-1}(\tau) - B_{m-2}(\tau) \qquad n > 2 \qquad (2)$$

with $B_0(\tau) = 1$, $B_1(\tau) = \tau$ and $B_2(\tau) = \tau^2 + 2$ Boubaker polynomials are not orthogonal but when applying the Gram-Schmit process on sets of Boubaker polynomials one can obtain orthogonal Boubaker polynomials $BO_m(\tau)$.

Orthogonal Boubaker polynomials are generated using

$$< \tau_i, \tau_j >= \int_0^1 \tau_i \tau_j d\tau$$

Define $BO_0(t) = 1$

The first orthogonal Boubaker polynomial $BO_1(\tau)$ is

$$BO_1(\tau) = \left[\tau - \frac{< \tau BO_0, BO_0 >}{< BO_0, BO_0 >}\right] BO_0$$

$$= \left[\tau - \frac{\int_0^1 tdt}{\int_0^1 BO_0 BO_0 d\tau}\right] BO_0 = \left(\tau - \frac{1}{2}\right) \Big/ \sqrt{\int_0^1 BO_0 BO_0 d\,\tau}$$

In order to construct higher order orthogonal Boubaker polynomial, the following recurrence relation is used

$$BO_{i+1}(\tau) = \left[\tau - \frac{< \tau BO_i, BO_i >}{< BO_i, BO_i >}\right] BO_i - \left[\frac{< BO_i, BO_i >}{< BO_{i-1}, BO_{i-1} >}\right] BO_{i-1}$$

Definition 2

The explicit representations of the orthogonal Boubaker polynomials of m^{th} degree are defined on the interval [0, 1] as

$$BO_m(\tau) = \frac{(m!)^2}{(2m)!} \sum_{k=0}^{m} (-1)^{m+k} \frac{(m+k)!}{(m-k)!(k!)^2} \tau^k \qquad (3)$$

The first six $BO_m(\tau)$ are given below and are plotted in Fig. 1

$$BO_0(\tau) = 1$$

$$BO_1(\tau) = \frac{1}{2}(2\tau - 1)$$

$$BO_2(\tau) = \frac{1}{6}(6\tau^2 - 6\tau + 1)$$

$$BO_3(\tau) = \frac{1}{20}(20\tau^3 - 30\tau^2 + 12\tau - 1)$$

$$BO_4(\tau) = \frac{1}{70}(70\,\tau^4 - 140\tau^3 + 90\tau^2 - 20\tau + 1)$$

$$BO_5(\tau) = \frac{1}{252}(252\tau^5 - 630\,\tau^4 + 560\tau^3 - 210\tau^2 + 30\tau - 1)$$

$$BO_6(\tau) = \frac{1}{924}(924\tau^6 - 2772\tau^5 + 3150\,\tau^4 - 1680\tau^3 + 420\tau^2 - 42\tau + 1)$$

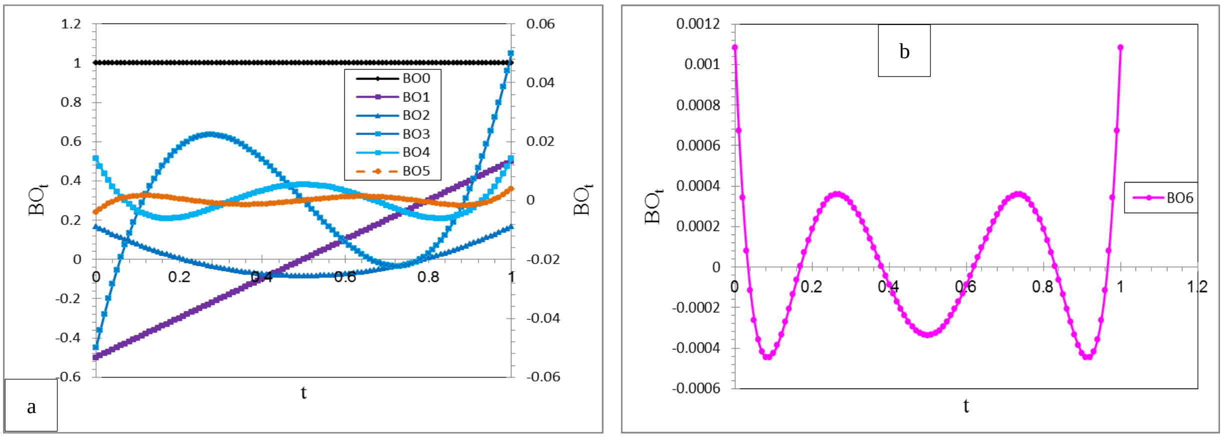

Figure 1. (a) Orthogonal Boubaker polynomials of orders 1, 2, 3, 4, 5 (b) 6

A recursive definition also can be used to generate orthogonal Boubaker polynomials over the interval [0,1]

