Some Results On Lie Ideals With (σ,τ)-derivation In Prime Rings

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Abstract

In this paper, we proved that if R is a prime ring, U be a nonzero Lie ideal of R , d be a nonzero (σ,τ)-derivation of R. Then if $Ua\subset Z(R)$ (or $aU\subset Z(R)$) for $a\in R$, then either or U is commutative Also, we assumed that Uis a ring to prove that: (i) If $Ua\subset Z(R)$ (or $aU\subset Z(R)$) for $a\in R$, then either a=0 or U is commutative. (ii) If $ad(U)=0$ (or $d(U)a=0$) for $a\in R$, then either a=0 or U is commutative. (iii) If d is a homomorphism on U such that $ad(U)\subset Z(R)$(or $d(U)a\subset Z(R)$, then a=0 or U is commutative.

Key words: R: prime ring, 6,J: R⟶R: automorphism mapping, U: lieideal

1.Introduction

Let $d: R\rightarrow R$ be an additive mapping If $d(xy)=d(x)\sigma(y)+\tau(x)d(y)$ for all $x,y\in R$, then d is called a (σ,τ)-derivation of R, where $\sigma,\tau:R\rightarrow R$ be two mappings on R [4]. On the other hand we said that d is an homomorphism or anti-homomorphism respectively if $d(xy)=d(x)d(y)$ or $d(xy)=d(y)d(x)$ for all x , $y\in R$.

Recall that a ring R is a prime if $aRb=0$, $a,b\in R$, implies that either a=0 or b=0 [4] . Also, we recall that U is a Lie ideal of a ring R if whenevere $u\in U$ and $r\in R$, $[u,r]\in U$ [3].Neset Aydin and Oznur Golbasi proved R is a prime ring and d is (σ,τ)-derivation of R , where $\sigma,\tau:R\rightarrow R$ automorphisms on R [2]. Then (i) If U is a nonzero left ideal of R which is a semiprime as a ring. If Ua=0 (or aU=0) for $a\in R$, then a=0. (ii) If U is a nonzero left ideal of R which is a semiprime as a ring such that $d(U)=0$, then $d(R)=0$.

In this paper we considered R *is* a prime ring ,U be a Lie ideal of R and d is a (σ,τ)-derivation of R, where $\sigma,\tau:R\rightarrow R$ be two automorphisms on R .

Also, we used the identities in this paper as follows: For all $x, y, z\in R$.

(i) $[xy,z]= x[y,z] + [x,z]y$

(ii) $[x,yz]= [x,y]z + z[x,z]$

(iii) $[xy,z]_{\sigma,\tau} = x[y,\sigma(z)]+[x,z]_{\sigma,\tau}$
$= x[y,z]_{\sigma,\tau} +[x,\tau(z)]y$

2.Results

Theorem(2. 1)

Let R be a prime ring ,U be nonzero Lie ideal of R . If $Ua=0$ ($aU=0$) for $a\in R$, then $a=0$.

Proof:

If $Ua=0$, then for all $u\in U$, $r\in R$, we have $0=[u,r]a=ura-rua=ura$. Hence $URa=0$. Since R is a Prime ring , then $a=0$. If $aU=0$, then for all $u\in U$, $r\in R$,we have $0=a[u,r]=aur-aru=-aru$ .Then $aru=0$, for all $u\in U$, $r\in R$. So, $aRU=0$. Since R is a prime ring, then $a=0$. Now, we can prove the first Theorem.

Theorem(2.2)

Let R be a prime ring , U nonzero Lie ideal of R and which ring . If $Ua\subset Z(R)$ ($aU\subset Z(R)$) for then either $a=0$ or U is commutative.

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Proof:

Assume $aU\subset Z(R)$, then for all we have $auv\in Z(R)$. So, for all $r\in R$ $0=[auv, r]= au[v,r]+[au,r]v = au[v,r]$ for all $u,v\in U$ $r\in R$ .Also, have $aUR[U,R]=0$. By a primeness of R , we have either $aU=0$ or $U\subset Z(R)$. If $aU=0$, then $a=0$ [by Lemma(2. 1)]. If $U\subset Z(R)$, then U is commutative. The same thing if we have $Ua\subset Z(R)$ so for all $u,v\in U$, $r\in R$ we have $0= [uva ,r]= u[va,r]+[u,r]va = [u,r]va$ Also, we have $[U,R]RUa=0$. By a primeness of R , we have either $U\subset Z(R)$ or $Ua=0$. Also, we have either $a=0$ or U *is* commutative.

