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Semi - α - $T_{½}$ - Space in Bitopological Spaces

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Abstract :

The objective of this research is to study some properties of ij - semi - α - $T_{1/2}$ - spaces in bitopological spaces and their relationship with ij - semi - α - T_0 - spaces and ij - semi - α - T_1 - spaces .

Key words : Bitopological space , ij - semi - α - open set , ij - semi - α - $T_{1/2}$ - space .

الخلاصة :

الهدف من هذا البحث هو دراسة بعض خصائص فضاءات ij- شبه ألفا $T_{½}$ في الفضاءات ثنائية التبولوجي وعلاقتها مع فضاءات ij- شبه ألفا T_0 و فضاءات ij- شبه ألفا T_1 .

1. Introduction :

In 1963 Kelly [2] defined : a set equipped with two topologies is called a bitopological space , denoted by (X,τ_1,τ_2) where (X,τ_1) and (X,τ_2) are two topological spaces . This notion was studied in different senses , one of these is the ij - semi - α - open sets ,that suggested by Qays , 2012 [4] . In 1997 , Kumar Sampath , S ., [3] introduced the concept of ij - α - open sets in bitopological spaces . In 1981 , Bose , S. , [1] introduced the notion of ij - semi - open sets in bitopological spaces . In this paper we first (in section 2) ij - semi - α - open sets , and then (in section 3) we study especial case of ij - semi - α - $T_{1/2}$ - space in bitopological spaces in the sense of ij - semi - α - open sets .

2. Preliminaries :

Throughout the paper , spaces always mean a bitopological spaces , the closure and the interior of any subset A of X with respect to τ_i, will be denoted by $\tau_i - cl(A)$, and $\tau_i - \text{int}(A)$ respectively, for $i = 1,2$.

Definition 2.1 : [4]

Let (X,τ_1,τ_2) be a bitopological space , $A \subseteq X$. Then A is a said to be ij - semi - α - open set if there exists an ij - α - open set U in X , such that $U \subseteq A \subseteq j - cl(U)$. The family of all ij - semi - α - open sets of X is denoted by ij - $S_\alpha O(X)$, where $i \neq j; i, j = 1,2$.

The following proposition will give an equivalent definition of ij - semi - α - open sets .

Proposition 2.2 : [4]

Let (X,τ_1,τ_2) be a bitopological space , $A \subseteq X$. Then A is an ij - semi - α - open set if and only if $A \subseteq j - cl(i - \text{int}(j - cl(i - \text{int}(A))))$.

Remark 2.3 : [4]

The intersection of any two ij - semi - α - open sets is not necessary ij - semi - α - open set as in the following example .

Example 2.4 :

Let $X = \{a,b,c\}$, $\tau_1 = \{X,\phi,\{a\},\{b\},\{a,b\}\}$, and $\tau_2 = \{X,\phi,\{a\}\}$. The family of all 12 - semi - α - open sets of X is : 12 - $S_\alpha O(X) = \{X,\phi,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\}\}$.
Hence $\{a,c\}$ and $\{b,c\}$ are two 12 - semi - α - open sets, but $\{a,c\} \cap \{b,c\} = \{c\}$ is not 12 - semi - α - open set.

Proposition 2.5 : [4]

The union of any family of ij-semi-α-open sets is ij-semi-α-open set .

Remark 2.6 : [4]

(i) Every τ_i – open set is ij-semi-α-open set ,but the converse need not be true .

(ii) If every τ_i – open set is τ_i – closed and every nowhere τ_i – dense set is τ_i – closed in any bitopological space , then every ij-semi-α-open set is an τ_i – open set .

Definition 2.7 :

A set A is said to be an ij-semi-α-neighborhood of a point x if there exists an ij-semi-α-open set U such that $x \in U \subseteq A$.

Definition 2.8 : [4]

The complement of ij-semi-α-open set is called ij-semi-α-closedset . Then family of all ij-semi-α-closedsets of X is denoted by $ij\text{-}S_\alpha C(X)$, where $i \neq j; i, j = 1,2$.

Remark 2.9 : [4]

The intersection of any family of ij-semi-α-closed sets is ij-semi-α-closedset .

