A Posteriori $L_\infty(L_2) + L_2(H^1)$–Error Bounds in Discontinuous Galerkin Methods For Semidiscrete Semilinear Parabolic Interface Problems

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Abstract:

The aim of this paper is to derive a posteriori error estimates for semilinear parabolic interface problems. More specifically, optimal order a posteriori error analysis in the $L_\infty(L_2) + L_2(H^1)$- norm for semidiscrete semilinear parabolic interface problems is derived by using elliptic reconstruction technique introduced by Makridakis and Nochetto in (2003). A key idea for this technique is the use of error estimators derived for elliptic interface problems to obtain parabolic estimators that are of optimal order in space and time.

Key words: A posteriori error estimates, Discontinuous Galerkin methods, Interface semilinear parabolic problems.

Introduction:

Whilst the topic of a posteriori error estimation for linear and nonlinear parabolic problems is now relatively well understood for both conforming and nonconforming methods, see, e.g., (1-15), there are comparatively few results for semi linear parabolic interface problems (16-21). Particularly, Cangiani et al (17) used a nonstandard elliptic projection of Douglas and Dupont (16) to derive optimal order a priori error estimates for these problems in $L_\yen(L_2)$-norm and extended this work to the fully discrete setting in (18). Metcalfe (19) derived optimal order a posteriori error estimates in the $L_\infty(L_2) + L_2(H^1)$ norm for fully discrete parabolic interface problems. Gupta and Sinha (20) used elliptic reconstruction techniques to derive a posterior error estimates for semi linear parabolic interface problems with a locally-Lipschtiz continuous nonlinearity in the forcing term. They used a Backward-Euler-Galerkin scheme to discretise in time with a conforming finite element method in space.

More recently, Sabawi (21) has derived an a posteriori error estimate for a class of nonlinear parabolic interface problems involving possibly curved interfaces, with flux balancing interface conditions, e.g., modelling mass transfer of solutes through semi-permeable membranes, in both the $L_\infty(L_2) + L_2(H^1)$ and $L_\yen(L_2)$ norms. Optimal order a posteriori error estimates $L_\infty(L_2) + L_2(H^1)$ were derived for semi and fully discrete nonlinear parabolic interface problems. The analysis revolved around a nonstandard elliptic reconstruction introduced by Douglas and Dupont (16).

The main contribution of this paper is to extend (21) to the case of semidiscrete semilinear parabolic interface problems in terms of $L_\infty(L_2) + L_2(H^1)$-norm. The main difficulty in constructing an optimal order a posteriori error estimator in $L_\infty(L_2) + L_2(H^1)$ is to deal with the nonlinear reaction term. These challenges are addressed by employing a continuation argument and the elliptic reconstruction technique, introduced by Makridakis and Nochetto (12) and extended to dG methods in (7).

It is worth noting the main reason of this technique is to lead us utilise ready elliptic interface a posteriori estimates that derived from elliptic interface problem (22-23) to bound the main part of the spatial error. There are some error estimators for semilinear parabolic problems available in the literature (24-30).

The rest of this paper is structured as follows. In Section 2, the model problem is introduced and discontinuous Galerkin method, with some necessary background results, are discussed. Section 3, $L_\infty(L_2) + L_2(H^1)$ error bounds for semi discrete semilinear parabolic interface problems, are presented. Conclusions are given in Section 4.

Model problem

Let $\Omega \subset \mathbb{R}^d, d = 2,3$ be a bounded open polygonal/polyhedral domain. The interface Γ^{tr} subdivides the domain Ω into two subdomains Ω_1, Ω_2. such that $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma^{tr}$, with $\partial\Omega :=$ $(\partial\Omega_1 \cap \partial\Omega_2)\backslash\Gamma^{tr}$; sFig. 1 for an illustration. Considered the semilinear parabolic interface problem

