On Monotonically T2-spaces and Monotonically normal spaces

Radhi I.M.Ali* ***Jalal Hatem Hussein****

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Abstract

In this paper we show that if ∏ Xi is monotonically T2-space then each Xi is monotonically T2-space, too. Moreover, we show that if ∏ Xi is monotonically normal space then each Xi is monotonically normal space, too. Among these results we give a new proof to show that the monotonically T2-space property and monotonically normal space property are hereditary property and topologically property and give an example of T2-space but not monotonically T2-space.

Keywords: topological space, continuous function, monotonically T_2-spaus, monotonically normal space.

1. Introduction

The property of monotonically T2-space first appeared by R.E. Buck, some weaker monotone separation and basis properties are presented in [1]. Unfortunately Buck's definition is not precise so we give a precise definition of monotonically T2-space and find several properties of such concept and other topological concepts. In order to make this work as self-contained as possible we give the following lemmas.

Lemma 1.1: [6] Let X, Y is two topological spaces and f: X→Y is a closed injective function then f is open function.

Proof: Let W be any open set in X, X\W is closed set. Hence f(X\W) is closed set in Y. Since f is injective f (X\W) =Y\ f (W) which implies that f (W) is open set in Y. Therefore f is open function□

Lemma 1.2: [6] Let X, Y is two topological spaces and f: X→Y is a continuous closed injective function and let B be a subset of Y then

f (B) = f (B)

Proof: Since f is continuous we have $f^{-1}(\overline{B}) \subseteq \overline{f^{-1}(B)}$.

Let $x \in f$ $(\overline{B})$, i.e. $f(x) \in \overline{B}$. If W is any open set in X containing x, then f (W) is open set in Y containing f(x) which implies that $f(W) \cap B \neq \varphi$ and have $W \cap f$ $(B) \neq \varphi$, therefore $x \in f$ (B)□

2. Monotonically T2-space

Let (x, τx) be a topological space, we begin with precise definition of monotonically T2-space.

Definition 2.1: Let Δ = {(x, x): x∈ X} and denote by (X x X)* = (X x X)\ Δ

A topological space X is monotonically T2-space if there is a function

g :(X x X)* → τx

* Department of mathematics- College of Science for Women- Baghdad University

Assigning to each order pair (x,y) in (X x X)* an open neighborhood g(x, y) ⊂ X of x such that

(i) $g(x , y) \cap g (y , x) = \phi$,
(ii) For each subset M of X and if $x \in \overline{U\{g(y, x): y\in M\}}$ then $x\in\overline{M}$.

We will call such a function g monotone T2-operator on X. Of course, every monotonically T2-space is T2-space, but later we find an example of T2-space but not monotonically T2-space.

Theorem 2.2: If(X, τx) is a topological space, (Y, τY) is a topological space which is monotonically T2-space and f: X→Y continuous closed injective function then X is a monotonically T2-space.

Proof: Let g′ be monotone T2-operator on Y, i.e.
$g':(Y \times Y)^* \rightarrow \tau Y$
Satisfies (i) and (ii) of definition (2.1). In order to define a monotone T2-operator on X, let (x,y) ∈ (X x X)* then (f(x) , f(y)) in (Y x Y)* because f is injective . So there is an open neighborhood g′(f(x),f(y))⊆ Y of f(x) .Since f is continuous from X into Y f(g′(f(x),f(y))) is an open neighborhood of x subset of X.
Define g: (X x X)* →τX as follows
g(x,y) = f (g′(f(x),f(y))) for all (x.y)∈(X x X)*.
We note that
g(x,y)∩g(y,x)=f (g′(f(x),f(y)))∩f (g′(f(y),f(x)))
= f (g′(f(x),f(y))∩g′(f(y),f(x))) = f (ϕ)=ϕ.
Let M be any subset of X and x ∈ U {g(y,x):y∈M} i.e.
x∈U{ f (g′(f(y),f(x))): y∈M} = f (U{g′(f(y),f(x)) : y∈M })
∴x∈ f (U{g′(f(y),f(x)) : y∈M }) which implies that
f(x)∈U{g′(f(y),f(x)) : y∈M }. i.e. f(x)∈U{g′(f(y),f(x)) : f(y)∈f(M)}.
Hence $f(x)\in\overline{f(M)}$ because Y is monotonically T-space.