Definition 2.10 : [4]

Let (X,τ_1,τ_2) be a bitopological space and $A \subseteq X$,the intersection of all ij-semi-α-closed sets containing A is called ij-semi-α-closureof A , and is denoted by $ij\text{-}S_\alpha\text{-}cl(A)$;

i.e $ij\text{-}S_\alpha\text{-}cl(A) = \bigcap\{B \subseteq X : B \text{ is } ij\text{-semi-}\alpha\text{-closedset}, A \subseteq X\}$.

Theorem 2.11 : [4]

Let (X,τ_1,τ_2) be a bitopological space , and let $A \subseteq X$, then :

(i) $ij\text{-}S_\alpha\text{-}cl(A)$ is the smallest ij-semi-α-closedset containing A .

(ii) A is ij-semi-α-closedset if and only if $ij\text{-}S_\alpha\text{-}cl(A) = A$.

Remark 2.12 : [4]

Let (X,τ_1,τ_2) be a bitopological space , and let A ,B be any subsets of X , then :

(i) $A \subseteq ij\text{-}S_\alpha\text{-}cl(A)$.

(ii) If $A \subseteq B$, then $ij\text{-}S_\alpha\text{-}cl(A) \subseteq ij\text{-}S_\alpha\text{-}cl(B)$.

(iii) $ij\text{-}S_\alpha\text{-}cl(ij\text{-}S_\alpha\text{-}cl(A)) = ij\text{-}S_\alpha\text{-}cl(A)$.

Lemma 2.13 :

If $ij\text{-}S_\alpha\text{-}cl(\{x\}) \bigcap A \neq \phi$, for each $x \in ij\text{-}S_\alpha\text{-}cl(A)$, then $(ij\text{-}S_\alpha\text{-}cl(A)) - A$ does not contain a non empty ij-semi-α-closedset .

Proof :

Suppose $(ij\text{-}S_\alpha\text{-}cl(A)) - A$ contains a non empty ij-semi-α-closedset , say B, then $x \in B$ implies $ij\text{-}S_\alpha\text{-}cl(\{x\}) \subseteq B \subseteq (ij\text{-}S_\alpha\text{-}cl(A)) - A$, and $ij\text{-}S_\alpha\text{-}cl(\{x\}) \bigcap A = \phi$, which contradicts the hypothesis . ■

Definition 2.14 :

A subset A of a bitopological space (X,τ_1,τ_2) is said to be ij-semi-α-generalized closed set (briefly ij-semi-α-g- closed set) if $ij\text{-}S_\alpha\text{-}cl(A) \subseteq U$ whenever $A \subseteq U$ and U is ij-semi-α-open set in (X,τ_1,τ_2) .

Lemma 2.15 :

A subset A of a bitopological space (X,τ_1,τ_2) is ij-semi-α-g- closed set if and only if $ij\text{-}S_\alpha\text{-}cl(\{x\}) \bigcap A \neq \phi$ for each $x \in ij\text{-}S_\alpha\text{-}cl(A)$.

Proof :

Suppose that A is ij-semi-α-g- closed set, and for some $x \in ij\text{-}S_\alpha\text{-}cl(A)$, $ij\text{-}S_\alpha\text{-}cl(\{x\}) \bigcap A = \phi$, then $A \subseteq X - (ij\text{-}S_\alpha\text{-}cl(\{x\}))$, where $X - (ij\text{-}S_\alpha\text{-}cl(\{x\}))$ is a

ij - semi - α - open set , so by definition (2.14) $ij\text{-}S_\alpha\text{-}cl(A) \subseteq X-(ij\text{-}S_\alpha\text{-}cl(\{x\}))$, hence $x \in X-(ij\text{-}S_\alpha\text{-}cl(\{x\}))$ i.e $x \notin X-(ij\text{-}S_\alpha\text{-}cl(\{x\}))$ which is a contradiction .

Conversely assume that for each $x \in ij\text{-}S_\alpha\text{-}cl(A)$, $ij\text{-}S_\alpha\text{-}cl(\{x\}) \cap A \neq \phi$; if there is a ij - semi - α - open set U such that $A \subseteq U$ but $ij\text{-}S_\alpha\text{-}cl(A) \not\subset U$ then there exists $x \in ij\text{-}S_\alpha\text{-}cl(A)$ and $x \notin U$, so $x \in X-U$ which implies $ij\text{-}S_\alpha\text{-}cl(\{x\}) \subseteq X-U$ (since $X-U$ is ij - semi - α - closed) i.e $ij\text{-}S_\alpha\text{-}cl(\{x\}) \cap A = \phi$, a contradiction . Therefore A is ij - semi - α - g - closed set . ■

3. Semi - α - $T_{1/2}$ - space in Bitopological Spaces :

In this section the notion of ij - semi - α - $T_{1/2}$ - space is introduced in bitopological spaces and their relationships with ij - semi - α - T_0 - space and ij - semi - α - T_1 - space are studied .

