Some Properties of a Fuzzy subgroup Homomorphism

بعض خواص التشاكل للزمر الجزئية الضبابية

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Abstract

In this paper aims to study some properties of homomorphism in group theory which gives the ability to apply on fuzzy subgroup and fuzzy normal subgroup homomorphisms . We introduce some propositions ,theorems and notes on some properties of fuzzy subgroup homomorphism and fuzzy normal subgroup homomorphism and we set some conditions carried over on some properties of fuzzy subgroup homomorphism .

الخلاصة

يهدف البحث الى دراسة بعض خواص التشاكلات في نظرية الزمر والتي يمكن تطبيقها على الزمر الضبابية الجزئية و الزمر الضبابية الجزئية السوية ،ولقد طرحنا بعض القضايا والنظريات حول التشاكلات على الزمر الجزئية الضبابية و الزمر الجزئية الضبابية السوية ولقد وضعنا بعض الشروط اللازمة من اجل تحقيق ذلك .

Introduction

Zadein in 1965 [12] *introduced the concept of fuzzy set* , *Anthony and Sherwood* [5] *introduced the concept of fuzzy group* ,sidky [10] *proved that if f is a homomorphism of group G and M is fuzzy subgroup of group G then f(M) is a fuzzy subgroup of* G' *. In this paper, we give some properties of fuzzy subgroup and fuzzy normal homomorphisms* . *We will use the symbol (.) to indicate the end of the proof.*

1- Fuzzy seat and Fuzzy subgroups:

This section contains the concepts of the fuzzy groups which was coined by Rosenfeld [8], who found mgany basic properties in group theory carried over on fuzzy group and in the same way applied to another algebraic structures like rings, ideals, modules and so on (See [9] , [2]) .

Definition (1.1): [1]

let G be anon – emptyset and I will denote the closed unit interval [0,1] of the real line (real numbers).Let $I^G = \{A: G \to I, A\ is\ a\ mapping\}$ be collection of all mappings from G into I , A member of I^G is called a fuzzy subset of G or a fuzzy set in G .

Definition (1.2) : **[1]**

Let M be afuzzy subset of G , for $t \in [0,1]$, Then $M_t = \{x \in G | M(x) \geq t\}$ is called a level subset of the fuzzy subset M.

Definition (1.3) : **[2]**

Let $(G,.)$be a group and $M \in I^G$ such that $M \neq \phi$. $(M(x) \neq o \forall x \in G)$. Then M is called a fuzzy subgroup of G if and only if for each $x, y \in$ G .

1) $M(xy) \geq \min\{M(x), M(y)\}$.

2) $M(x) = M(x^{-1})$.

Definition (1.4) : **[3]**

Let G be a group and M be a fuzzy subgroup of G we define the following sets :

1)$\Gamma(M) = \{x_t | x \in M^* and\ M(x) = t\}$.

2)$\lambda\Gamma(M) = \{x | x\ is\ afuzzy\ subset\ of\ G\ and\ x \subseteq M\}$.

3) $\lambda(M) = \{N | N\ is\ a\ fuzzy\ subgroup\ of\ G\ and\ N \subseteq M\}$.

Definition (1.5) : **[4]**

Let G be a group and M be fuzzy subgroup of G. Then we define the following :

1)$M^* = \{x \in G | M(x) > 0\}$ is called the support of M .

Also $M^* = \cup_{t \in (0,1]} M_t$.

2) $M_* = \{x \in G | M(x) = M(e)\}$.

It is easy to show that $M^* and M_*$ are subgroups of G.

Theorem (1.6) : **[5]**

Let G be a group and M be a fuzzy subgroup of G. Then the level subset of M_t $(t \in [0,1] and\ M(e) \geq t)$ are called level subgroups of M.

Definition (1.7) : **[6]**

Let M be fuzzy subgroup of (G,.) ,$x_t \subseteq M$ and N$\in \lambda(M)$.Then the fuzzy subset $x_t \cdot N, (N \cdot x_t)$is called left(right) coset of N in M with representative x_t .

Proposition (1.8) : **[6]**

Let $x, y \in G\ and\ s \in$ [0,1] then:

1) $x_t \cdot N = y_s \cdot N$ iff $t = \inf\{s, N(y^{-1}x)\}\ and\ s = \inf\{t, N(x^{-1}\ y)\}$.

2) $x_t \cdot N = y_t \cdot N\ iff\ (y^{-1}x)_t \subseteq N$.

3) $x_t \cdot N = y_s \cdot N\ iff\ t = s \leq N(x^{-1}y)$.

4) $x_t \cdot N = x_s \cdot N\ iff\ t = s$.