$$
\begin{aligned}
\frac{\partial u}{\partial t} - a\Delta u &= f(u) & \text{in} & \quad (\Omega_1 \cup \Omega_2) \times (0,T], \\
u &= 0 & \text{on} & \quad \partial\Omega \times (0,T], \\
a\nabla u_1.n^1 &= C_{tr}(u_2 - u_1) & \text{on} & \quad \Gamma^{tr} \times (0,T], \\
a\nabla u_2.n^2 &= C_{tr}(u_1 - u_2) & \text{on} & \quad \Gamma^{tr} \times (0,T], \\
u &= u_i & \text{in} & \quad (\Omega_1 \cup \Omega_2) \times \{0\}.
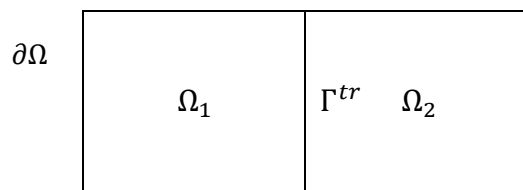\end{aligned} \tag{1}
$$

Figure 1. The interface Γ^{tr} subdivides the domain Ω into two subdomains Ω_1, Ω_2.

Here, $C_{tr} > 0$ is the interface transmission coefficient, and $u^i, i = 1,2$ represent the concertation of two compounds present in Ω_1 and Ω_2 respectively. Let $f: \Omega_1 \cup \Omega_2 \to \mathbb{R}$ be a given data function, and let $u_i, i = 1,2$ denote the respective outward unit normal vectors of Ω_i. This model is presented in (17, 18) to describe the mass transfer of solutes through semi-permeable membranes. For simplicity in our analysis will make some abbreviations. The standard Lebesgue and Hilbertian Sobolev spaces are denoted by $L_p(\omega), 0 \le p \le \infty$ $\quad L_p(W), 0 \pounds p \pounds \yen$ and $H^r(\omega), r \in \mathbb{R}$, respectively. In the special case when $r = 0$, will be donated by $L_2(\omega) \equiv H^0(\omega), \omega \subset \Omega$. The norm and standard $L_2(\omega)$-inner product will be denoted by $\|\cdot\| \equiv \|\cdot\|_\Omega$ and $\langle\cdot,\cdot\rangle \equiv \langle\cdot,\cdot\rangle_\Omega$ respectively when $\omega = \Omega$. Let the nonlinear forcing term $f(u)$ satisfy the following growth condition. There exist real numbers $C_g > 0$ and $\sigma \ge 0$ such that

$$|f(u) - f(v)| = C_g(1 + |u| + |v|)^\sigma |u - v|. \tag{2}$$

Next, the definition of the norm is given by

$$\|v\|_{L_P(0,T;X)} := \left(\int_0^T \|v\|_X^p \, dt\right)^{1/p}$$
$$< \infty \quad for \ 1 \le p < \infty,$$
$$\|v\|_{L_P(0,T;X)} := \text{ess sup}_{0 \le t \le T}\|v\|_X < \infty \ for \ p = \infty.$$

With this notation, picking $H^1(0,T; X) :=$ $\left\{u \in L^2(0,T; X) : \frac{\partial u}{\partial t} \in L^2(0,T; X)\right\}$.

Also, setting $\mathcal{H}^1 := H^1(\Omega_1 \cup \Omega_2)$, and

$\mathcal{H}_0^1 := \{v \in \mathcal{H}^1 : v = 0 \ on \ \partial\Omega\}$.

Multiply (1) by a test function $v \in \mathcal{H}_0^1$ and integration by parts on each sub-domain and applying the interface condition in (1), such that

$$\int_\Omega \frac{\partial u}{\partial t} \cdot v \, dx + \int_\Omega a\Delta u \cdot v \, dx = \int_\Omega f(u) \cdot v \, dx$$

$$= \int_\Omega \frac{\partial u}{\partial t} \cdot v + \int_{\Omega_1 \cup \Omega_2} a\nabla u \cdot \nabla v \, dx$$

$$- \sum_{i=1}^2 \int_{\Gamma^{tr}} n^i \cdot a\nabla u_i v_i \, ds = \int_\Omega f(u) \cdot v \, dx$$