Definition 3.1 :

Let (X,τ_1,τ_2) be a bitopological space ;two subsets A and B of X are ij - semi - α - separated if each is disjoint from the other's ij - semi - α - closure . (i.e $A \cap ij\text{-}S_\alpha\text{-}cl(B) = \phi$ and $ij\text{-}S_\alpha\text{-}cl(A) \cap B = \phi$) . Two points x and y in X are ij - semi - α - distinguishable if they do not have exactly the same ij - semi - α - neighborhoods (i.e there exists a ij - semi - α - open set containing x but not containing y or containing y but not containing x) . Two points x and y are ij - semi - α - separated if the singletons $\{x\}$ and $\{y\}$ are ij - semi - α - separated .

Definition 3.2 :

A bitopological space (X,τ_1,τ_2) is called ij - semi - α - T_0 - space if any two distinct points are ij - semi - α - distinguishable .

Remark 3.3 :

Every i - T_0 - space is ij - semi - α - T_0 - space .

Proof :

Follows from remark (2.6) (i) . ■

Remark 3.4 :

The converse of remark (3.3) is not true ,see the following example .

Example 3.5 :

Let $X = \{a,b,c,d\}$, $\tau_1 = \{X,\phi,\{a\},\{a,d\},\{b,c\},\{a,b,c\}\}$, $\tau_2 = \{X,\phi,\{a\}\}$. Then $12\text{-}S_\alpha O(X) = \tau_1 \cup \{\{a,b\},\{a,b,d\},\{a,c\},\{a,c,d\}\}$ and it is clear that (X,τ_1,τ_2) is 12 - semi - α - T_0 - space but (X,τ_1) is not 1 - T_0 - space since the points b and c are not distinguishable .

Theorem 3.6 :

A bitopological space (X,τ_1,τ_2) is ij - semi - α - T_0 - space iff for each distinct points x , y in X , $ij\text{-}S_\alpha\text{-}cl(\{x\}) \neq ij\text{-}S_\alpha\text{-}cl(\{y\})$.

Proof :

Suppose that (X,τ_1,τ_2) is ij - semi - α - T_0 - space and let $x \neq y$ be two points of X such that $ij\text{-}S_\alpha\text{-}cl(\{x\}) = ij\text{-}S_\alpha\text{-}cl(\{y\})$, therefore $x \in ij\text{-}S_\alpha\text{-}cl(\{y\})$ and $y \in ij\text{-}S_\alpha\text{-}cl(\{x\})$. If U is ij - semi - α - open set such that $x \in U$ and $y \notin U$, then $y \in X-U$ (ij - semi - α - closedset) so $ij\text{-}S_\alpha\text{-}cl(\{y\}) \subseteq X-U$ which means $ij\text{-}S_\alpha\text{-}cl(\{x\}) \subseteq X-U$ and so $x \in X-U$ i.e $x \notin U$ a contradiction . Similarly the assumption that $x \notin V$ and $y \in V$ (for some ij - semi - α - open set V) leads to a contradiction , that is (X,τ_1,τ_2) is not an ij - semi - α - T_0 - space .

On the other hand, suppose that for each $x, y \in X$ and $x \neq y$ we have $ij\text{-}S_\alpha\text{-}cl(\{x\}) \neq ij\text{-}S_\alpha\text{-}cl(\{y\})$, therefore either $x \notin ij\text{-}S_\alpha\text{-}cl(\{y\})$ and so $x \in X-(ij\text{-}S_\alpha\text{-}cl(\{y\}))$ but $y \notin X-(ij\text{-}S_\alpha\text{-}cl(\{y\}))$,or $y \notin ij\text{-}S_\alpha\text{-}cl(\{x\})$ and so $y \in X-(ij\text{-}S_\alpha\text{-}cl(\{x\}))$ but $x \notin X-(ij\text{-}S_\alpha\text{-}cl(\{x\}))$ where $X-(ij\text{-}S_\alpha\text{-}cl(\{x\}))$ and $X-(ij\text{-}S_\alpha\text{-}cl(\{y\}))$ are ij-semi-α-open sets in (X,τ_1,τ_2) i.e x and y are ij-semi-α-distinguishable , hence (X,τ_1,τ_2) is ij-semi-α-T_0- space . ■

Theorem 3.7 :

Every subspace of ij-semi-α-T_0- space is ij-semi-α-T_0- space .