$$BO_{m+1}(\tau) = \frac{((m+1)!)^2}{(2(m+1))!}\left[\frac{(2m+1)}{m+1}\frac{(2m)!}{(m!)^2}(2\tau - 1)BO_m(\tau) - \frac{m}{m+1}\frac{2(m-1)!}{((m-1)!)^2}BO_{m-1}(\tau)\right] \quad (4)$$

$m \geq 2$

with $BO_0(\tau) = 1$ and $BO_1(\tau) = \frac{1}{2}(2\tau - 1)$

The other property of the orthogonal Boubaker polynomials is

$$\dot{B}O_m(\tau) = \frac{(m!)^2}{(2m)!}\frac{(2m-4)!}{((m-2)!)^2}BO_{m-2}(\tau) + mBO_{m-1}(\tau) \quad (5)$$

Definition 3

The m[th] degree of orthonormal Boubaker polynomials are defined below over the interval [0,1]

$$BOr_m(\tau) = \sqrt{2m+1}\,\frac{(2m)!}{(m!)^2}BO_m(\tau) \quad (6)$$

Boubaker Wavelet

Wavelet functions are constructed from dilation and translation of a definite function, named mother wavelet. Wavelet functions may be defined as

$$\sigma_{a,b}(\tau) = |a|^{-\frac{1}{2}}\sigma\left(\frac{\tau-b}{a}\right) \qquad a, b \in \mathbb{R}, a \neq 0.$$

where a and b are dilation and translation parameters respectively while τ is the normalized time.

Consider the Boubaker wavelets as $\eta_{m,n}(\tau) = \eta(m,n,\tau)$ where $n = 0, 1, 2, \ldots, 2^{k+1}$ $(k = 0,1,2,3,\ldots)$, while m represents the order of orthogonal Boubaker polynomials.

Then the Boubaker wavelets can be defined as below

$$\eta_{nm}(\tau) = \begin{cases} \sqrt{2m+1}\,\frac{(2m)!}{(m!)^2}\, 2^{\frac{k}{2}}BO_m(2^k\tau - n) & \frac{n}{2^k} \leq \tau < \frac{n+1}{2^k} \\ 0 & otherwise \end{cases} \quad (7)$$

In Eq. 7, $BO_m(\tau)$ describes orthogonal Boubaker polynomial of order m.

Therefore, the total Boubaker wavelet approximation can be presented as below

$$u(\tau) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} a_{nm}\,\eta_{n,m}(\tau) \quad (8)$$

By truncating the infinite series in Eq. 8, then the result can be written as

$$u(\tau) \cong \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} a_{nm}\eta_{nm} \quad (9)$$

where $a_{nm} = \big(u(\tau), \eta_{nm}(\tau)\big)$, in which (0 , 0) denoted the inner product in $L^2[0,1]$.

Eq. 9 can be written in a matrix form as,

$$u(\tau) = a^T\eta(\tau) \quad (10)$$

where a and $\eta(\tau)$ are $2^{k-1}M{\times}1$ matrices given by

$$a = [a_{10}, a_{11}, \ldots, a_{1(M-1)}, a_{20}, \ldots, a_{2(M-1)}, \ldots, a_{(2^{(k-1)}0)}, \ldots, a_{(2^{(k-1)}(M-1))}]^T \quad (11)$$

and

$$\eta(\tau) = [\eta_{10}, \eta_{11}, \ldots, \eta_{1(M-1)}, \eta_{20}, \ldots, \eta_{2(M-1)}, \ldots, \eta_{(2^{(k-1)}0)}, \ldots, \eta_{(2^{(k-1)}(M-1))}]^T \quad (12)$$

Bounded of Boubaker Wavelet Coefficients

Theorem 1

Let $x(\tau)$ be a continuous function defined on [0,1] and $\alpha^*(\tau)$ be the approximation of $\alpha(\tau)$ by applying Boubaker wavelets. Also suppose that $x(\tau)$ is bounded by a positive constant, that is