∴ $f(x)\in f(\overline{M})$ because f is closed function ,which implies x∈M because is injective function . Thus X is monotonically T2-space and g is monotone T2-operator on X□

Corollary 2.3: If topological space X is a homeomorphic to monotonically T2-space Y then X is monotonically T2-space.

Proof: Since a Homeomorphism function is a continuous closed injective function then the result can be deduced from theorem 2.2.□

Theorem 2.4 [2]: If the topological space is monotonically T2-space then X is regular space.

We are going to give an example of T2-space but not monotonically T2-space.

Example 2.5 [6]**:** Let X denote the closed interval [0, 1] and D the subset {1/n: n=1, 2, 3…}. Define on X the smallest topology which contains every open set of X \{0} as a subspace of the real line R and every set Ba, (0<a≤1), defined by Ba= {t∈X: t<a and t∉D}. The space X is T2-space but not regular which implies it is not monotonically T2-space.

Theorem 2.6[2]**:** If X is a monotonically T2-space and A is a subspace of X then A is monotonically T2-space.
In the following theorem we are going to show that if the product
X x Y is monotonically T2-space then X and Y are monotonically T2-spaces.

Theorem 2.7: Let X ,Y be two topological spaces .If the product space X x Y is monotonically T2-space then X , Y are monotonically T2-spaces.

Proof: For each fixed $y \in Y$, X x {y} is a subspace of X x Y then by theorem 2.6 X x {y} is a monotonically T2-space but X x {y} homeomorphic to space X hence, by corollary 2.3, X is monotonically T2-space. Using same argument we can show that Y is monotonically T2-space, too.□
Use mathematical induction and apply theorem 2.7 we can get the following result □

Corollary 2.8: Let Xi be indexed family of topological spaces .If the product space ∏Xi is monotonically T2-space then each Xi is monotonically T2-space .

3. Monotonically Normal spaces

The property of monotone normality first appears without name, in Lemma 2.1 of C.R. Borge's paper on startifiable spaces [3]. In [4], Zenor and others gave properties and characterizations of monotonically normal spaces.

Definition 3.1 A T1-space X is monotonically normal if there is a function G which assigns to each order pair (H, K) of disjoint closed subsets of X an open set G(H,K) such that

(i) $H \subset G(H,K) \subset \overline{G(H,K)} \subset X \setminus K$

(ii) If (H′,K′) is a pair of disjoint closed subsets of X such that $H \subset H'$ and $K \supset K'$, then $G(H,K) \subset G(H',K')$.

The function G is called a monotone normality operator for X.

Lemma 3.2: [4] Any monotonically normal has a monotone normality operator G satisfying $G(H,K) \cap G(K,H)= \phi$ for any pair (H,K) of closed sets . Furthermore, each of the following properties is equivalent to monotone normality of a space X:

(a) There is a function G which assigns to each order pair (S, T) of separated subsets of X an open set G (S,T) satisfying

(i) $S \subset G(S, T) \subset \overline{G(S, T)} \subset X \setminus T$.

(ii) If (S′, T′) is a pair of separated sets having $S \subset S'$ and $T \supset T'$ then $G(S, T) \subset G(S', T')$.

(b) There is a function F which assigns to each order pair (p, C) with C closed set and $p \in X \setminus C$, an open set F (p ,C) satisfying

(i) $p \in F(p, C) \subset X \setminus C$.

(ii) If D is closed and $C \supset D$ then $F(p, C) \subset F(p, D)$.

(iii)If $p \neq q$ are points of X then $F(p, \{q\}) \cap F(q, \{p\}) =\phi$.

Remarks 3.3: [4] (a) The property described in lemma 3.2 (a) was originally called complete monotone normality and because condition (i) in (a) the space is T5-space. Hence the space which is normal but it is not T5-space is a normal space but not monotonically normal space, see [5].