Example(2.3)

Let $R = \left\{ \begin{pmatrix} x & y \\ z & t \end{pmatrix} ,x,y,z,t \in Z ,\text{where } Z \text{ is the number of integers} \right\}$ be 2x 2 matrices with respect to the usual operation of addition and multiplication, then R is a prime ring see[1].Let $\sigma,\tau: R\to R$ be automorphisms $\sigma\begin{pmatrix} x & y \\ z & t \end{pmatrix}=\begin{pmatrix} x & -y \\ -z & t \end{pmatrix}$, $\tau\begin{pmatrix} x & y \\ z & t \end{pmatrix}=\begin{pmatrix} x & -y \\ -z & t \end{pmatrix}$ .Let d: $R\to R$, defined by $d\begin{pmatrix} x & y \\ z & t \end{pmatrix}=\begin{pmatrix} 0 & -y \\ z & 0 \end{pmatrix}$ is (σ,τ)-derivation of R . Let $U = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} a \in Z\right\}$ be an additive subgroup of R. So, U be a Lie ideal of R and which as is a ring .By the hypothesis of Theorem(2.2), we have $a\in R$ such that $a=\begin{pmatrix} a_1 & 0 \\ 0 & a_1 \end{pmatrix}$, $a_1\in Z(R)$ and so by the hypothesis $aU\subset Z(R)$(*or* $Ua\subset Z(R)$) is satisfied and we get U is commutative.

Lemma(2.4)

Let R be a prime ring , U be a nonzero Lie ideal of R , d be (σ,τ)derivation of R. If $d(U)=0$, then $d(R)=0$ or U is commutative.

Proof:

Assume that $d(U)=0$, then for all $u\in U$, $r\in R$, we have
$0=d(ur-ru)=d(ur)-d(ru)$
$=d(u)\sigma(r)+\tau(u)d(r)-d(r)\sigma(u)-\tau(r)d(u)$
$=\tau(u)(r)-d(r)\sigma(u)$
$=-[d(r),u]_{\sigma,\tau}$, so we have $[d(r),u]_{\sigma,\tau}=0$. Take rv , $v\in U$ instead of r, then
$0=[d(rv) u]_{\sigma,\tau}=[d(r)\sigma(v)+\tau(r)d(v),u]$
$=[d(r)\sigma(v),u]_{\sigma,\tau}$
$= d(r)[\sigma(v),\sigma(u)]+[d(r),u]_{\sigma,\tau}\sigma(v)$
$= d(r)[\sigma(v),\sigma(u)]$
Take xr instead of r, $x\in R$, then
$0=d(xr)[\sigma(v),\sigma(u)]$
$=d(x)\sigma(r)[\sigma(v),\sigma(u)]+\tau(x)d(r)[\sigma(v),\sigma(u)]$
$= d(x)\sigma(r)[\sigma(v),\sigma(u)]$ for all $u,v\in U$ $x,r\in R$. . Hence ,$d(R)R[\sigma(U),\sigma(U)]=0$. Since R is a prime ring , then we have either $d(R)$ or U is commutative.

Lemma (2.5)

Let R be a prime ring, Ube a nonzero Lie ideal of R. If $aUb=0$, for $a, b\in R$ then $a=0$ or $b=0$.

Proof :

Assume that $aUb=0$.So, for all $u\in U$, $r\in R$ we have $0=a[u,r]b=aurb-arub$. Take bx , $x\in R$ instead of r, then we have $aubxb-abxub=0$.Also,. $-abxub=0$.Then we have $abxub=0$ for all $u\in U$, $x\in R$ Hence , $abRUb=0$.By primeness of R, we have either $ab=0$ or $Ub= 0$.If $Ub=0$, then by Lemma (2.1), we have $b=0$. If $ab=0$ we have $b\neq 0$,then $a=0$.

Theorem (2.6)

Let R be a prime ring ,U be a nonzero Lie ideal of R which as is a ring, d be a nonzero (σ,τ)-derivation of R such that if $ad(U)=0$ (or $d(U)a=0$) for $a\in R$ then either a=0 or U is commutative .

Proof:

If $ad(U)=0$, then for all $u,v\in U$ we have $0=ad(uv)= ad(u)\sigma(v)+a\tau(u)d(v) = a\tau(u)d(v)$.*Since* τ is an automorphism of R, then $\tau(a)u\tau(d(v))=0$ for all $u,v\in U$ Hence , $\tau(a)U\tau(d(v))=0$. By Lemma (2.5) we have either a=0 or d(v)=0.If $d(v)=0$, then $d(U)=0$.So , by Lemma(2.4), we get U is commutative. If d(U)a=0, then for all $u,v\in U$ we have $0=d(uv)a= d(u)\sigma(v)a+\tau(u)d(v)a =d(u)\sigma(v)a$.Since σ is an automorphism *of* R, then $\sigma(d(u))v\sigma(a)=0$ for all $u,v\in U$. Hence, $\sigma(d(u))U\sigma(a)=0$. Therefore, we get either $d(u)=0$ or a= 0. If $d(u)=0$, then $d(U)=0$.So , by Lemma(2.4), we get U is commutative.

Example(2.7)

From Example(2.3) , we have R is a prime ring, U be a Lie ideal of R , d be a nonzero (σ,τ)-derivation of R. By the hypothesis of Theorem (2.6) , we have $ad(U)=0$ (or $d(U)a=0$) for $a\in R$, is satisfied and we get U is commutative.