Proof :

Let (Y,τ_{1Y},τ_{2Y}) be a subspace of a bitopological space (X,τ_1,τ_2) and let X is ij-semi-α-T_0- space . To prove that Y is ij-semi-α-T_0- space , let $y_1 \neq y_2 \in Y$. Since $Y \subseteq X$, then $y_1 \neq y_2 \in X$ and X is ij-semi-α-T_0- space . There exists ij-semi-α-open set G in X , such that $y_1 \in G$ and $y_2 \notin G$. So $G \cap Y$ is ij-semi-α-open set in Y and $y_1 \in G \cap Y$, $y_2 \notin G \cap Y$.

Hence (Y,τ_{1Y},τ_{2Y}) is ij-semi-α-T_0- space . ■

Definition 3.8 :

A bitopological space (X,τ_1,τ_2) is said to be ij-semi-α-$T_{1/2}$- space if every ij-semi-α-g-closed set in (X,τ_1,τ_2) is ij-semi-α-closed.

Theorem 3.9 :

A bitopological space (X,τ_1,τ_2) is ij-semi-α-$T_{1/2}$- space if and only if , for each $x \in X$, $\{x\}$ is ij-semi-α-closedor ij-semi-α-open .

Proof :

Assume that (X,τ_1,τ_2) is ij-semi-α-$T_{1/2}$- space and $\{x\}$ is neither ij-semi-α-closednor ij-semi-α-open then $X-\{x\}$ is not ij-semi-α-closedso $ij\text{-}S_\alpha\text{-}cl(X-\{x\}) = X \subseteq X$ i.e $X-\{x\}$ is ij-semi-α-g- closed set , by definition of ij-semi-α-$T_{1/2}$- space , $X-\{x\}$ must be ij-semi-α-closed, a contradiction with the assumption .

On the other hand, suppose that for each x in(X,τ_1,τ_2) ,$\{x\}$ is ij-semi-α-closedor ij-semi-α-open. Let A be ij-semi-α-g- closed set in(X,τ_1,τ_2), then by lemma (2.13) and lemma (2.15) $(ij\text{-}S_\alpha\text{-}cl(A))-A$ does not contain a non empty ij-semi-α-closedset , so if $x \in (ij\text{-}S_\alpha\text{-}cl(A))-A$ then $ij\text{-}S_\alpha\text{-}cl(\{x\}) \not\subset (ij\text{-}S_\alpha\text{-}cl(A))-A$ i.e $ij\text{-}S_\alpha\text{-}cl(\{x\}) \neq \{x\}$ which means $\{x\}$ is not ij-semi-α-closed, so it must be ij-semi-α-open, but $\{x\} \cap A = \phi$ implies $x \notin ij\text{-}S_\alpha\text{-}cl(A)$,
a contradiction , hence $(ij\text{-}S_\alpha\text{-}cl(A))-A=\phi$. Therefore A is ij-semi-α-closed, and so (X,τ_1,τ_2) is
ij-semi-α-$T_{1/2}$- space . ■

Theorem 3.10 :

If (X,τ_1,τ_2) is ij-semi-α-$T_{1/2}$- space then it is ij-semi-α-T_0- space .

Proof :

Suppose (X,τ_1,τ_2) is ij-semi-α-$T_{1/2}$- space by theorem (3.9) every singleton is either ij-semi-α-closedor ij-semi-α-open. Let $x \neq y$ (in X) , if $\{x\}$ is ij-semi-α-closedthen $X-\{x\}$ is ij-semi-α-open set containing y but not containing x ; and if $\{x\}$ ij-semi-α-open then it is containing x but not containing y . So (X,τ_1,τ_2) is ij-semi-α-T_0- space . ■

Remark 3.11 :

The converse of theorem (3.10) is not true , see the following example .

Example 3.12 :

Let $X=\{a,b,c\}$, $\tau_1=\{X,\phi,\{a\},\{a,b\}\}$, $\tau_2=D$ (the discrete topology on X) then $12\text{-}S_\alpha O(X)=\tau_1$ and (X,τ_1,τ_2) is a 12-semi-α-T_0- space but not 12-semi-α-$T_{1/2}$- space (since $\{b\}$ is neither 12-semi-α-closed nor 12-semi-α-open) .