$$\int_\Omega \frac{\partial u}{\partial t} \cdot v + \int_{\Omega_1 \cup \Omega_2} a\nabla u \cdot \nabla v \, dx$$

$$- \int_{\Gamma^{tr}} n^1 \cdot a\nabla u_1 \cdot v_1 \, ds - \int_{\Gamma^{tr}} n^2 \cdot a\nabla u_2 v_2 \, ds$$

$$= \int_\Omega f(u) \cdot v \, dx$$

$$= \int_\Omega \frac{\partial u}{\partial t} \cdot v \, dx + \int_{\Omega_1 \cup \Omega_2} a\nabla u \cdot \nabla v \, dx$$

$$+ \int_{\Gamma^{tr}} C_{tr}[\![u]\!] \cdot [\![v]\!] \, ds = \int_\Omega f(u) \cdot v \, dx.$$

In weak form, the above equation becomes the following: find $u \in L_2(0,T,\mathcal{H}_0^1) \cap \mathcal{H}_0^1(0,T,L_2(\Omega_1 \cup \Omega_2))$ such that for almost every $t \in [0,T]$, reads

$$\left\langle \frac{\partial u}{\partial t}, v \right\rangle + D(u,v) = \langle f(u), v \rangle$$
$$\forall v \in \mathcal{H}_0^1, \qquad u(.,0) = u_I, \qquad (3)$$

with

$$D(u,v) := \int_{\Omega_1 \cup \Omega_2} a\nabla u \cdot \nabla v \, dx \qquad (4)$$
$$+ \int_{\Gamma^{tr}} C_{tr} \llbracket u \rrbracket \cdot \llbracket v \rrbracket \, ds,$$
$$\left\langle \frac{\partial u}{\partial t}, v \right\rangle = \int_\Omega \frac{\partial u}{\partial t} v \, dx,$$
$$\langle f(u), v \rangle = \int_\Omega f(u)v \, dx,$$

where $D(u,v)$ is a bilinear form and $\llbracket u \rrbracket = u_1|_K \cdot \boldsymbol{n^2} + u_2|_K \cdot \boldsymbol{n^2}$ is the jump across the interface. The Cauchy-Schwarz inequality provides the coercivity and continuity of the bilinear form D ,viz.,

$$D(v,v) := \int_{\Omega_1 \cup \Omega_2} a|\nabla v|^2 dx + \int_{\Gamma^{tr}} C_{tr} |\llbracket v \rrbracket|^2 \, ds$$
$$=: \le \vert\vert\vert v \vert\vert\vert^2 \qquad \forall v \in \mathcal{H}_0^1 \qquad (5)$$

$$D(v,w) \le \vert\vert\vert v \vert\vert\vert \, \vert\vert\vert u \vert\vert\vert \quad \forall v \in \mathcal{H}_0^1.$$

Discontinuous Galerkin method. Given a mesh $\mathcal{T} = \{K\}$ (with K representing a generic element), the discontinuous finite element space V_h^p is constructed by Fig.2
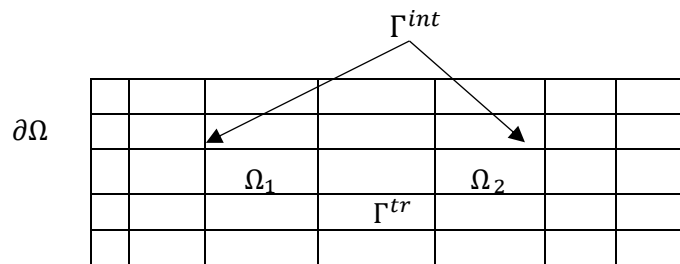$$V_h^p = \{L_2(\Omega): v|_K \in P_p(K)\}, \qquad (6)$$

Figure 2. The mesh skeleton Γ discretises the domain into $\partial\Omega$, Γ^{int}, Γ^{tr}.