$|x(\tau)| < \epsilon$. Then the Boubaker wavelet coefficients of $x(\tau)$ are bounded and

$$|c_{nm}| \le \frac{\epsilon}{2^{\frac{k-1}{2}}} \frac{(2m)!}{(m!)^2} \sqrt{m+\frac{1}{2}}$$

Proof

$$x^*(\tau) = \sum_{m=0}^{M-1} \sum_{n=1}^{2^{k-1}} c_{nm} \eta_{nm}(\tau)$$

$$c_{nm} = (x, \eta_{nm}) = \int_0^1 x(\tau)\, \eta_{nm}(\tau) d\tau = \frac{(2m)!}{(m!)^2} 2^{\frac{k}{2}} \sqrt{2m+1} \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} x(\tau)\, BO_m(2^k\tau - n) d\tau$$

Let $2^k\tau - n = u$ then $du = 2^k d\tau$

$$c_{nm} = \frac{(2m)!}{(m!)^2} 2^{\frac{k}{2}} \sqrt{2m+1} \quad 2^{-k} \int_0^1 x\left(\frac{u+n}{2^k}\right) BO_m(u) du$$

$$= \frac{(2m)!}{(m!)^2} 2^{-\frac{k}{2}} \sqrt{2m+1} \int_0^1 x\left(\frac{u+n}{2^k}\right) BO_m(u) du$$

Since $|x(\tau)| < \epsilon$

$$|c_{nm}| = \frac{(2m)!}{(m!)^2} \sqrt{2m+1}\ 2^{-\frac{k}{2}} \left| \int_0^1 x\left(\frac{u+n}{2^k}\right) BO_m(u) du \right|$$

$$\le \frac{(2m)!}{(m!)^2} \sqrt{2m+1}\ 2^{-\frac{k}{2}} \int_0^1 \left| x\left(\frac{u+n}{2^k}\right) \right| |BO_m(u)|\, du$$

$$\le \frac{(2m)!}{(m!)^2} \sqrt{2m+1}\ 2^{-\frac{k}{2}} \epsilon \int_0^1 |BO_m(u)|\, du$$

$|BO_m(u)| \le 1 \ \forall\ u \in [0,1]$

$$\int_0^1 |BO_m(u)| du \le 1$$

$$|c_{nm}| \le \frac{(2m)!}{(m!)^2}\ 2^{-\frac{k}{2}} \sqrt{2m+1} \ = \epsilon \frac{(2m)!}{(m!)^2}\ 2^{-\frac{k}{2}} \sqrt{2} \sqrt{m+\frac{1}{2}}$$

$$= \epsilon\ 2^{-\frac{k}{2}}\ 2^{\frac{1}{2}} \frac{(2m)!}{(m!)^2} \sqrt{m+\frac{1}{2}}$$

$$|c_{nm}| \le \frac{\epsilon}{2^{\frac{k-1}{2}}} \frac{(2m)!}{(m!)^2} \sqrt{m+\frac{1}{2}}.$$

This is the required result.

The Derivative of Boubaker Wavelet in Terms of Boubaker Wavelets

In the next theorem, a relation between orthogonal Boubaker polynomials and their derivatives is derived, which is very important in deriving the derivative of Boubaker wavelets.

Theorem 2

Let $BO_m(\tau)$ be the orthogonal Boubaker polynomials into [0, 1], then the following relation is satisfied

$$\dot{BO}_m(\tau) = 2\frac{(m!)^2}{(2m)!} \sum_{k=1}^{m} \frac{(2(m-k))!}{((m-k)!)^2} (2(m-k)+1) BO_{m-k}(\tau) \qquad (13)$$

$(m-k)$ is odd for m even, $(m-k)$ is even for m odd

where $m = 1,2,\dots$, and $\dot{BO}_0(\tau) = 0$

Theorem 3

Let $\eta(\tau)$ be the Boubaker wavelet into [0, 1], then the following relation is satisfied

$$\dot{\eta}(\tau)_m = 2^{k+1} \cdot (2m+1)^{-\frac{1}{2}} \sum_{k=1}^{m} \frac{(2s)!}{(s!)^2} (2s+1) \eta_s(\tau) \qquad (14)$$

s is odd for m even, s is even for m odd

where $s = m-k, m = 1,2,3,\dots$ and $\dot{\eta}_0(\tau) = 0$

Proof

Consider the vector Boubaker wavelet defined in Eq.7

$$\eta_{nm}(\tau) = \begin{cases} \sqrt{2m+1}\ \frac{(2m)!}{(m!)^2}\ 2^{\frac{k}{2}} BO_m(2^k\tau - n) & \frac{n}{2^k} \le \tau < \frac{n+1}{2^k} \\ 0 & otherwise \end{cases}$$

