

## MINIMUM COST QUALITY CONTROL TESTS

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### ABSTRACT

An expected cost model of a process whose mean is controlled by an  $\bar{X}$  chart is developed.

Tow- stage numerical procedure is used to calculate the sample size, the number of units produced between sample, and the control limits of optimal control charts.

### 1- INTRODUCTION

Statistical testing procedures are used to control the quality of products produced by many types of industrial processes.

A product is considered characteristic of the product falls within prescribed limits, otherwise, the product is considered to be unacceptable or defective.

In most processes, the measurable characteristic which describes product quality is a random variable whose density function depends upon one or more parameters.

A process is said to be in control when these parameters equal some prescribed values called the control values. Control values of parameters which maximize the percent of acceptable quality product are generally selected.

The function of quality control procedure is to determine if the process is in or out of control.

This function is accomplished by periodically testing the null hypothesis that the process parameters are equal to the control values. The test is conducted by measuring the quality of a sample of several units of product.

From these measurements, the value of a test statistic is calculated.

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The set of all possible values of the test statistic is divided in to two subsets.

One subset includes these values of the test statistics considered likely to occur if the null hypothesis is true.

The other subset includes those values considered unlikely to occur if the null hypothesis is true.

This later subset is generally called the test critical region. If a value of the test statistic falls in the critical region, the null hypothesis is rejected and the process is investigated in order to determine and correct the condition which caused the process to go out of control.

If the value of the test statistic is not in the critical region, the null hypothesis is not rejected, the process is assumed to be in control, and it is allowed to continue to operate.

As in any hypothesis testing procedure, two types of errors may be made.

One type (generally called type I errors) involves rejecting the null hypothesis when the process is in control.

The second type (generally called type II errors) involves the acceptance of the null hypothesis when the process is out of control.

Type II error leads to costs associated with the increase in defective products produced by an out-of-control process.

Cost of unnecessary investigation and loss of production arise from type I error.

Both of these costs can be decreased by increasing the sample size and the frequency of sampling.

This reduction in error cost is, of course, accompanied by an increase in sampling and testing costs.

Type I error costs can also be decreased by decreasing the size of the critical region.

Decreasing the size of the critical region increasing the probability of accepting the null hypothesis when it is true, thus decreasing the type I error costs.

However, if some alternative hypothesis is true (that is, if the process is out of control) the probability of accepting the null hypothesis is also increased, thus increasing type II error costs.

## 2- *Purpose of the article*

The purpose of this article is to develop a method for choosing the test parameters (that is, the sample size, the frequency of sampling, and the critical region) in a manner that will minimize the total cost.

The investigation will be limited to quality control tests involving a single process parameter.

## 3- *The Content*

The method presented in most introductory quality control texts (for example, Burr (2), Duncan (5), and Grant (8)) involves the selection of the sample size and the critical region such that the power of the test to detect some specified shift in the process parameter (that is, the probability of rejection the null hypothesis given some specific alternative hypothesis) and the type I error level are some arbitrarily selected values.

With approach, the problem of how frequently a sample should be taken is ignored.

It has also been suggested Wetler (13) that the sample size should be selected to minimize the total amount of sampling required to detect a process parameter shift of some specified size with some specific probability.

Several investigations on minimum cost selection of quality control test parameters have been reported.

Cowden (3) carried out some numerical experiments based on the assumption that the process mean is out of control at the beginning of the day.

Duncan (4) assumed that the process may shift from the in-control state to a single out-of-control state any time during the day.

He further assumed that the process remains in the in-control state before going out-of-control is an exponential random variable with mean ( $\lambda^{-1}$ ) hours.

Girshick and Rubin (7) considered the problem of minimizing the running and repair costs for a machine which could be considered to be in one of four states.

The first two states represent levels of performance while the last two represent overhaul states.

Bather (1) developed a stopping rule which indicates when production on a machine should be halted and the machine overhauled. Tayler (11) shows that inspection of a fixed number of items at a fixed interval of time is non-optimal.

Instead, sampling should be determined at each stage by the current posterior probabilities.

Using this approach, Taylor (12) develops an optimal control procedure based on the assumption that the process has only two states, in-control and out-of-control.

Despite the non-optimal nature of fixed sample size, fixed time increment sampling plans, such plans are still widely used because of their ease of administration.

In light of the widespread use, it is desirable to obtain an optimal sampling plan within the class of fixed size –fixed time sampling plans. It is the purpose of this article to develop a method for selecting the optimal sampling plan from this class of plans.

In place of the assumption of one in-control state and one out-of-control state made by other authors DUNCAN (4), TAYLOR (12), it is assumed that the process parameter,  $\mu$ , is a continuous random variable which can be satisfactorily approximated by a discrete random variable.

One value,  $\mu_0$ , of the discrete random variable is associated with the in-control value of the process parameter and the remaining values,  $\mu_1, \mu_2, \dots, \mu_5$ , are associated with out-of-control values of the process parameter.