Proposition (1.9) : **[6]**

Let $x_t, y_s \subseteq M$ .Then :

1) $x_t \cdot N = y_s \cdot N$iffinf $\{t, N(e)\} = \inf\{s, N(y^{-1}x)\}$ and $\inf\{s, N(e)\} = \inf\{t, N(x^{-1}y)\}$.

2) $N \cdot x_t = N \cdot y_s$ iff inf $\{t, N(e)\} = \inf\{s, N(xy^{-1})\}$ and $\inf\{s, N(e)\} = \inf\{t, N(yx^{-1})\}$.

Definition (1.10) : **[6]**

Let M be fuzzy subgroup of (G,.) and N$\in \lambda(M)$ .Then N is said to be fuzzy normal in M iff $N.x_t = x_t.N\ \forall x_t \subseteq M$.

proposition (1.11) : **[6]**

1) Let $x_t, y_s \subseteq M$ and N be a fuzzy normal in M ,then $(x_t \cdot N) \cdot (y_t \cdot N) = (x_t \cdot y_t) \cdot N$.

2)N is fuzzy normal in M iff N_t normal in $M_t, \forall t \in [0,1]$.

Definition (1.12) : **[7]**

Let M be a fuzzy subgroup of (G,.), N$\in \lambda(M)$ and N be a normal fuzzy subgroup in M .Then $(M/N)_t = \{x_t \cdot N | x_t \subseteq M, x \in G\}, \forall t \in [0,1]$ is a group under "." .

Definition (1.13) : [8]

Let M be a fuzzy subgroup of G .Then the set given by $N(M) = \{x \in G: M^x = M\}$. Is called the normalizer of M.

Definition (1.14) : [7]

A fuzzy map $f: G \to G'$ that maps a group G to a group G' is said to be a fuzzy homomorphism $\forall x_1, x_2 \in G$and $x' \in G'$,

$M_f(x_1x_2, x') = sup x' = x'x'[M_f(x_1, x_1')M_f(x_2, x_2')]$.

Definition (1.15) : [7]

Let f be a homomorphism from a group G to a group G' and let M_1, M_2 be fuzzy sets in G and G' respectively .Then the homomorphic image (direct image)$f(M_1)$ and the inverse image $f^{-1}(M_2)$ are fuzzy sets in G' and G respectively ,defined by:

1) $f(M)(y) = \begin{cases} \sup\{M(x): x \in f^{-1}(y)\} & if\ f^{-1}(y) \neq \emptyset \\ 0 & iff^{-1}(y) = \emptyset \end{cases}$

2) $f^{-1}(M_2)(x) = M(f(x)), \forall\ x \in G$. If $\forall s \subseteq x, \exists s_o \in S$ such that ,

$M(s_o) = \sup\{M(s): s \in S\}$.Then M_1 is said to have a sup – property . **<u>Theorem(1.16)</u>**: [10]

If f is homomorphism from a group G into a group G',if M is a fuzzy subgroup of G , then f(M) is a fuzzy subgroup of G'.

Theorem(1.17): [6]
If f is homomorphism from a group G into a group G' , if N is a fuzzy subgroup of G', then $f^{-1}(N)$ is a fuzzy subgroup of G .

Definition (1.18) : [11]
Let G be a group and M,N be fuzzy subgroups of G ,then M is called to be fuzzy isomorphic to N if there exists an isomorphism $f: M^* \to N^*$ such that M(x) > M(y) ⇔ N(f(x)) > N(f(y)) .
∀x,y ∈M^* and is denoted as M≃N .

Proposition (1.20) :
If M is a fuzzy subgroup of a group G ,then $N(M)$is a subgroup of G.
Proof :
Let$x, y \in N(M)$, we want to show that $xy \in N(M)$ and $x^{-1} \in N(M)$.
Let$x, y \in N(M)$,then$M^{xy} = M^{yx} = M^y = M$.
Hence $x, y \in N(M)$ implies $xy \in N(M)$.
Let $u = x^{-1}$,then ∀g ∈G .
$M^u(x) = M(u^{-1}gu) = M(xgx^{-1}) = M((x^{-1}g^{-1}x)^{-1})$
$= M(x^{-1}g^{-1}x) = M^x(g^{-1}) = M(g^{-1}) = M(g)$.
So that $x^{-1} \in N(M)$is a subgroup of G .

Proposition (1.21) :
If G is a commutative group , then for every $N \in \lambda(M)$is fuzzy normal in$M$.
Proof :
Let G commutative group .
Let N be fuzzy subgroup of G , $N \subseteq M$.
$x_t \cdot N = x_t \cdot N, \forall x_t \subseteq M$,(by definition (1.8)) .
Then$x_t \cdot N = N \cdot x_t \, , \forall x_t \subseteq M$.
Hence N fuzzy normal in M .

2- Fuzzy Homomerphism:
In this section , we introduce some propositions of fuzzy subgroup homomorphism .