where $P_p(K)$ denotes the space of polynomials of total degree p on an element K. Suppose that K_1 and K_2 are two elements sharing the same face $E \subset \Gamma^{int} \cup \Gamma^{tr}$, where $E \subset \partial K_1 \cap \partial K_2$ with n_{K_1} and n_{K_2} denoting the outward unit normal vectors on E of ∂K_1 and ∂K_2, respectively. Then, subdividing Γ the mesh skeleton into three disjoint subsets $\Gamma = \partial\Omega \cup \Gamma^{int} \cup \Gamma^{tr}$, where $\Gamma^{int} = \Gamma \setminus (\partial\Omega \cup \Gamma^{tr})$ is the interior points.
Let v be a discontinuous function across Γ. Setting $v_i = v|_{K_i}$ and defining its jump $\llbracket v_h \rrbracket$ and average $\{v_h\}$ across E by
$$\llbracket v_h \rrbracket = v_h|_{K_1} + v_h|_{K_2}, \quad \{v_h\} = \frac{v_h|_{K_1} + v|_{K_2}}{2}.$$

Similarly, for a vector valued function w_h, piecewise smooth on $\mathcal{T}$ with $w_i = w|_{K_i}$, such that
$$\llbracket w_h \rrbracket = w|_{K_1} + w_h|_{K_2}, \quad \{v_h\} = \frac{w_h|_{K_1} + w_h|_{K_2}}{2}.$$

Thus, setting $\{v_h\} = v$, $\llbracket v_h \rrbracket = vn$ and $\llbracket w_h \rrbracket = w_h \cdot n$ with n denoting the outward unit normal on the boundary $\partial\Omega$. For the mesh size, using the function $h: \Omega \to \mathbb{R}$, where $h|_K = h_K, K \in \mathcal{T}$ and $h = \{h\}$ on each $(d-1)$ dimensional open face $E \subset \Gamma$. Further, assuming that $h_{max} := max_{x \in \Omega} h$ and $h_{min} := min_{x \in \Omega} h$. Without loss of generality, assuming that h_{max} remains uniformly

bounded throughout this work, to avoid having estimation constants depend on $max\{1, h_{max}\}$. To introduce the interior penalty discontinuous Galerkin method, Multiplying (1) by a test function in $v \in \mathcal{H}_0^1 + V_h^p(\mathcal{T})$ and, integrate over each subdomain, so that

$$\int_\Omega \frac{\partial u}{\partial t} \cdot v - \int_\Omega a\Delta u \cdot v \, dx = \int_\Omega f(u) \cdot v \, dx.$$

Then, splitting the integral into element contributions and integrate by parts:

$$\int_\Omega \frac{\partial u}{\partial t} \cdot v \, dx + \sum_{K \in \mathcal{T}} \int_K a \nabla u \cdot \nabla v \, dx$$
$$+ \sum_{K \in \mathcal{T}} \int_{\partial K} a \, \mathbf{n} \cdot \nabla u v \, ds = \sum_{K \in \mathcal{T}} \int_K f(u)v \, dx.$$

The next step is to decompose the face integrals:

$$\sum_{K \in \mathcal{T}} \langle a \, n \cdot \nabla u, v \rangle$$
$$\overline{\overline{=}} \left(\sum_{E \in \Gamma \setminus \Gamma^{tr}} \langle a \, \mathbf{n} \cdot \nabla u, v \rangle_{\partial K_1 \cap E} \right.$$
$$\left. + \sum_{E \in \Gamma \setminus \Gamma^{tr}} \langle a \, \mathbf{n} \cdot \nabla u, v \rangle_{\partial K_2 \cap E} \right)$$