#### 4- GENERAL COST MODEL

The expected total cost  $E(c)$ , per unit of product, associated with a quality control test procedure can be written as

$$E(c) = E(c_1) + E(c_2) + E(c_3) \dots\dots\dots[1]$$

Where  $E(c_1)$  is the expected cost per unit associated with carrying out the test procedure,  $E(c_2)$  is the expected cost per unit associated with investigating and carrying the process when the test indicates that the process is out of control (that is, when the null hypothesis is rejected), and  $E(c_3)$  is the expected cost per unit associated with the production of defective product.

Both Cowden (3) and Duncan (4) consider the cost of sampling and testing to consist of a constant amount independent of the number of units sampled plus a constant amount for each unit sampled.

In view of the difficulty of obtaining accurate cost estimates, more complex cost functions appear to be unwarranted.

Thus, the expected sampling and testing cost per unit is

$$E(c_1) = \frac{a_1}{k} + \frac{a_2 N}{k n} \dots\dots\dots[2]$$

Where

$n$  is the sampling size,

$a_1$  is the fixed cost per sample,

$a_2$  is the cost per unit of product sampled,

$k$  is the number of units produced between samples.

The cost of rejecting the null hypothesis involves the costs of determining and correcting the cause of an apparent shift of the process parameter from  $\mu_0$  to some new value of  $\mu$ .

There may be some reason to suppose that the costs of determining the cause of a shift will depend upon the true value

of the parameter  $\mu$ , since it is likely that the cause of small shifts will be more difficult to find than the cause of large shift.

However, the cost of correcting process after the cause has been determined is often larger for large shifts than for small shifts.

It is also difficult to conceive of a situation prior information will be available concerning the cost of correcting a process as a function of the true value of the parameter  $\mu$ .

Prior information is generally available concerning how often the process goes out of control, how long the process is inoperative, and the cost per hour of an inoperative process.

From this information, the average cost of getting the process back into operation can be determined with reasonable accuracy.

Thus, it will be assumed that the cost of investigating and correcting a process that is apparently out of control is a random variable,  $(\nu)$ , with mean  $a_3$  whose distribution does not depend on the parameter  $\mu$ .

If  $(u)$  is a random variable which takes on the value one if the null hypothesis is rejected and zero otherwise, and if  $(\nu)$  equals zero when  $(u)$  equals zero (that is, investigation and correction costs are not incurred unless the null hypothesis is rejected), then the expected cost per unit for rejecting the null hypothesis is

$$E(c_2) = \frac{a_3 p(u=1)}{k}$$

Let  $(q)$  be the row vector of probabilities  $(q_i)$ , where  $(q_i)$  is the conditional probability of rejecting.

Ho given  $(\mu = \mu_i)$ , and let  $(\alpha)$  be the row vector of probabilities  $(\alpha_i)$ , where  $(\alpha_i)$  is the probability that  $(\mu = \mu_i)$  at the time the test is performed, then the expected cost of rejecting the null hypothesis is

$$E(c_2) = \frac{a_3}{k} \sum_{i=0}^s q_i \alpha_i = \frac{a_3}{k} q \alpha^t \dots\dots\dots [3]$$

where  $\alpha^t$  is the transpose of  $\alpha$ .

There is some intuitive appeal to the argument that the relationship between the number of defectives produced and the cost of producing defectives is nonlinear, since a small number of defectives may go unnoticed by the custom while a large number of defective may cause loss of future business.

However, in view of the inherent difficulties involved in determining the nature of this relationship, a simple linear relationship is assumed.

If  $a_4$  is the cost associated with producing a defective unit of product, if  $f$  is the row vector of probabilities  $f_i$ , where  $f_i$  is the conditional probability of producing a defective unit given  $\mu = \mu_i$ , and if  $\gamma$  is the row vector of probabilities  $\gamma_i$  where  $\gamma_i$  is the probability that the process is in state  $(\mu_i)$ , then the expected cost per unit associated with accept the null hypothesis is

$$E(c_3) = a_4 \sum_{i=0}^s f_i \gamma_i = a_4 f \gamma^t \dots\dots\dots [4]$$

Combining equations 1, 2, 3, and 4, the total expected cost becomes

$$E(c) = \frac{a_1}{k} + \frac{a_2 n}{k} + \frac{a_3}{k} q \alpha^t + a_4 f \gamma^t \dots\dots\dots [5]$$

In the above function, the  $a$ 's are cost coefficients which are assumed to be functionally independent of the test parameter.

The vector  $f$  depends only on the nature of the process parameter and the definition of defective unit and, thus, does not functionally depend on the test parameters.

The vector  $q$ ,  $\alpha$  and  $\gamma$  are, however, functionally dependent on the test parameters.

The form of this dependency is developed in later sections. Thus far, only two (that is,  $n$  and  $k$ ) of the three test parameters have been defined.

In order to express the third test parameter (that is, the test critical rejoin) as a single parameter, some restrictions must be placed on the nature of the test.