Theorem 3.4: If X is monotonically normal and A is a closed subspace of X then A is monotonically normal space, too.

Proof : Let G be a monotone normality operator for X and H1,K1 be any two disjoint closed sets in A , so $H=H1 \cap A$, $K=K1 \cap A$ are two disjoint closed sets in X .Hence there is an open set G(H,K) in X ,we can define a function G1 assigns to each order pair (H1,K1) an open set G1 (H1,K1) in A such that
$G1 (H1,K1) = G(H, K) \cap A$,
It is easy to show that G1 is a monotone normality operator for A□

Theorem 3.5: If (X, τx) is a topological space, (Y, τY) is a monotone normal space with G is a monotone normality operator for Y and f: $X \to Y$ continuous closed injective function then X is a monotonically normal space, too.

Proof: Let H, K be any two disjoint closed sets in X. Since f is closed and injective function , f (H) , f(K) are two disjoint closed sets in Y and hence there is an open set G(f(H) , f(K)) in Y assigns with (f(H) , f(K)) . f (G(f(H) , f(K))) is an open set in X therefore we can define a function G′ assigns with H,K an open set G′ (H,K) in X as follows

G′(H,K) =f (G(f(H),f(K))) .

Since G is a monotone normality operator in Y and f is closed we obtain from

f (H) $\subset$ G (f(H) , f(K)) $\subset$ G (f(H) , f(K)) $\subset$ Y \ f(K) ,

H $\subset$ f (G (f(H) , f(K))) $\subset$ f (G(f(H), f(K))) =f (G(f(H),f(K))) $\subset$ f (Y\ f(K)) = X \ K.

Moreover, let (H′, K′) be a pair of disjoint closed sets in X having H $\subset$ H′ and K $\supset$ K′. Since f is closed and injective function we have f (H) $\subset$ f (H′) and f (K) $\supset$ f (K′). Hence G(f(H) , f(K)) $\subset$ G (f(H′). f(K′)) implies f (G(f(H) , f(K)))$\subset$ f (G(f(H′) , f(K′))) i.e. G′(H,K) $\subset$ G′(H′,K′)

$\therefore$ G′ is a monotone normality operator for X□

Corollary 3.6: The monotonically normal property of space is topological property.

Proof: Homeomorphism is continuous closed injective function and use theorem 3.5□

In the following theorem we are going to show that if the product space X x Y is a monotonically normal space then X , Y are monotonically normal spaces.

Theorem 3.7: let X and Y be two topological spaces. If the product space X x Y is monotonically normal space then X, Y are monotonically normal spaces.

Proof: For each fixed y $\in$ Y, X x {y} is closed subspace of X x Y and then by theorem 3.4 X x {y} is monotonically normal space .But X x {y} homeomorphic to space X hence X is monotonically normal space. By using same argument we can show that Y is monotonically normal space□

Use mathematical induction and apply theorem 3.7 we can get the following result.

Corollary 3.8: Let Xi indexed family of topological spaces .If the product space $\prod$Xi is monotonically normal space then Xi is monotonically normal space.

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الفضاءات$-T_2$ الرتيبية والفضاءات العادية الرتيبية

*راضي إبراهيم محمد** *جلال حاتم حسين**

* جامعة بغداد – كلية العلوم للبنات – قسم الرياضيات

الخلاصة:

أثبتنا في هذا البحث انه اذا كان $\prod$ Xi فضاءاً T2 رتيباً فأن كل من Xi فضاء T2 رتيب اضافة الى ذلك اثبتنا انه اذا كان $\prod$ Xi فضاءاً عاديا رتيباً فأن كل من Xi فضاء عادي رتيب ايضاً.و من بين هذه النتائج اعطينا اثباتاً جديداً بأن خاصية كون الفضاء T2رتيب بانها خاصية وراثية وتوبولوجية وأن خاصية كون الفضاء الرتيب العادي خاصية وراثية وتوبولوجية واعطينا مثالاً لفضاء T2 لكنه ليس فضاء T2رتيباً.