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$$+ \left(\sum_{E \in \Gamma^{tr}} \langle a\, \mathbf{n} \cdot \nabla u, v \rangle_{\partial \Omega_1 \cap E} \right.$$

$$\left. + \sum_{E \in \Gamma^{tr}} \langle a\, \mathbf{n} \cdot \nabla u, v \rangle_{\partial \Omega_1 \cap E} \right)$$

$$= \int_{\Gamma \backslash \Gamma^{tr}} \{ a \nabla u \} \cdot [\![v]\!]\, ds$$

$$+ \int_{\Gamma^{int} \backslash \Gamma^{tr}} \{ a \nabla u \} \cdot [\![v]\!]\, ds$$

$$- \int_{\Gamma^{tr}} C_{tr} [\![u_h]\!] \cdot [\![v_h]\!]\, ds,$$

where K_1 and $K_2 \in \mathcal{T}$, for $E \in \partial K_1 \cap \partial K_2$ n is a corresponding unit normal on E (exterior to K_2). Finally,
it is ready to introduce the interior penalty discontinuous Galerkin method for (3), which reads: find $u_h \in V_h^p$ such that

$$\left\langle \frac{\partial u_h}{\partial t}, v_h \right\rangle + D_h(t;\, u_h, v_h) = \langle f(u_h), v_h \rangle, \quad (7)$$

$for\ all\ v_h \in V_h^p,$
where

$$\left\langle \frac{\partial u_h}{\partial t}, v_h \right\rangle = \int_{\Omega} \frac{\partial u_h}{\partial t}\, v_h\ dx,$$

$$D_h(t;\, u_h, v_h) = \sum_{K \in \mathcal{T}} \int_{\Omega} a\, \nabla u_h \cdot \nabla v_h\, dx$$

$$- \int_{\Gamma \backslash \Gamma^{tr}} (\{ a \nabla u_h \} \cdot [\![v_h]\!] + \{ a \nabla v_h \} \cdot [\![u_h]\!])\, ds$$

$$+ \int_{\Gamma \backslash \Gamma^{tr}} \frac{\gamma_0}{h} [\![u_h]\!] \cdot [\![v_h]\!] ds + \int_{\Gamma^{tr}} C_{tr} [\![u_h]\!] \cdot [\![v_h]\!] ds,$$

where γ_0 and C_{tr} donate by the discontinuity penalization parameter and permeability coefficient, respectively. Here, γ_0 has to be chosen large enough to ensure the stability of the discontinuous Galerkin scheme.

Then, extending the norm $||| v |||$ to functions in $v \in \mathcal{H}_0^1 + V_h^p(\mathcal{T})$ by setting

$$||| v ||| := \left(\sum_{K \in \mathcal{T}} \| \sqrt{a} \nabla v \|_K^2 \right.$$
$$+ \frac{1}{2} \| \sqrt{a/h} [\![v_h]\!] \|_{\Gamma \backslash \Gamma^{tr}}^2$$
$$\left. + \frac{1}{2} \| \sqrt{C_{tr}} [\![v_h]\!] \|_{\Gamma^{tr}}^2 \right)^{1/2}$$

$\left(L_\infty(L_2) + L_2(H^1) \right)$**-norm a posteriori error bounds**
The aim of this section is to derive an a posteriori error estimate for the $\left(L_\infty(L_2) + L_2(H^1) \right)$ norm

$$\| e \|_* \equiv \| e(t) \|_*$$

$$:= \left(\| e \|_{L_\infty(0,t;L(\Omega))}^2 + \int_0^t ||| e |||^2\, ds \right)^{\frac{1}{2}}.$$

The estimator is derived using the elliptic reconstruction techniques in (2003). For each $t \in [0, T]$ the elliptic reconstruction $\mathcal{R}_h(u_h) \in \mathcal{H}_0^1$ to be the unique solution of the problem

$$D(t; \mathcal{R}_h(u_h), \phi) = \left\langle f(u_h) - \frac{\partial u_h}{\partial t}, \phi \right\rangle \forall \phi \in \mathcal{H}_0^1$$
(8)