Differentiating $\eta_{nm}(\tau)$ with respect to τ, yields

$$\dot{\eta}_{nm}(\tau) = \begin{cases} \sqrt{2m+1}\ \frac{(2m)!}{(m!)^2} 2^{\frac{k}{2}} \dot{BO}_m(2^k\tau - n) & \frac{n}{2^k} \le \tau < \frac{n+1}{2^k} \\ 0 & otherwise \end{cases}$$

(15)

Using the result in Eq. 13, one can get Eq. 15

$$\dot{\eta}_{nm}(\tau) = \sqrt{2m+1}\ \frac{(2m)!}{(m!)^2}\ 2^{\frac{k}{2}} \cdot 2^k \cdot 2\ \frac{(m!)^2}{(2m)!} \sum_{k=1}^{m} \frac{(2(m-k))!}{((m-k)!)^2} BO_m(2^k - n)$$

$$= 2^{k+1}\ (2m+1)^{-1/2} \sum_{k=1}^{m} \frac{(2(m-k))!}{((m-k)!)^2} (2(m-k)+1) \eta_{m-k}(\tau)$$

Since $s = m-k$

Therefore;

$$\dot{\eta}_{nm}(\tau) = 2^{k+1}\ (2m+1)^{-1/2} \sum_{k=1}^{m} \frac{(2s)!}{(s!)^2} (2s+1) \eta_s(\tau)$$

Application of Solutions in the Boubaker Wavelets Basis

In this section, the solution of Lane-Emden equations are obtained by applying Boubaker wavelets collocation method based on making use of the previously introduced derivative of Boubaker wavelet.

Consider the Lane-Emden of the form

$\ddot{y}(\tau) + \frac{\alpha}{\tau}\dot{y}(\tau) + y(\tau) = g(\tau)$
(16)

where $\tau \in (0,1] and\ \alpha \geq 0$

with the conditions

$y(0) = a,\ \dot{y}(0) = 0$
(17)

Consider an approximate solution to Eq. 16 which is given in terms of Boubaker wavelets as

$y_{nm}(\tau) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} a_{n,m}\eta_{n,m}(\tau)$
(18)

Then one can obtain the following residual after substituting of Eq. 18 into Eq. 16

$$R(\tau) = \tau\sum_{n=1}^{2^{k-1}}\sum_{m=2}^{M-1} a_{nm}\ddot{\eta}_{nm}(\tau) + \alpha\sum_{n=1}^{2^{k-1}}\sum_{m=1}^{M-1} a_{nm}\dot{\eta}_{nm}(\tau)$$

$+x\sum_{1=0}^{2^{k-1}}\sum_{m=0}^{M-1} a_{nm}\eta_{nm}(\tau) - \tau g(\tau)$
(19)

Using the collocation method yields

$R(\tau_i) = 0,\ \ i = 1,2,3,\dots,2^k(M+1) - 2\,,$

Moreover, uses of initial conditions Eq. 17 give

$\sum_{n=0}^{2^{k-1}}\sum_{m=0}^{M-1} a_{nm}\eta_{nm}(0) = a$
(20)

$\sum_{n=0}^{2^{k-1}}\sum_{m=0}^{M-1} a_{nm}\dot{\eta}_{nm}(0) = 0$
(21)

Additional $2^{k-1}M$ equations are obtained in the unknown expansion coefficients a_{nm}.

Numerical Examples

Example 1

The first test example is

$\ddot{y}(\tau) + \frac{2}{\tau}\dot{y}(\tau) + 1 = 0$, $0 < \tau \leq 1$

with $y(0) = 1\,,\ \ \dot{y}(0) = 0$

Applying the collocation technique with Boubaker wavelet presented in section 6 to obtain

$y_{nm}(\tau) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} a_{n,m}\eta_{n,m}(\tau)$
(22)