Theorem 3.13 :

Every i-$T_{1/2}$-space is ij-semi-α-$T_{1/2}$- space .

Proof :

It follows the fact that $\tau_i\subseteq ij\text{-}S_\alpha O(X)$ and theorem (3.9) . ■

Remark 3.14 :

The converse of theorem (3.13) is not true , see the following example .

Example 3.15 :

Let $X=\{a,b,c\}$, $\tau_1=\{X,\phi,\{a\},\{a,b\}\}$, $\tau_2=I$ (the indiscrete topology on X) then $12\text{-}S_\alpha O(X)=\tau_1\cup\{\{a,c\}\}$ and (X,τ_1) is not 1-$T_{1/2}$-space but (X,τ_1,τ_2) is 12-semi-α-$T_{1/2}$-space .

Definition 3.16 :

A bitopological space (X,τ_1,τ_2) is said to be ij-semi-α-T_1- space if any two distinct points in X are ij-semi-α-separated .

Proposition 3.17 :

A bitopological space (X,τ_1,τ_2) is ij-semi-α-T_1- space iff every singleton subset $\{x\}$ of X is ij-semi-α-closed.

Proof :

Suppose (X,τ_1,τ_2) is ij-semi-α-T_1- space , and $x\in X$, if $y\in ij\text{-}S_\alpha\text{-}cl(\{x\})$ but $x\neq y$ then $ij\text{-}S_\alpha\text{-}cl(\{y\})\subseteq ij\text{-}S_\alpha\text{-}cl(\{x\})$. On the other hand by definition (3.16) we have $\{y\}\cap ij\text{-}S_\alpha\text{-}cl(\{x\})=\phi$ which is a contradiction ,so $ij\text{-}S_\alpha\text{-}cl(\{x\})=\{x\}$ i.e $\{x\}$ is ij-semi-α-closedset .

Conversely if for each x , $\{x\}$ is ij-semi-α-closed then $ij\text{-}S_\alpha\text{-}cl(\{x\})=\{x\}$ and any two distinct points of X are ij-semi-α-separated i.e (X,τ_1,τ_2) is ij-semi-α-T_1- space . ■

Remark 3.18 :

Every i-T_1-space is ij-semi-α-T_1-space.

Proof :

It follows from the fact that in i-T_1-space every singleton is an i-closed set in (X,τ_1) also any i-closed set in (X,τ_1) is ij-semi-α-closedset in (X,τ_1,τ_2) . ■

Remark 3.19 :

The converse of remark (3.18) is not true , see the following example .

Example 3.20 :

Let $X=\{a,b,c,d\}$, $\tau_1=\{X,\phi,\{a\},\{d\},\{a,d\},\{b,c\},\{a,b,c\},\{b,c,d\}\}$, and $\tau_2=\{X,\phi,\{a\},\{d\},\{a,d\}\}$ then $12\text{-}S_\alpha O(X)=\tau_1\cup\{\{a,b\},\{a,c\},\{b,d\},\{c,d\},\{a,c,d\},\{a,b,d\}\}$. So (X,τ_1,τ_2) is 12-semi-α-T_1-space since all singletons are 12-semi-α-closed sets but (X,τ_1) is not a 1-T_1-space since $\{b\}$ and $\{c\}$ are not 1-closed sets .

Theorem 3.21 :

Every subspace of ij-semi-α-T_1- space is ij-semi-α-T_1- space .

Proof :

Let (Y,τ_{1Y},τ_{2Y}) be a subspace of a bitopological space (X,τ_1,τ_2) and let (X,τ_1,τ_2) is ij-semi-α-T_1- space .To prove that Y is ij-semi-α-T_1- space , let $y_1 \neq y_2 \in Y$. Since $Y \subseteq X$, then $y_1 \neq y_2 \in X$ and since X is ij-semi-α-T_1- space . Then there exist two ij-semi-α-open sets G , H in X , such that $y_1 \in G$,but $y_2 \notin G$; and $y_2 \in H$,but $y_1 \notin H$ .Then we obtain two sets $G_1 = G \bigcap Y$, $H_1 = H \bigcap Y$ are ij-semi-α-open sets in Y , we have $y_1 \in G_1$, but $y_2 \notin G_1$; and $y_2 \in H_1$,but $y_1 \notin H_1$. Hence (Y,τ_{1Y},τ_{2Y}) is ij-semi-α-T_1- space . ■

Remark 3.22 :

Every ij-semi-α-T_1- space is ij-semi-α-T_0- space , but the converse is not true .