Lemma 3.1. Let $\mathcal{R}_h(u_h) \in \mathcal{H}_0^1$ be the exact solution of elliptic problem (8). Then, the following a posteriori bound holds:

$$||| \mathcal{R}_h(u_h) - u_h ||| \le C \left(\eta_{S^c}(u_h) + \eta_{S^d}(u_h) \right),$$
where

$$\eta_{S^c}(u_h) := \left(\sum_{K \in \mathcal{T}} \left(\left\| \sqrt{\frac{h^2}{a}} \left(f(u_h) - \frac{\partial u_h}{\partial t} + \right. \right. \right. \right.$$
$$\left. a \Delta u_h \right) \Big\|_K^2 \left. + \left\| \sqrt{\frac{h}{a}} (a \nabla u_h + C_{tr} [\![u_h]\!] \cdot n_K) \right\|_{\partial K \cap \Gamma^{tr}}^2 + \right.$$
$$\left. \frac{1}{2} \| \sqrt{ah} [\![\nabla u_h]\!] \|_{\partial K \cap \Gamma^{int}}^2 + \frac{1}{2} \left\| \sqrt{\frac{h}{a}} [\![u_h]\!] \right\|_{\partial K \cap \Gamma \backslash \Gamma^{tr}}^2 \right)^{\frac{1}{2}}$$
$$\eta_{S^d}(u_h) := \left(C_1 \| \sqrt{h^{-1}} [\![u_h]\!] \|_{\Gamma \backslash \Gamma^{tr}}^2 \right)^{1/2}.$$

Proof. See (21, Lemma 6.2).

Lemma 3.2. (Error relation) Let u and u_h are the exact and approximate solutions defined by (3) and (7) respectively, and let $\mathcal{R}_h(u_h)$ be given by (8), and set
$e := \rho + \varepsilon, \quad \rho := u - \mathcal{R}_h(u_h), \varepsilon := \mathcal{R}_h(u_h) - u_h, e^c := u - u_h^c, \varepsilon^c = \mathcal{R}_h(u_h) - u_h^c, u_h^d := u_h^c - u_h.$
So that, the identity
$$\left\langle \frac{\partial e^c}{\partial t}, v \right\rangle + D(e^c, v) = \langle f(u_h) - f(u_h), v \rangle +$$
$$D(t, \varepsilon^c, v) + \left\langle \frac{\partial u_h^d}{\partial t}, v \right\rangle. \quad (9)$$

Proof: Going back in the definition (8), lead to
$$\left\langle \frac{\partial u_h}{\partial t}, v \right\rangle + D(t; \mathcal{R}_h(u_h), v) = \langle f(u_h), v \rangle,$$
$\forall\, v \in \mathcal{H}_0^1. \quad (10)$
Subtracting (3) from above equation, gives
$$\left\langle \frac{\partial u_h}{\partial t} - \frac{\partial u}{\partial t}, v \right\rangle + D(t; \mathcal{R}_h(u_h) - u, v)$$
$$= \langle f(u_h) - f(u), v \rangle$$

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$$\left\langle \frac{\partial u_h}{\partial t} + \frac{\partial u_h^c}{\partial t} - \frac{\partial u_h^c}{\partial t} - \frac{\partial u}{\partial t}, v \right\rangle$$
$$+ D(t; \mathcal{R}_h(u_h) - u_h^c + u_h^c - u, v)$$
$$= \langle f(u_h) - f(u), v \rangle$$

$$\left\langle \frac{\partial u_h^c}{\partial t} - \frac{\partial u}{\partial t}, v \right\rangle + D(t; u_h^c - u, v)$$
$$\quad = \langle f(u_h) - f(u), v \rangle + D(t; \mathcal{R}_h(u_h) - u_h^c, v)$$
$$\quad + \left\langle \frac{\partial u_h}{\partial t} - \frac{\partial u_h^c}{\partial t}, v \right\rangle.$$

Using $\quad \frac{\partial e^c}{\partial t} := \frac{\partial u_h^c}{\partial t} - \frac{\partial u}{\partial t}, \; \varepsilon^c = \mathcal{R}_h(u_h) - u_h^c \quad$,
$\frac{\partial u_h^d}{\partial t} := \frac{\partial u_h}{\partial t} - \frac{\partial u_h^c}{\partial t}$ the result follows.