It will be assumed that the test statistic, ( $T$ ), is normally distributed with mean ( $\mu$ ) and variance ( $\sigma^2 / N$ ), and that the critical rejoin is symmetric and defined by the critical rejoin parameter  $L$  such that the null hypothesis is rejected if

$$T > \mu_0 + (L\sigma / \sqrt{N})$$

Of if

$$T < \mu_0 - (L\sigma / \sqrt{N})$$

#### 4-1 PROBABILITY VECTOR $q$

On the basis of the assumption that ( $T$ ) is normally distributed with mean ( $\mu$ ) and variance ( $\sigma^2 / N$ ), the probability of rejecting ( $H_0$ ) when ( $\mu = \mu_i$ ) can be written as

$$\begin{aligned} q_i &= p(T > \mu_0 + \frac{L\sigma}{\sqrt{N}}) + p(T < \mu_0 - \frac{L\sigma}{\sqrt{N}}) \\ &= \phi(\frac{\mu_0 - \mu_i}{\sigma} \sqrt{N} + L) + 1 - \phi(\frac{\mu_0 - \mu_i}{\sigma} \sqrt{N} - L) \dots\dots\dots[6] \end{aligned}$$

Where

$$\phi(a) = \int_{-\infty}^a \frac{1}{\sqrt{2\Pi}} \left[ \exp\left(-\frac{Z^2}{2}\right) \right] dz.$$

When  $i=0$  the above reduces to

$$q_0 = 2\phi(L) \dots\dots\dots[7]$$



#### 4-2 PROBABILITY VECTOR $\alpha$

The elements,  $\alpha_i$ , of the probability vector  $\alpha$  represent the steady-state probability that the process is in state  $i$  (that is,  $\mu = \mu_i$ ) at the time a sample is selected.

To obtain these steady-state probabilities, the transition probability matrix,  $(\beta)$ , is required-the elements,  $(b_{ij})$ , of the matrix  $(\beta)$  represent the probability of the process shifting from state  $(i)$  to state  $(j)$  during the production of the  $(k)$  unit between sample.

To obtain the probabilities  $(b_{ij})$  the a priori probabilities vector  $(p)$  must be defined. The elements  $(p_i)$  of the vector  $(p)$  represent the probability that the process will shift from the in-control state (that is,  $\mu = \mu_0$ ) to the out-of-control state ( $\mu = \mu_i$ ) during the production of  $(k)$  units.

If it is assumed that the time the process remains in the in-control state before going out-of-control is an exponential random variable with mean  $(\lambda^{-1})$  hours, then the probability of remaining in state  $(\mu_0)$  for  $(h)$  hours is

$$1 - \int_0^h \lambda e^{-\lambda \tau} d\tau = e^{-\lambda h}$$

If a production rate of  $(R)$  units per hour is assumed and the production of a fraction of a unit is allowed,  $(p_0)$  becomes

$$p_0 = e^{-\lambda K / R} = e^{-\lambda' K} \dots\dots\dots [8]$$

Where  $(\lambda' = \lambda / R)$  (that is,  $\lambda'^{-1}$  is the average number of units produced before an out-of-control shift occurs).

Same method is needed to assign the remaining probability  $(1 - e^{-\lambda' K})$ , to the  $(s)$  states  $(\mu_1, \mu_2, \dots\dots\dots, \mu_5)$ . Many such

assignments exist; however, one simple family of distribution, which is indexed by single parameter, is presented.

Recall that the binomial probability of  $(i)$  successes in  $(s)$  trials is

$$p_i^* = \frac{s!}{i!(s-i)!} \pi^i (1-\pi)^{s-i},$$

Where  $0 < \pi < 1$ . recall also that

$$\sum_{i=1}^s p_i^* = 1 - p_0^* = 1 - (1-\pi)^s.$$

But probabilities,  $(p_i)$ , are desired such that

$$\sum_{i=1}^s p_i = 1 - e^{-\lambda'K}$$

Thus, let

$$p_i = \frac{(1 - e^{-\lambda'K}) s! \pi^i (1-\pi)^{s-i}}{[1 - (1-\pi)^s] i!(s-i)!} \dots\dots\dots [9]$$

For  $i = 1, 2, 3, \dots, s$

It is easily show that  $(p_0)$  defined by equation (8) and  $(p_i)$  defined by equation (9) define a probability distribution for a discrete random variable wick takes on the values  $(0, 1, 2, \dots, s)$ . To show this, note that all  $(p_i)$  must be non-negative and that

$$\begin{aligned} \sum_{i=0}^s p_i &= e^{-\lambda'K} + \frac{1 - e^{-\lambda'K}}{1 - (1-\pi)^s} \sum_{i=1}^s \frac{s! \pi^i (1-\pi)^{s-i}}{i!(s-i)!} \\ &= e^{-\lambda'K} + \frac{1 - e^{-\lambda'K}}{1 - (1-\pi)^s} [1 - (1-\pi)^s] \\ &= e^{-\lambda'K} + 1 - e^{-\lambda'K} = 1 \end{aligned}$$

While it is difficult to justify this a priori distribution on an intuitive basis, by appropriate choice of  $(\lambda'), (\pi)$  and  $(s)$  should be possible approximate almost any desired a priori distribution.

Several additional assumptions must be made before the transition probabilities are defined.