For M = 5 and k =1 and by using Eq. 19 with the collocation points $\tau_1 = 0.25$, $\tau_2 = 0.5$, and $\tau_3 = 1$, one can obtain

$$4\sqrt{3}a_1 - 6\sqrt{5}a_2 + \frac{6072}{423}a_3 + \frac{57}{7}a_4 = -0.25$$

$$4\sqrt{3}a_1 - \frac{2505}{263}a_3 + 12a_4 = -0.5$$

$$4\sqrt{3}a_1 + 12\sqrt{5}a_2 + \frac{9639}{253}a_3 + \frac{240}{7}a_4 = -1$$

Applying the initial conditions from Eqns. 20-21 to get

$$a_0 - \frac{3}{2}\sqrt{3}a_1 + \frac{13}{6}\sqrt{5}a_2 - \frac{63}{20}\sqrt{7}a_3 = 1$$

$$2\sqrt{3}a_1 - 6\sqrt{5}a_2 + \frac{66}{5}\sqrt{7}a_3 + \frac{540}{7}a_4 = 0$$

After solving such system, one can get

a_0= 0.902777777777778, a_1= -0.072168783648703, a_2= -0.018633899812498, a_3= 0, a_4= 0.

Consequently, the solution presented in Eq. 22 can be determined.

Table 1 gives the comparison between the approximate solution and the exact solution $u(\tau) = 1 - \frac{1}{3!}\tau^2$

Table 1. Results of Example 1

T	$y_{appr}(\tau)$	$y_{exact}(\tau)$	error
0	1.000000000000000	1.000000000000000	0.000000000000000
0.1	0.998333333333334	0.998333333333333	0.000000000000001
0.2	0.993333333333334	0.993333333333333	0.000000000000001
0.3	0.985000000000000	0.985000000000000	0.000000000000000
0.4	0.973333333333334	0.973333333333333	0.000000000000000
0.5	0.958333333333334	0.958333333333333	0.000000000000000
0.6	0.940000000000000	0.940000000000000	0.000000000000000
0.7	0.918333333333333	0.918333333333333	0.000000000000000
0.8	0.893333333333334	0.893333333333333	0.000000000000000
0.9	0.865000000000000	0.865000000000000	0.000000000000000
1.0	0.833333333333334	0.833333333333333	0.000000000000001

Example 2

The second test Lane-Emden differential equation is

$\ddot{y}(x) + \frac{2}{\tau}\dot{y}(x) + y(\tau) = 0$, $0 < \tau \leq 1$ with

$y(0) = 1\,,\ \ \dot{y}(0) = 0$

where the exact solution is $y(\tau) = \frac{\sin(\tau)}{\tau}$.

Applying the collocation technique with Boubaker wavelet presented in section 6 to obtain

$$y_{nm}(\tau) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} a_{n,m}\eta_{n,m}(\tau)$$

For $M = 5$ and k =1and by using Eq. 19 with the collocation points $\tau_1 = 0.25$, $\tau_2 = 0.5$, and $\tau_3 = 1$, one can obtain

$$\frac{1}{2}a_0 + \frac{15}{4}\sqrt{3}a_1 + \frac{311}{1609}a_2 - \frac{7126}{743}a_3 + \frac{1683}{140}a_4 = 0$$

$$\frac{1}{4}a_0 + \frac{15}{4}\sqrt{3}a_1 - \frac{3897}{302}a_2 + \frac{7126}{743}a_3 - \frac{2045}{149}a_4 = 0$$

$$a_0 + \frac{9}{2}\sqrt{3}a_1 + \frac{1986}{73}a_2 + \frac{8602}{225}a_3 + \frac{2403}{70}a_4 = 0$$

Applying the initial conditions from Eqns. 20-21 to get

$$a_0 - \frac{3}{2}\sqrt{3}a_1 + \frac{13}{6}\sqrt{5}a_2 - \frac{63}{20}\sqrt{7}a_3 + \frac{963}{70}a_4 = 1$$

$$2\sqrt{3}a_1 - 6\sqrt{5}a_2 + \frac{66}{5}\sqrt{7}a_3 + \frac{540}{7}a_4 = 0$$

After solving such system, one can get a_0=0.905936447152376, a_1= -0.067954491278279, a_2= - 0.015573553854944, a_3=0.001100514472143, a_4=0.000155220642522.

The comparison between the approximate solution and the exact solution can be seen in Table 2.