Proposition (2.1) :
Let $f: X \to X'$ be a map and let U_i *and* V_i are the families of fuzzy subsets of X and X',then $\forall u_i \in U_i$ *and* $v_i \in V_i$ and indexing set I,the following are true :
1) $f(\cup_{i\in I} u_i) = \cup_{i\in I} f(u_i)$.
2) $f^{-1}(\cup_{i\in I} v_i) = \cup_{i\in I} f^{-1}(v_i)$.
3) $ff^{-1}(v_i) = v_i$ if f is surjective .
4) $ff^{-1}(u_i) = u_i$ if f is f- invariant .

Proof :
1)Since $\cup_{i\in I} u_i = sup\{u_i | i \in I\}$
$f(\cup_{i\in I} u_i) = f(sup\{u_i | i \in I\})$
$= sup\{f(u_i) | i \in I\}$
$= \cup_{i\in I} f(u_i)$.
2) Since $\cup_{i\in I} v_i = sup\{v_i | i \in I\}$
$f^{-1}(\cup_{i\in I} v_i) = f^{-1}(sup\{v_i | i \in I\})$
$= sup\{f^{-1}(v_i) | i \in I\}$
$= \cup_{i\in I} f^{-1}(v_i)$.
3)Since $f: X \to X'$ is a map and surjective .
Then $\exists f^{-1}: X' \to X$,now let $v_i \in V_i$, $\exists u_i \in U_i$, $f^{-1}(v_i) = u_i$.
$ff^{-1}(v_i) = f(u_i)$
$(v_i) = f(u_i)$
Then $ff^{-1}(v_i) = v_i$.
4)Let u_i be fuzzy subsets of X and X' , and let f is f- invariant .

$$f^{-1}(u_i) = v_i$$
$$ff^{-1}(u_i) = f(v_i)$$
$$u_i = f(v_i)$$

Then $ff^{-1}(u_i) = u_i$.

Proposition (2.2) :

If $f: G \to G'$ homomorphism from a group G onto a group $G'$and M_i families of fuzzy subgroup of G, then we have :

1)If $\cup_i (M_i)$ is a families of fuzzy subgroup of G,then $\cup_i f(M_i)$ is a fuzzy subgroup of G'.

2)If $\cup_i f(M_i)$ is a fuzzy subgroup of G', then $\cup_i (M_i)$ is a fuzzy subgroup of G such that f is f-invariant .

Proof :

1)suppose $\cup_i (M_i)$ is a families of fuzzy subgroup of $G, \forall i$.

Then $f(M_i)$ is a fuzzy subgroup of G' , $f(M_i) \in V_i$,where be a families of fuzzy subgroup of G',(by theorem (1.16)).

Then $f(\cup_i M_i) = \cup_i f(M_i)$ be fuzzy subgroup of G', (by proposition (2.1)).

2) suppose $\cup_i f(M_i)$ is a fuzzy subgroup of G' .

Then $f^{-1}(\cup_i f(M_i))$ is a fuzzy subgroup of G , (by proposition (2.1)).

Proposition (2.3) :

If f a group homomorphism of a group G onto a group G' and M_i is a family of fuzzy subgroups of G ,then the following are equivalent :

1) $\cup_i (M_i)$ is a fuzzy subgroup of G .

2) $\cup_i f^{-1}(M_i)$ is a fuzzy subgroup of G.

Proof :

$(1) \Rightarrow (2)$

Since the homomorphic preimage of a fuzzy subgroup is a fuzzy subgroup and ,(by proposition (2.1)).

$f^{-1}(\cup_i (v_i)) = \cup_i f^{-1}(v_i)$.

$(2) \Rightarrow (1)$

Since the homomorphic preimage of a fuzzy subgroup is a fuzzy subgroup and ,(by proposition (2.1)).

$f(\cup_i f^{-1}(v_i)) = \cup_i ff^{-1}(v_i) = \cup_i (v_i)$.

Proposition (2.4) :

If $f: G \to G'$ homomorphism from a group G onto a group $G'$and if $M \sim N$ in G then $f(M) \sim f(N)$ in G' .

Proof :

Suppose $f: G \to G'$ *be* homomorphism and suppose also that $M \sim N$ in .

Hence $M(x) > M(y)$ iff $N(x) > N(y)$ and $M(x) = 0$ iff $N(x) = 0$, $\forall x, y \in G$. Now ,implies that the $M^* = N^*$.

Then $f(M^*) = f(N^*)$ since f is a homomorphism .

$[f(M)]^* = [f(N)]^*$.

Let $x_1, y_2 \in G'$ in such that away that $f(M)(x) > f(M)(y)$ .We show that $f(N)(x) > f(N)(y)$. Now , $\forall y \in G$ such that $f(y) = y_1$,there exists an $x \in G$ such that $f(x) = x_1$ and $M(x) > M(y)$.