First it is assumed that when the process goes out-of-control (that is,  $\mu$  shifts from  $\mu_0$  to  $\mu_i$ ) it stays out-of-control until detected (that is, until  $H_0$  is rejected). In practice this means that the process will not correct itself.

It will be further assumed that when the process goes out-of-control it will not improve, but it may get worse.

This means that if  $(\mu)$  shifts to  $(\mu_i)$  it cannot shift back to  $(\mu_i - j)$  but it may shift to  $(\mu_i + j)$ . However, only one shift is allowed in each testing period.

On the basis of these two assumptions, the elements of  $(\beta)$  can now be defined.

When  $(j)$  is less than  $(i)$  the probability,  $(b_{ij})$  that the process is in state  $(\mu_i)$  at time  $(t)$  and in state  $(\mu_j)$  at time  $(t + (k / R))$  is simply the probability of rejecting  $(H_0)$  (that is,  $q_i$ ) at time  $(t)$  multiplied by the probability  $(p_i)$  of shifting to  $(\mu_j)$  during the production of the next  $(k)$  units. Thus, for  $(j < i)$ ,

$$b_{ij} = q_i p_j \dots\dots\dots [10]$$

For the case  $(j > i)$ , the probability of failing to reject  $(H_0)$  at time  $(t)$  multiplied by the probability,  $(p_{ij})$  of shifting directly (that is, without returning to  $\mu_0$ ) from  $(\mu_i)$  to  $(\mu_j)$  in the time required produce  $(k)$  units must be added to the above equation.

Thus far, the  $(p_{ij})$ 's have not been defined. It seems reasonable to let  $(p_{ij})$  be proportional to the probability of shifting from  $(\mu_0)$  to  $(\mu_j)$  (that is,  $p_j$ ).

One way to do this, while staying within the restriction that the process cannot get better by itself (that is, the process cannot shift directly from  $(\mu_i)$  to  $(\mu_j)$  when  $i > j$ ), is to let

$$p_{ij} = \frac{\sum_{i=1}^j p_i}{1 - p_0},$$

$$p_{ij} = \frac{p_i}{1 - p_0}$$

for  $(i > j)$ , and  $p_{ij} = 0$  for  $(i > j)$ .

Thus, for  $(j > i)$ ,  $b_{ij}$  becomes

$$b_{ij} = q_i p_j + \frac{(1 - q_i) p_j}{1 - p_0} \dots\dots\dots [11]$$

for  $(i \neq 0)$ , the probability,  $(b_{ij})$ , of being in state  $(\mu_i)$  at time  $(t)$  and also time  $(t + (k / R))$  is equal to the probability of rejecting  $(H_0)$  at time  $(t)$ , multiplied by the probability of returning to  $(\mu_i)$  during the production of the next  $(k)$  units, plus the probability of failing to reject  $(H_0)$  at time  $(t)$ , multiplied by the probability of remaining in state  $(\mu_i)$  during the production of the next  $(k)$  units.

Thus, for  $(i = 0)$ ,  $(b_{ij})$  is

$$b_{ii} = q_i p_i + (1 - q_i) p_{ij},$$

and hence,

$$b_{ii} = q_i p_i + \frac{(1 - q_i) \sum_{j=1}^s p_j}{1 - p_0} \dots\dots\dots [12]$$

Finally, the probability of shifting from state  $(\mu_0)$  at time  $(t)$  to state  $(\mu_j)$  at time  $(t + (k / R))$  is simple  $(p_j)$ . Hence

$$b_{0j} = p_j \dots\dots\dots [13]$$

It is easily shown that the elements of each row of the matrix  $(\beta)$  defined above sum to one.

It is also clear from the definitions of these vectors are greater than zero but less than one.

Thus, it is easily shown that

$$0 < b_{ij} < 1$$

for  $(i, j = 0, 1, 2, \dots, s)$ . It is shown in many texts on stochastic processes (6), (10) that the above conditions define a transition matrix of an irreducible a periodic positive recurrent markov chain. Thus, there exists a vector  $(\alpha)$  such that

$$\alpha\beta = \alpha,$$

where

$$\alpha = (\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_s) \text{ and } \sum_{i=0}^s \alpha_i = 1.$$

Further more,  $(\alpha_i)$  is the long-run (or steady-state) un conditional probability of being in state  $(\mu_i)$  regardless of the initial state of the process.

To solve for  $(\alpha)$ , rewrite the above equation as

$$\alpha\beta - \alpha = 0$$

and, hence,

$$\alpha(\beta - I) = 0 \dots\dots\dots [14]$$

where 0 is a row vector of  $(s + 1)$  zeroes and  $(I)$  is an  $(s + 1) \times (s + 1)$  identity matrix.

Adding the condition

$$\sum_{i=0}^s \alpha_i = 1$$

to equation (14), the result is

$$\alpha(\beta - I \setminus 1) = (0, 1) \dots\dots\dots [15]$$

Where  $(\beta - I \setminus 1)$  is the  $(\beta - I)$  matrix augmented by a column vector of  $(s + 1)$  ones.