Table 2. Results of Example 2 with *M* = 5 and *k* =1

	M=5, k=1		
T	$y_{exact}(\tau)$	$y_{appr}(\tau)$	**error**
0	1.000000000000000	1.000000000000000	0.000000000000000
0.1	0.998334166468282	0.998332108068162	0.000002058400119
0.2	0.993346653975306	0.993341139895023	0.000005514080283
0.3	0.985067355537799	0.985059567977624	0.000007787560175
0.4	0.973545855771626	0.973537746231033	0.000008109540593
0.5	0.958851077208406	0.958843909988337	0.000007167220069
0.6	0.941070788991726	0.941064176000641	0.000006612991084
0.7	0.920310981768130	0.920302542437070	0.000008439331060
0.8	0.896695113624403	0.896680888884764	0.000014224739639
0.9	0.870363232919426	0.870338976348886	0.000024256570540
1.0	0.841470984807897	0.841434447252614	0.000036537555283

The values of exact solution and approximate solution at some points are reported in Table 3 with $M = 6$, $M = 7$ and $k = 1$. In addition, the maximum absolute error has been listed in Table 4 for $M = 5, 6$ *and* 7 and $k = 1$.

Table 3. Results of Example 2 for *M* = 6, *M* = 7 and *k* =1

	M=6, k=1		M=7, k=1	
t	$y_{exact}(\tau)$	**error**	$y_{appr}(\tau)$	**error**
0	1	0	1	0
0.1	0.9983341664682	0.000000760382	0.9983341664682	0.00000000380
0.2	0.9933466539753	0.000001898197	0.9933466539753	0.00000000587
0.3	0.9850673553779	0.000002594718	0.9850673555378	0.00000000637
0.4	0.9735458557716	0.000002782978	0.9735458557716	0.00000000962
0.5	0.9588510772084	0.000002710436	0.9588510772084	0.00000001531
0.6	0.9410707889170	0.000002639264	0.9410707889917	0.00000001872
0.7	0.9203109817861	0.000002680435	0.9203109817681	0.00000001626
0.8	0.8966951136244	0.000002756771	0.8966951136244	0.00000001135
0.9	0.8703632391940	0.000002689089	0.8703632329194	0.00000001486
1.0	0.6411470984807	0.000002398682	0.8414709848078	0.00000003333

Table 4. Maximum absolute error of Example 2 for *M* = 5,6, 7 and *k* =1

Maximum error		
M=5, k=1	**M=6, k=1**	**M=7, k=1**
0.000036537556	0.000002782978	0.00000003333

It is clear from Table 3 that only a small number of Boubaker wavelets basis functions are needed to obtain the approximate solution, which agrees with the actual one.

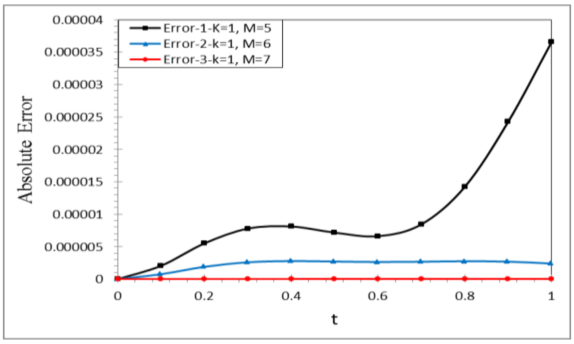

Figure 2. Comparison between the maximum absolute error for Example 2 with $M = 5, 6$ *and* 7 and $k = 1$.

Example 3

Consider the third Lane-Emden type equation:
$\ddot{y}(x) + \frac{2}{\tau}\dot{y}(x) + y(\tau) = 6 + 12\tau + \tau^2 + \tau^3 \quad , 0 < \tau \leq 1$

Subject to $y(0) = 0$, $\dot{y}(0) = 0$ with exact solution is $y(\tau) = \tau^2 + \tau^3$

This problem is solved with Boubaker wavelets using $M = 5$, and $k = 1$ the linear system of 5-equations is obtained

$$\frac{1}{2}a_0 + \frac{15}{4}\sqrt{3}a_1 + \frac{311}{1609}a_2 - \frac{7126}{743}a_3 + \frac{1683}{140}a_4 = \frac{99}{8}$$
$$\frac{1}{4}a_0 + \frac{15}{4}\sqrt{3}a_1 - \frac{3897}{302}a_2 + \frac{7126}{743}a_3 - \frac{2045}{149}a_4 = \frac{851}{64}$$
$$a_0 + \frac{9}{2}\sqrt{3}a_1 + \frac{1986}{73}a_2 + \frac{8602}{225}a_3 + \frac{2403}{70}a_4 = 20$$
$$a_0 - \frac{3}{2}\sqrt{3}a_1 + \frac{13}{6}\sqrt{5}a_2 - \frac{63}{20}\sqrt{7}a_3 + \frac{963}{70}a_4 = 0$$
$$2\sqrt{3}a_1 - 6\sqrt{5}a_2 + \frac{66}{5}\sqrt{7}a_3 + \frac{540}{7}a_4 = 0$$

The following unknown parameters are obtained a_0=1.052083235111342, a_1= 0.930977360202020, a_2= 0.363361161797005, a_3 = 0.047245598838191, a_4= 0.000000000195274.