Proof :

This follows directly from the definitions (3.2) and (3.16) . ■

But the converse is not true as the following example .

Example 3.23 :

Let $X = \{a,b,c\}$, $\tau_1 = \{X,\phi,\{a\}\}$,and $\tau_2 = \{X,\phi,\{a\},\{a,b\}\}$, the family of all 12-semi-α-open sets of X is : $12\text{-}S_\alpha O(X) = \{X,\phi,\{a\},\{a,b\},\{a,c\}\}$ .If we take a and b ,$a \neq b$, then we can not find two 12-semi-α-open sets, such that one of them contains a but not b and the other contains b but not a . Therefore (X,τ_1,τ_2) is not 12-semi-α-T_1- space , but it is clear that (X,τ_1,τ_2) is ij-semi-α-T_0- space .

Theorem 3.24 :

If (X,τ_1,τ_2) is ij-semi-α-T_1- space then it is ij-semi-α-$T_{1/2}$- space .

Proof :

It follows from theorem (3.9) and proposition (3.17) . ■

Remark 3.25 :

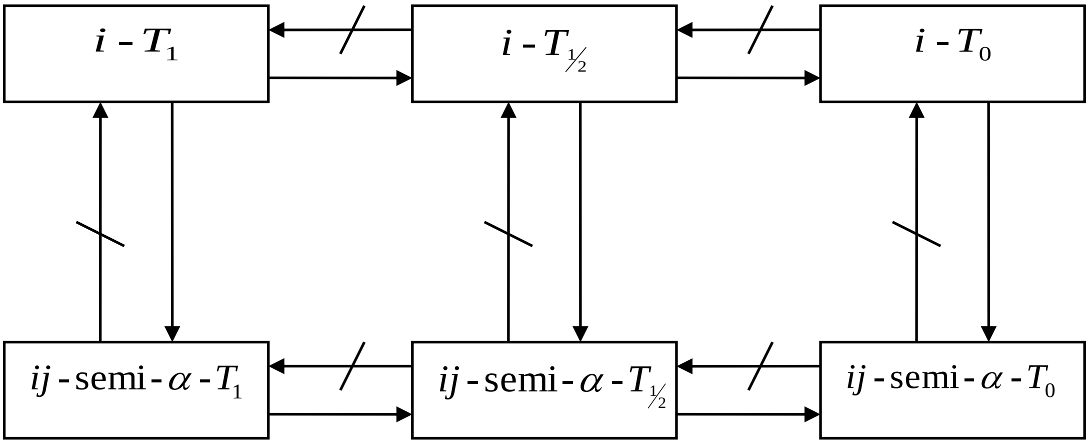
The converse of theorem (3.24) is not true , see the following example .

Example 3.26 :

Let $X = \{a,b\}$, $\tau_1 = \{X,\phi,\{a\}\}$, $\tau_2 = D$ (the discrete topology on X) , then $12\text{-}S_\alpha O(X) = \tau_1$, and (X,τ_1,τ_2) is 12-semi-α-$T_{1/2}$- space (since $\{a\}$ is 12-semi-α-open and $\{b\}$ is 12-semi-α-closed) , but not 12-semi-α-T_1- space since $\{a\}$ is not 12-semi-α-closed .

Remark 3.27 :

The following diagram shows the relations between $i\text{-}T_k$ and $ij\text{-semi-}\alpha\text{-}T_k$ spaces , where k = 0, ½ ,1 :



References :

[1] Bose , S. , " Semi - open sets , semi - continuity and semi - open mappings in bitopological spaces " , Bull . Cal . Math . Soc ., 73 (1981) , 237 – 246 .

[2] Kelly, J .C ., " Bitopological Spaces ", Proc . London Math . Soc . , 13 (1963) , 17 – 89 .

[3] Kumar Sampath , S .," On a Decomposition of Pairwise Continuity ", Bull. Cal . Math . Soc . , 89 (1997) , 441 – 446 .

[4] Qaye , H . I. Al-Rubaye , " Semi - α - Separation Axioms in Bitopological Spaces " , Al-Muthanna Journal of Pure Sciences , Vol .1 , No .1 , September , (2012) , 190 – 206 .