It can be shown that the sum of any  $(s)$  of the first  $(s + 1)$  restrictions represented by Equation (15) implies the  $(s + 1)$  st restriction.

Thus to solve for  $(\alpha)$ , any one of the first  $(s + 1)$  restrictions represented by Equation (15) can be eliminated by eliminating any one of the columns on the left side of the equation and the corresponding zero on the right side.

The column labeled '0' will be arbitrarily eliminated. The resulting equation is

$$\alpha\beta^* = (0, 1), \dots\dots\dots [16]$$

where  $(0)$  is now a vector of  $(s)$  zeroes and the elements of  $(\beta^*)$  (that is,  $(b_{ij}^*)$ ,  $i = 0, 1, 2, \dots\dots\dots, s$  and  $j = 1, 2, 3, \dots\dots\dots, s + 1$ ) are defined as

$$b_{ij}^* = b_{ij}$$

for  $i \neq j - 1$  and  $j \neq s + 1$ .

$$b_{i, s+1}^* = 1$$

for  $j = s + 1$ , and

$$b_{j-1, j}^* = b_{jj} - 1$$

for  $j \neq s + 1$ .

Since, according to Parzen (10) , a unique solution for  $(\alpha)$  exists,  $(\beta^*)$  must have an inverse. Thus  $(\alpha)$  becomes

$$\alpha = (0,1)\beta^{*-1}$$

If  $(b_{ij}^{*-1})$  are the elements of  $(\beta^{*-1})$ , then for  $(i = 0,1,2,\dots,s)$

$$\alpha_i = b_{s,i+1}^{*-1} \dots\dots\dots [17]$$

#### 4-3 PROBABILITY VECTOR $\gamma$

The vector  $(\alpha)$ , where  $(\alpha_i)$  is the steady-state probability that the process is in state  $(\mu_i)$  at the time a sample is taken, was developed.

In order to determine the cost of producing defectives, as defined in Equation (4), the steady-state probability,  $(\gamma_i)$ , of the process being in state  $(\mu_i)$  at any point in time is required.

In order to define the elements of  $(\gamma)$ , it is necessary to analyze further those intervals in which the process shifts out-of-control. Duncan (4) shows that, given a shift between the  $(n\text{th})$  and the  $(n+1\text{st})$  sample, the average fraction of the interval that elapses before the shift occurs is

$$F = \frac{1 - (1 + \lambda h) e^{-\lambda h}}{\lambda h(1 - e^{-\lambda h})}$$

where  $(h)$  is the number of hours between samples.

Again let  $(\lambda' = \lambda / R)$  and  $(k = Rh)$ , the above equation becomes

$$F = \frac{1 - (1 + \lambda' k) e^{-\lambda' k}}{\lambda' k(1 - e^{-\lambda' k})} \dots\dots\dots [18]$$

where  $(k)$  is the number of units produced between samples.

Since the process is assumed to be unable to correct itself,  $(\gamma_0)$  depends only on the probability of the event that the process is in state  $(\mu_0)$  at the time a sample is taken and remains there until the next sample is taken, and the probability of the event that the process is in state  $(\mu_0)$  at the time the sample is taken but shifts to a new state sometime during the production of the next  $(k)$  units. Thus,  $(\gamma_0)$  is

$$\gamma_0 = \alpha_0 p_0 + F \alpha_0 (1 - p_0) \dots\dots\dots [19]$$

where  $(F)$  is the fraction of time spent in state  $(\mu_0)$  before the shift occurs,  $(\alpha_0)$  is the steady-state probability of being in state  $(\mu_0)$  at the time a sample is taken, and  $(p_0)$  is the probability of remaining in state  $(\mu_0)$  during the production of the  $(k)$  units.

The probability  $(\gamma_i, i \neq 0)$  depends upon the probability of the event that the process is in state  $(\mu_i)$  at the time a sample is taken and stays there during the production of the next  $(k)$  units,

The probability of the event that the process is in state  $(\mu_0)$  at the time a sample is taken and shifts to state  $(\mu_i)$  during the production of the next  $(k)$  units, the probability of the event that the process is in some lower state (say,  $\mu_m$  where  $m < i$ ) at the time a sample is taken and shifts to state  $(\mu_i)$  during the production of the next  $(k)$  units, and the probability of the event that the process is in state  $(\mu_i)$  at the time a sample is taken and switches to some higher state (say,  $\mu_n$  where  $n > i$ ) during the production of the next  $(k)$  units.

It is assumed that the fraction of time a process spends in the lower state when a switch is made to a higher state is on the



average the same fraction,  $(F)$ , of time the process spends in state  $(\mu_0)$ , given that a shift to some other state has occurred.

Thus,  $(\gamma_i)$ , for  $i = 0$ , is

$$\gamma_i = \alpha_i \frac{(\sum_{j=1}^i p_j)}{1 - p_0} + \alpha_0 (1 - F) p_i + \sum_{m=1}^{i-1} \alpha_m \left( \frac{p_i}{1 - p_0} \right) (1 - F) + \frac{\alpha_i F}{1 - p_0} \sum_{n=i+1}^s p_n, \dots [20]$$

Where the third term is zero when  $(i = 1)$  and the last term is zero when  $(i = s)$  [that is, it is impossible to shift to state  $(\mu_1)$  from any other state except  $(\mu_0)$ , and it is impossible to shift from state  $(\mu_s)$  to some higher state].