Theorem 3.3. $\left(L_2(H^1) + L_\infty(L_2) \right)$-norm. For each $t \in [0, T]$, let r be as in, (17, lemma 5.1), with $h_{\max}$ small enough. Then, the error bound is

$$\left(\|e\|^2_{L_\infty(0,t;L(\Omega))} + \int_0^t ||| e |||^2 \, ds \right)^{1/2}$$
$$\leq 4\theta^2(\varepsilon) e^{\mathcal{M}(\varepsilon)T}$$
$$+ 2 \left\| \sqrt{h} [\![u_h]\!] \right\|_{L_\infty(0,t;L(\Omega))},$$

where

$$\theta^2(\varepsilon) = \Big(\|u_0 - u_h(0)\|$$
$$+ C_0 \left\| \sqrt{h} \, [\![u_h(0)]\!] \right\|_{\Gamma \backslash \Gamma^{tr}} \Big)^2$$
$$+ C \int_0^T \left(\left(\eta_{S^c}^2(u_h) + \eta_{S^d}^2(u_h) \right) dt \right.$$
$$+ \left(\eta_{S^c}^{2r+2}(u_h) + \eta_{S^d}^{2r+2}(u_h) \right) dt$$
$$+ \left\| \sqrt{h} \left[\!\!\left[\frac{\partial u_h}{\partial t} \right]\!\!\right] \right\|_{\Gamma \backslash \Gamma^{tr}}^2 \right) dt$$
$$+ \left\| \sqrt{h} [\![u_h]\!] \right\|_{\Gamma \backslash \Gamma^{tr}}^2 + \left\| \sqrt{h} [\![u_h]\!] \right\|_{2+2r}^{2+2r}$$

Proof. Testing with $v = e^c$ in (9), gives

$$\left\langle \frac{\partial e^c}{\partial t}, e^c \right\rangle + D(e^c, e^c)$$
$$\quad = \langle f(u_h) - f(u_h), e^c \rangle + D(t, \varepsilon^c, e^c)$$
$$\quad + \left\langle \frac{\partial u_h^d}{\partial t}, e^c \right\rangle.$$

To simplify the left-hand side and using the identity

$$\left\langle \frac{\partial e^c}{\partial t}, e^c \right\rangle = \int_\Omega \frac{de^c}{dt} \cdot e^c dx = \frac{1}{2} \frac{d}{dt} \left(\int_\Omega (e^c)^2 \, dx \right)$$
$$= \frac{1}{2} \frac{d}{dt} \|e^c\|^2.$$

Therefore,

$$\frac{1}{2} \frac{d}{dt} \|e^c\|^2 + ||| e^c |||^2$$
$$\quad = \langle f(u_h) - f(u_h), e^c \rangle + D(t, \varepsilon^c, e^c)$$
$$\quad + \left\langle \frac{\partial u_h^d}{\partial t}, e^c \right\rangle.$$

Applying Cauchy-Schwarz inequality on the right-hand side of above equation, this becomes

$$\frac{1}{2} \frac{d}{dt} \|e^c\|^2 + ||| e^c |||^2$$
$$\leq \left(\|f(u_h) - f(u_h)\| + \left\| \frac{\partial u_h^d}{\partial t} \right\| \right) \| e^c \|$$
$$+ ||| \varepsilon^c ||| \; ||| e^c |||. \tag{11}$$

The first term on the right-hand side of (11) can be handled using (2) as follows