Lemma(2.8)

Let R be a prime ring and let d be a (σ,τ)-derivation and is a homomorphism on U, when U be a nonzero Lie ideal of R which as is a ring. If $d(U)\subset Z(R)$, then $d=0$ or U is commutative.

Proof:

Since $d(U)\subset Z(R)$, then for all $u,v\in U$ and we have $d(uv)\in Z(R)$. Hence, $d(uv)=d(u)d(v)$

$d(u)\sigma(v)+\tau(u)d(v)\in Z(R)$ ….(1)

Replace u by urn ,$rn\in U$, then $d(urn)d(v)=$

$d(u)d(rn)\sigma(v)+\tau(u)\tau(rn)d(v)$

$d(u)d(rn)d(v)=d(u)d(rn)\sigma(v)+\tau(u)\tau(rn)d(v)$

$d(u)d(rnv)=d(u)d(rn)\sigma(v)+d(u)\tau(rn)d(v)$

$= d(u)\ d(rn)\sigma(v)+\tau(u)\tau(rn)d(v)$ for all $u ,v ,rn \in U$. Then $d(u)\tau(rn)d(v)= \tau(u)\ \tau(rn)d(v)$.Also, we have $[d(u)-\tau(u)]\ \tau(rn)d(v)=0$. Since $d(U)\subset Z(R)$ then $[d(u)-\tau(u)]\tau(m)Rd(v)=0$. By a primeness of R, we have either $d(U)=0$ or $\tau^{-1}(d(u)-\tau(u))m=0$.If $d(U)=0$, then by Lemma(2.4) we get either $d=0$ or U is commutative.

If $\tau^{-1}(d(u)-\tau(u))m = 0$ then $\tau^{-1}(d(u)-\tau(u))U=0$. Hence, by Lemma(2.1) , we have $d(u)-\tau(u)=0$ and so we have $d(u)=\tau(u)$ for all $u,v\in U$. From (1) , we have $d(u)\sigma(v)=0$. Also, we have $\sigma^{-1}(d(u))v=0$ for all $u,v\in U$. Hence , $\sigma^{-1}(d(u))U=0$. Then we have $d(u)=0$ [by Lemma (2. 1)] . Hence, $d(U)=0$. And so we have either U is commutative or $d(R)=0$, by Lemma(2.3)

Theorem(2.9)

Let R be a prime ring and let d be a nonzero (σ,τ) -derivation and is a homomorphism on U , when U be a nonzero Lie ideal of R which as is a ring. If $ad(U)\subset Z(R)$ $(d(U)a)\subset Z(R)$, then $a=0$ or U is commutative.

Proof:

Assume that $ad(U)\subset Z(R)$. So, for all $u,v\in U$ we have $ad(uv)=ad(u)d(v)\in Z(R)$. Then for all $r\in R$ we have $0=[ad(u)d(v),r]$

$= ad(u)[d(v),r]+[ad(u),r]d(v)$

$= ad(u)[d(v),r]$.

Also, we have $ad(u)R[d(v),r]=0$. Since R is a prime ring , then we have either $ad(u)=0$ or $d(U)\subset Z(R)$. If $ad(u)=0$ for all $u\in U$, then $ad(U)=0$.So, by Theorem (2.6) , we get either $a=0$ or U is commutative. If $d(U)\subset Z(R)$, then U is commutative. Assume that $d(U)a\subset Z(R)$. So, for all $u,v\in U$, we have $d(uv)a=d(u)d(v)a\in Z(R)$. Then for all $r\in R$ we have $0=[d(u)d(v)a,r]=d(u)\ [d(v)a,r]+[d(u),r]d(v)a$

$= [d(u),r]d(v)a$ .Also, we have $[d(u),r]Rd(v)a=0$.

Since R is a prime ring , then we have either $d(u)a=0$ or $d(U)\subset Z(R)$. If d(u)a=0 for all $u \in U$ then $d(U)a=0$.So, by Theorem (2.6) , we get either a=0 or U is commutative. If $d(U) \subset Z(R)$, then U is commutative.

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بعض النتائج على امثلة لي مع مشتقة(σ,τ) في الحلقات الاولية

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الخلاصة:

في هذا البحث، سوف نبرهن على انه أذا كانت R حلقة اولية ، U مثالي لي غير صفري في R ، d مشتقة (σ,τ) غير صفرية في R وكان العنصر $a \ni R$. لقد اعتبرنا ان U تمثل بحد ذاتها حلقة لبرهنة التالي:

1. اذا كان $Z(R) \supset Ua$ او $Z(R) \supset aU$)، فانه اما a=0 او U ابدالية.
2. اذا كان $ad(U)$=0 او $d(U)a$=0) و a ∍ R ، فانه اما a=0 او U ابدالية.
3. اذا كانت d متشاكلة على U بحيث ان $Z(R) \supset ad(U)$ أو $Z(R) \supset d(U)a$)، فانه اما a=0 او U ابدالية .