## 5- SOULTION METHOD

Tabulation of optimal quality central test parameter for the model as shown in equation (5) would require rather extensive tables in order to include several values of each of the cost coefficients  $(a_1, a_2, a_3, \text{ and } a_4)$  and a prior distribution some simplification is possible by letting

$$k = \lambda' k \dots \dots \dots [21]$$

$$A_i = \frac{a_i \lambda'}{a_4} \dots \dots \dots [22]$$

and

$$E^*(c) = \frac{E(c)}{a_4} \dots \dots \dots [23]$$

By dividing Equation [5] by  $(a_4)$  and making the above substitutions, the simplified model

$$E^*(c) = \frac{A_1}{K} + \frac{A_2 n}{K} + \frac{A_3}{K} q \alpha^t + f \gamma^t \dots\dots\dots [24]$$

is obtained.

A two-stage numerical procedure (9) was developed for choosing the values of (n, k and  $l$ ) which minimize  $[E^*(c)]$ .

In the first stage, the expected costs  $[E^*(c)]$  are computed for a wide variety of test parameters ( that is, n, k and  $l$ ), cost coefficients (that is,  $A_1, A_2$ , and  $A_3$ ), and for the desired values of the a priori distribution parameters (that is,  $\pi$  and  $s$ ).

From these results, the general behavior of the model can be studies and preliminary estimates of the optimal values of (n, k and  $l$ ) can be obtained.

In the second stage, the preliminary estimates obtained from the first stage are used as the starting point for a search method designed to locate the optimal values of the test parameters within any desired accuracy.

## 6- SOME NUMERICAL RESULTS

To illustrate the design of an optimal sampling plan, consider the following example:

$a_1 = 10$  Dr. per sample.

$a_2 = 1$  Dr. per unit sampled.

$a_3 = 100$  Dr. per investigation.

$a_4 = 10$  Dr. per defective unit produced,

$\lambda' = 0.001$  (that is, on the average, the process shifts out of control every 1000 units), and

$\pi = 0.376$  (that is ,on the average shift is  $2.4 \sigma$  )

From Equation 22

$$A1 = \frac{a_1 \lambda'}{a_4} = 0.001$$

$$A2 = \frac{a_2 \lambda'}{a_4} = 0.0001 \text{ and}$$

$$A3 = \frac{a_3 \lambda'}{a_4} = 0.01$$

From Table 1, the optimal sampling plan is shown as

$$E^*(c) = 0.0737, n=3$$

$$K=0.046, \text{ and } L=2.75.$$

Since  $K=\lambda'k$ , the optimal value of  $k$  is  $k = \frac{0.046}{0.001} = 46$

Thus, the optimal testing procedure is to take a sample of three every (46) units and reject  $H_0$  if  $(\bar{x} > \mu_0 + 2.75 \sigma \text{ or if } \bar{x} < \mu_0 - 2.75 \sigma)$ .

Since  $E^*(c) = \frac{E(c)}{a_4}$ , the expected cost per unit associated with the optimal testing procedure is

$$E(c) = 10(0.0737) = 0.737 \text{ Dr.}$$

As an aid to understanding the meaning and the interrelationships that exist between several of the variables used in the model, some of the intermediate calculations performed in the above sample problem will be presented.

First, consider the effect of  $k, \lambda'$  and  $\pi$  on the a priori distribution of the process mean  $\mu$

In the sample problem, it was assumed that, on the average, 1000 units are produced before the process goes out of control (that is,  $\lambda' = 0.001$ ). It was also assumed that when the process goes out of control the mean  $\mu$  shifts on the average  $2.4 \sigma$ . using Equations 8 and 9, and the fact that  $\mu_i = \mu_0 + i \sigma$ , it can be shown that  $\pi$  must equal 0.376 in order for the expected value of  $\mu$ , given that  $\mu$  is out of control (that is, given  $\mu \neq \mu_0$ ), to equal  $\mu_0 + 2.4 \sigma$ . If a sample is taken every 40 units (that is,  $k=40$ ), then  $K=\lambda'k=(0.001)(40)=0.04$  and, thus, the a priori distribution

of  $\mu$  from Equations 8 and 9 is (  $p_0, p_1, p_2, p_3, p_4, p_5, p_6$  )=(0.961, 0.009, 0.013, 0.011, 0.005, 0.001, 0.000 ).