But $M \sim N$ hence $N(x) > N(y)$.

From this it follows that $f(N)(x) > f(N)(y)$.

Thus $f(M) \sim f(N)$ in G' .

Proposition (2.5) :

If $f: G \to G'$ homomorphism from a group G onto a group G' and if $M \sim N$ in G' then $f^{-1}(M) \sim f^{-1}(N)$ in G .

Proof :

Assume $f: G \to G' be$ homomorphism and assume that $M \sim N$ in G' .
Then $M^* = N^*$ in G' .
Since f is a homomorphism then $f^{-1}(M^*) = f^{-1}(N^*)$ in G .
Let $x_1, y_2 \in G'$such that $f^{-1}(M)(x_1) > f^{-1}(M)(x_2)$.
Then we have : $M(f(x_1)) > M(f(x_2))$.
Since $M \sim N$ we have : $N(f(x_1)) > N(f(x_2))$.
It follows that $f^{-1}(N)(x_1) > f^{-1}(N)(x_2)$.
Thus $f^{-1}(M) \sim f^{-1}(N)$ in G .

Example(2.6) :

Consider $G = G_1 \times G_2$and let M and N be fuzzy subgroups of G such that

$$M(x) = \begin{cases} 1 & x = \{e\} \\ \rho & x \in c_2 \times \{o\}/\{e\} \\ \gamma & other\ wise \end{cases}$$

$$N(x) = \begin{cases} 1 & x = \{e\} \\ \rho & x \in \{o\} \times c_2 /\{e\} \\ \gamma & other\ wise \end{cases}$$

We define f in G given by $f: (x, y) \to (y, x)$. Then f is an isomorphism. Thus M and N are not equivalent even though they are
isomorphic .

Proposition (2.7) :

Let $f: G \to G'$is a homomorphism from a group G onto a group G',if M is a normal fuzzy subgroup of G ,then $f(M)$ is a normal fuzzy subgroup of G'.

Proof :

Since $f(M)$ is a fuzzy subgroup of G', (by theorem (1.16)) .
To prove $f(M)$ is a normal fuzzy subgroup of G'.
Since M be normal fuzzy subgroup of .
Then $x_t \cdot M = M \cdot x_t, \forall x_t \subseteq M$.
Since f homomorphism and onto $\exists x_t \subseteq M \ni f(x_t) \subseteq f(M)$.
Then $f(x_t) \cdot f(M) = f(M) \cdot f(x_t), \forall f(x_t) \subseteq f(M)$.
Thus $f(M)$ is a normal fuzzy subgroup of G'.

Proposition (2.8) :

Let $f: G \to G'$is a homomorphism from a group G into a group G',if N is a normal fuzzy subgroup of G' ,then $f^{-1}(N)$ is a normal fuzzy subgroup of G.

Proof :

Since $f^{-1}(N)$ is a fuzzy subgroup of G, (by theorem (1.17)) .
To prove $f^{-1}(N)$ is a normal fuzzy subgroup of G.
Since N be normal fuzzy subgroup of G' .
Then $y_t \cdot N = N \cdot y_t, \forall y_t \subseteq N$.
Since f homomorphism and onto $\forall y_t \subseteq N$.
Then $f(x_t) \cdot N = N \cdot f(x_t), \forall f(x_t) \subseteq f(N)$.
$f^{-1}f(x_t) \cdot f^{-1}(N) = f^{-1}(N) \cdot f^{-1}f(x_t)$.
$x_t \cdot f^{-1}(N) = f^{-1}(N) \cdot x_t$.
Thus $f^{-1}(N)$ is a normal fuzzy subgroup of G.

Proposition (2.9) :

Let $f: G \to G'$ be a homomorphism from a group G into a group G',if N is a fuzzy subgroup of G' and let $\{N_{t_j}: j \in J\}$ be a collection of all of all level subgroups of $f^{-1}(N)$

.

Proof :

Let M=$f^{-1}(N)$ and $t \in [0,1]$. we show that $M_t = f^{-1}(N_t), \forall t \in [0,1]$.

$x \in M_t \iff f^{-1}(N) \geq t$

$\iff N(f(x)) \geq t$

$\iff f(x) \in N_t$

$\iff x \in f^{-1}(N_t)$.

Thus ,in particular ,we have $M_{t_j} = f^{-1}\left(N_{t_j}\right), \forall j \in J$.

Suppose M has a level subgroup .

$M_t \notin \{N_{t_j}: j \in J\}$. This means that N must have a level subgroup N_t which does not belong to $\{N_{t_j}: j \in J\}$ such that $M_t = f^{-1}(N_t), t \in [0,1]$ holds (contradiction).

Thus $\{f^{-1}(N_{t_j}): j \in J\}$ is the collection of all level subgroups of $f^{-1}(N)$.

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