As it can be shown in Table 5 that only a few number of Boubaker wavelets basis functions are utilized to reach the approximate solution with a satisfying result.

Table 5. Results of Example 3

t	$y_{appr}(\tau)$	$y_{exact}(\tau)$	error
0	0.000000000000001	0	0.000000000000001
0.1	0.010999992538870	0.011000000000000	0.000000007461130
0.2	0.047999973416433	0.048000000000000	0.000000026583567
0.3	0.116999947540987	0.117000000000000	0.000000052459013
0.4	0.223999919843328	0.224000000000000	0.000000080156672
0.5	0.374999895276744	0.375000000000000	0.000000104723256
0.6	0.575999878817022	0.576000000000000	0.000000121182978
0.7	0.832999875462443	0.833000000000000	0.000000124537557
0.8	1.151999890233785	1.152000000000000	0.000000109766216
0.9	1.538999928174318	1.539000000000000	0.000000071825682
1.0	1.999999994349811	2.000000000000000	0.000000005650189

Conclusion:

In this paper, has presented for the first time the exact expression for orthogonal Boubaker polynomials and then defined the Boubaker wavelet. The basic shapes of the first six orthogonal Boubaker polynomials are plotted in Fig.1. These polynomials can be used to present complicated functions. In addition, some important properties are derived and employed for obtaining the approximate solution of Lane-Emden equations. Only a few number of Boubaker wavelet basis is needed to achieve the high accuracy. The approximate solutions obtained using the collocation Boubaker wavelets are compared with the exact solution and the agreement between them is obtained. This method has reasonably shown good performance for all of the Lane -Emden type equations.

Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Technology.

References:

1. Miguel G F, Patrick K P. Marcelo G V. Electromagnetic device modeling using a new adaptive wavelet finite element method. Math Comput Simul. In press. 2020
2. Nighila V P, Johney I, Jacob P. Application of Wavelet Transform to the study of Lattice Dynamics of two-dimensional Nanostructures. Mater Today PR. 2019; 18(3): 1524-1531.
3. Dilshad A H. A well-balanced adaptive Haar wavelet finite volume scheme for 1D free surface water flows. Ain Shams Eng. J. 2019; 10(4): 891-895.
4. Zhaoming X, Hao Z, Kongchao S, Yongquan Q, Zheng J. Wavelet analysis of extended X-ray absorption fine structure data: Theory, application J. Phys. Condens. Matter. 2018; 542(1): 12-19.
5. Seyed A T, Seyed M T. Wavelet based damage identification and dynamic pull-in instability analysis of electrostatically actuated coupled domain microsystems using generalized differential quadrature method. Mech Syst Signal Process. 2019; 133(1): 106256.
6. Muhammad A I, Umar K, Ayyaz A, Syed T M D. Modified Chebyshev Wavelet-Picard Technique for Thin Film Flow of Non-Newtonian Fluid Down an