If the number of units between samples should be increased ( say, from 40 to 80 ) it would be expected that the probability of finding the process in control ( that is ,  $p_0$  ) should decrease and the other values of  $p_i$  should increase. Using  $K=(0.001)(80)=0.08$  in equations 8 and 9, the following a priori distribution is obtained:

( $p_0, p_1, p_2, p_3, p_4, p_5, p_6$  )=(0.923, 0.018, 0.026, 0.021, 0.010, 0.002, 0.000).

in order to obtain the transition probability matrix  $B$ , the probability of rejecting  $H_0$ , given that  $\mu = \mu_i$  (that is,  $q_i$  ), is required for all values of  $i$ . these probabilities are a function of the control limits  $L$  and the sample size  $n$ . For example, when  $n=4$  and  $L=3$ , the vector  $q$  is calculated, using Equations ( 6 ) and ( 7 ), as

( $q_0, q_1, q_2, q_3, q_4, q_5, q_6$  )=(0.003, 0.159, 0.841, 0.998, 1.000, 1.000, 1.000).

Using the above values for the vectors  $p$  and  $q$  for (  $k=40, n=2$ , and  $L=3$  ), the modified transition probability matrix  $B^*$  defined in Equation ( 16 ) is calculated as

$$B^* = \begin{bmatrix} -0.8078 & 0.2895 & 0.2326 & 0.1051 & 0.0253 & 0.0025 \\ & 1.0 & & & & \\ 0.0075 & -0.8984 & 0.0527 & 0.0238 & 0.0057 & 0.0006 \\ & 1.0 & & & & \\ 0.0089 & 0.0134 & -0.9879 & 0.0051 & 0.0012 & 0.0001 \\ & 1.0 & & & & \\ 0.0089 & 0.0134 & 0.0108 & -0.9951 & 0.0012 & 0.0001 \\ & 1.0 & & & & \end{bmatrix}$$

The steady-state vector ( $\alpha$  ) of probabilities that the process mean is  $\mu_i$  at the time a test is conducted is obtained by finding the last row of the inverse of the above matrix.

The results for (  $k=40, n=2$  and  $L=3$  ) are

(  $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$  ) =(0.949, 0.011, 0.018, 0.014, 0.006, 0.002, 0.000 ).

The steady-state vector (  $\gamma$  ) of probabilities that the process mean is  $\mu_i$  at any point in time is calculated using Equation (19) and ( 20 ). The result for (  $k=40$ ,  $n=2$ , and  $L=3$  ) are  $(\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6) = (0.931, 0.011, 0.022, 0.022, 0.011, 0.003, 0.000)$ .

The only additional information required in order to calculate the expected cost is the vector (  $f$  ) of probabilities of producing a defective given that the process mean is  $\mu_i$ . Since it has been assumed that a defective is any unit whose measurement falls outside the range(  $\mu_i - 3\sigma$  ) to (  $\mu_i + 3\sigma$  ), the vector (  $f$  ) can easily be obtained from tables of the standard normal deviates as  $(f_0, f_1, f_2, f_3, f_4, f_5, f_6) = (0.003, 0.023, 0.159, 0.500, 0.841, 0.977, 0.999)$ .

Using the cost coefficients of the sample problem and values of (  $k=40$ ,  $n=4$ , and  $L=3$  ), the expected sampling and testing cost per unit can be calculated as

$$E(c_1) = \frac{a_1}{k} + \frac{a_2 n}{k} = \frac{10}{40} + \frac{(1)(4)}{40} = 0.350.$$

The expected cost per unit associated with rejecting the null hypothesis can be calculated as

$$E(c_2) = \frac{a_3}{k} q \alpha^i = \frac{100}{40} (0.0417) = 0.104,$$

and the expected cost per unit associated with accepting the null hypothesis can be calculated as

$$E(c_3) = a_4 f \gamma^i = 10(0.0297) = 0.297.$$

To illustrate the behavior of the three cost components as the number of units produced between samples is increased. As would be expected, using a fixed sample size, (  $n=4$  ) and fixed control limits (  $L=3$  ), the expected sampling and inspection costs per unit  $E(c_1)$  and the expected cost per unit associated with rejecting the null hypothesis  $E(c_2)$  both decrease as the number of units produced between samples increases. The expected cost per unit  $E(c_3)$ , of course, increases as the number of units between samples increases. Since  $n=4$  and  $L=3$  are close to the optimal values of  $n=3$  and  $L=2.75$  of the sample problem, it is not

surprising that the total expected cost achieves a minimum at a value of  $k$  close to the optimal value of 46.

To illustrate the effect of changing  $k$ ,  $n$ , and  $L$ , the total expected cost was calculated using the five values of  $k$  ( 20, 40, 60, 80, 100 ), three values of  $n$  ( 2,3,and 4), and two values of  $L$  ( 2 and 3)

The minimum total expected cost was obtained with a sample size of three (  $n=3$  ), three sigma control limits (  $L=3$ ), and a frequency of sampling of 40 units between samples ( $k=40$ ).The total expected cost associated with this scheme is 0.745 Dr. per unit.

| Cost<br>$A_2$ | Cost<br>$A_3$ | Param-eter | A priori Distribution Parameter $\pi$ |      |      |
|---------------|---------------|------------|---------------------------------------|------|------|
|               |               |            | .376                                  | .597 | .800 |

|       |      |       | Cost $A_1$ |       |       | Cost $A_1$ |       |       | Cost $A_1$ |       |       |
|-------|------|-------|------------|-------|-------|------------|-------|-------|------------|-------|-------|
|       |      |       | .0001      | .001  | .01   | .001       | .01   | .1    | .001       | .01   | .1    |
| .0001 | .001 | E*(c) | .0341      | .0610 | .1543 | .0782      | .2065 | .5796 | .0846      | .2335 | .6711 |
|       |      | n     | 2          | 2     | 4     | 2          | 3     | 4     | 2          | 2     | 2     |
|       |      | K     | .022       | .046  | .15   | .026       | .105  | .36   | .03        | .09   | .29   |
|       |      | L     | 2.25       | 1.75  | 1.0   | 3.5        | 1.75  | 3.0   | 2.75       | 2.25  | 3.75  |
|       | .01  | E*(c) | .0460      | .0737 | .1662 | .0875      | .2205 | .5830 | .0941      | .2430 | .6789 |
|       |      | n     | 2          | 3     | 8     | 2          | 3     | 9     | 2          | 2     | 2     |
|       |      | K     | .02        | .046  | .15   | .034       | .085  | .42   | .03        | .085  | .29   |
|       |      | L     | 3.0        | 2.75  | 2.25  | 3.0        | 3.75  | 2.0   | 3.5        | 3.25  | 3.75  |
|       | .1   | E*(c) | .1378      | .1635 | .2517 | .1784      | .3044 | .6571 | .1836      | .3297 | .7480 |
|       |      | n     | 2          | 4     | 10    | 3          | 4     | 12    | 2          | 3     | 5     |
|       |      | K     | .02        | .05   | .16   | .036       | .095  | .44   | .03        | .095  | .37   |
|       |      | L     | 3.75       | 3.5   | 3.0   | 3.75       | 4.0   | 3.0   | 4.0        | 3.75  | 3.5   |
| .001  | .001 | E*(c) | .0777      | .0907 | .1665 | .1201      | .2228 | .5859 | .1305      | .2521 | .6690 |
|       |      | n     | 2          | 2     | 2     | 2          | 2     | 2     | 2          | 2     | 2     |
|       |      | K     | .062       | .076  | .16   | .046       | .115  | .38   | .048       | .10   | .36   |
|       |      | L     | 1.5        | 1.25  | 0.5   | 3.5        | 1.25  | 2.5   | 2.5        | 2.25  | 1.5   |
|       | .01  | E*(c) | .0923      | .1058 | .1842 | .1285      | .2404 | .5918 | .1401      | .2615 | .6776 |
|       |      | n     | 2          | 2     | 2     | 2          | 2     | 2     | 2          | 2     | 2     |
|       |      | K     | .06        | .072  | .16   | .054       | .09   | .41   | .048       | .10   | .36   |
|       |      | L     | 2.25       | 2.25  | 1.75  | 2.75       | 3.5   | 2.25  | 3.25       | 2.75  | 2.25  |
|       | .1   | E*(c) | .1844      | .1978 | .2738 | .2194      | .3249 | .6673 | .2290      | .3486 | .7546 |
|       |      | n     | 2          | 2     | 3     | 2          | 2     | 3     | 2          | 2     | 2     |
|       |      | K     | .062       | .074  | .17   | .056       | .115  | .45   | .05        | .10   | .37   |
|       |      | L     | 3.0        | 3.0   | 2.75  | 3.25       | 3.0   | 2.5   | 3.75       | 3.5   | 2.75  |
| .01   | .001 | E*(c) | .2174      | .2196 | .2516 | .2888      | .3392 | .6186 | .3393      | .3869 | .7166 |
|       |      | n     | 2          | 2     | 2     | 2          | 2     | 2     | 2          | 2     | 2     |
|       |      | K     | .14        | .15   | .21   | .15        | .18   | .48   | .095       | .17   | .40   |
|       |      | L     | 0.25       | 0.25  | 0.25  | 2.0        | 1.5   | 0.25  | 3.75       | 1.75  | 1.25  |
|       | .01  | E*(c) | .2266      | .2307 | .2658 | .3010      | .3521 | .6302 | .3480      | .3962 | .7250 |
|       |      | n     | 2          | 2     | 2     | 2          | 2     | 2     | 2          | 2     | 2     |
|       |      | K     | .22        | .23   | .29   | .15        | .18   | .47   | .095       | .17   | .40   |
|       |      | L     | 1.5        | 1.5   | 1.25  | 2.5        | 2.25  | 1.75  | 3.75       | 2.5   | 2.0   |
|       | .1   | E*(c) | .3158      | .3195 | .3526 | .3895      | .4394 | .7136 | .4309      | .4806 | .8015 |
|       |      | n     | 2          | 2     | 2     | 2          | 2     | 2     | 2          | 2     | 2     |
|       |      | K     | .23        | .24   | .30   | .16        | .20   | .48   | .10        | .17   | .39   |
|       |      | L     | 2.5        | 2.25  | 2.25  | 2.75       | 2.75  | 3.5   | 3.75       | 3.25  | 3.75  |

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