Inclined Plane. Proc. Nat. Acad. Sci. A., 2018, 89(3): 533-538.
7. Hariharan G. An efficient wavelet based approximation method to water quality assessment model in a uniform channel. Ain Shams Eng. J. 2014; 5(2): 525-532.
8. Keshavarz E, Ordokhani Y. A fast numerical algorithm based on the Taylor wavelets for solving the fractional integro-differential equations with weakly singular kernels. Math. Methods Appl. Sci. 2019; 42(13): 4427–4443.
9. Jing G, Yao L. Trigonometric Hermite wavelet approximation for the integral equations of second kind with weakly singular kernel. J. Comput. Appl. Math. 2008; 215(1): 242-259.
10. Saeed U. Hermite wavelet method for fractional delay differential equations. J. Differ. Equ. 2014.
11. Mohsen R, Sohrabali Y. Legendre wavelets method for constrained optimal control problems. Math. Methods Appl. Sci.2002; 25(7): 529-539.
12. Mujeebur R, Rahmat A K. The Legendre wavelet method for solving fractional differential equations. Commun Nonlinear Sci Numer Simul. 2011; 16(11): 4163-4173.
13. Nanshan L E, Bing L. Legendre wavelet method for numerical solutions of partial differential equations. Numer. Methods Partial Differential Equations. 2010; 26(1): 81-94.
14. Sohrab A Y. Legendre wavelets method for solving differential equations of Lane–Emden type. Appl. Math. Comput. 2006; 181(2): 1417-1422.
15. Moustafa A S. Approximate solution for the Lane-Emden equation of the second kind in a spherical annulus. J. King Saud Univ. Sci. 2019; 31(1): 1-5.
16. Joachim M, Herbert S. Statistical mechanics of the isothermal lane-emden equation. J. Stat. Phys. 1982; 29(3): 561-578.
17. Asghar G, Mojtaba B. A variational iteration method for solving nonlinear Lane–Emden problems. New Astron. 2017; 54: 1-6.
18. Boubaker K, Chaouachi A, Amlouk M, Bouzouita H. Enhancement of pyrolysis spray disposal performance using thermal time-response to precursor uniform deposition. Eur. Phys. J. Appl. Phys. 2007; 37(1): 105-109.
19. Boubaker K. On modified Boubaker polynomials: some differential and analytical properties of the new polynomials issued from an attempt for solving bi-varied heat equation. Trends Appl. Sci. Res. 2007: 2(6): 540-544.
20. Tuğçe A, Salih Y. Boubaker polynomial approach for solving high-order linear differential-difference equations. AIP Conf. Proc. 2012; 26: 1493.

خواص وتطبيقات دوال بوبكر الموجية

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الخلاصة:

تم في هذا البحث تقديم شرح تفصيلي لدوال متعددة حدود بوبكر المتعامدة مع بعض الخواص ذات الاهمية، كذلك استنتاج تعريف متعددات حدود بوبكر الموجية في الفترة [0، 1) وذلك بالاستفادة من بعض الخواص المهمة لمتعددة حدود بوبكر. تمتلك هذه الدوال الاساسية خاصية العيارية المتعامدة بالإضافة الى ضرورة تواجد المنطلق المرصوص. لهذه الدوال الموجية العديد من المزايا وقد استخدمت في المجال النظري بالإضافة الى المجال العملي وتم استخدامها مع متعددات الحدود المتعامدة لغرض طرح طريقة جديدة للتعامل مع العديد من المسائل في العلوم والهندسة ولذلك تعتبر طريقة استخدام الموجبات ذات اهمية كبيرة عند الاستفادة منها في المجالات ذات العلاقة. بالإضافة الى الاستفادة من موجبات بوبكر للحصول على خاصية جديدة وهي مشتقات دالة بوبكر الموجية. استخدمت موجية بوبكر مع طريقة الترصيف للحصول على حل عددي تقريبي لمعادلات لان ايمدن من النوع الخطي المنفرد. تصف معادلات لان ايمدن العديد من الظواهر المهمة في علم الرياضيات والفيزياء السماوي مثل الانفجارات الحرارية الكونية وتكوين النجوم. وتعتبر احدى حالات مسائل القيمة الابتدائية المنفردة للمعادلات التفاضلية اللاخطية من الرتبة الثانية. تقوم هذه الطريقة المقترحة بتحويل معادلة لان ايمدن الى نظام من المعادلات التفاضلية الخطية والتي يمكن حلها بسهولة باستخدام الحاسبة. بناءً على هذا فقد ظهر تطابق الحل العددي مع الحل التحليلي بالرغم من استخدام عدد قليل من متعددات حدود بوبكر الموجية لغرض ايجاد هذا الحل. كذلك، تم في هذا البحث البرهنة على قيمه قيد الخطأ المستخرج من هذه الطريقة. وتضمن هذا البحث على ثلاث امثلة عددية من نوع معادلات لان ايمدن لتوضيح قابلية استخدام الطريقة المقترحة. تم توضيح النتائج الحقيقة مع النتائج التقريبية في شكل جداول ورسوم هندسية لغرض المقارنة.

الكلمات المفتاحية : حدود متعددات بوبكر، طريقة التجميع، صيغة الاقتراب، تحليل الخطأ، المتعددات الموجية.