$$\|f(u_h) - f(u_h)\|^2 = \int_\Omega \|f(u_h) - f(u_h)\|^2 dx$$
$$\leq C_g^2(t) \int_\Omega (1 + \|u\| + \|u_h\|)^{2r} \, | u - u_h|^2$$
$$\leq C_g^2(t) \int_\Omega (1 + \|\rho\| + \|\varepsilon\| + 2\|u_h\|)^{2r} \, (|\rho|^2$$
$$\qquad + | \varepsilon|^2) dx$$
$$\leq \max\{1, 16^r\} C_g^2(t) \int_\Omega (1 + \|\rho\|^{2r} + \|\varepsilon\|^{2r} \text{hp}$$
$$- \text{adaptive discontinuous Galerkin methods for non}$$
$$- \text{stationary convection– diffusion problems. Comput}$$
$$+ 4^r |u_h|^{2r})^{2r} + (|\rho|^2 + |\rho|^2) \, dx$$
$$\leq \alpha^2(t) \left[\|\rho\|^2 + \|\rho\|^{2+2r}_{2+2r} + \|\varepsilon\|^2 + \|\varepsilon\|^{2+2r}_{2+2r} \right.$$
$$\qquad + \int_\Omega 4^r \, |u_h|^{2r} (\|\rho\|^2 + \|\rho\|^2$$
$$\qquad + \|\varepsilon\|^2 \|\rho\|^{2r} + \|\varepsilon\|^{2r} \|\rho\| \Big] \, dx$$

$$\leq \alpha^2(t) \left[(1 + 4^r |u_h|^{2r}_\infty) \|\rho\|^2 + \|\rho\|^{2+2r}_{2+2r} \right.$$
$$\qquad + (1 + 4^r |u_h|^{2r}_\infty) \|\varepsilon\|^2 + \|\varepsilon\|^{2+2r}_{2+2r}$$
$$\qquad + \int_\Omega 4^r \, |u_h|^{2r} (\|\rho\|^2 + \|\rho\|^2$$
$$\qquad + \|\varepsilon\|^2 \|\rho\|^{2r} + \|\varepsilon\|^{2r} \|\rho\| \Big] \, dx,$$

where $\alpha^2(t) = \max\{1, 16^r\} C_g^2(t)$, having used the inequality, $\quad \int_\Omega |\alpha|^{2r} |\beta|^2 dx \leq \frac{\|\alpha\|^{2+2r}_{2+2r}}{r+1} + \frac{r\|\beta\|^{2+2r}_{2+2r}}{r+1}$ from [17]. Therefore,

$$\int_\Omega (\|\varepsilon\|^2 \|\rho\|^{2r} + \|\varepsilon\|^{2r} \|\rho\|)$$

$$\leq \left(\frac{1}{r+1} + \frac{r}{r+1}\right) \|\rho\|_{2+2r}^{2+2r}$$

$$+ \left(\frac{1}{r+1} + \frac{r}{r+1}\right) \|\varepsilon\|_{2+2r}^{2+2r}$$

$$= \|\rho\|_{2+2r}^{2+2r} + \|\varepsilon\|_{2+2r}^{2+2r}$$

$$\leq C\left(\|e^c\|_{2+2r}^{2+2r} + \|u_h^d\|_{2+2r}^{2+2r}\right)$$

$$+ \|\varepsilon\|_{2+2r}^{2+2r}.$$

Applying the Hölder's inequality with $m = \frac{2}{2-r}, n = \frac{r}{2}, \frac{1}{m} + \frac{1}{n} = 1$, along with the Sobolev imbedding inequality $\|\rho\|_{L_{2m}} \leq C_P ||| \rho |||$, on the first term on the above equation, which gives

$$\|e^c\|_{2+2r}^{2+2r} = \int_\Omega |e^c|^{2r+2} dx = \int_\Omega |e^c|^{2r} |e^c|^2 dx$$

$$\leq \left(\int_\Omega |e^c|^2\right)^{\frac{2r}{2}} \left(\int_\Omega |e^c|^{2\left(\frac{2}{2-r}\right)}\right)^{2\left(\frac{2-r}{4}\right)}$$

$$= \left(\int_\Omega |e^c|^2\right)^{\frac{2r}{2}} \left(\int_\Omega |e^c|^{\left(\frac{4}{2-r}\right)}\right)^{\left(\frac{2-r}{2}\right)}$$

$$= \|e^c\|^{2r} \|e^c\|_{L_{4/(2-r)}}^2$$

$$\leq C_P \|e^c\|^{2r} ||| e^c |||^2$$

Hence,

$$\int_\Omega (\|\varepsilon\|^2 \|\rho\|^{2r} + \|\varepsilon\|^{2r} \|\rho\|) \, dx$$

$$= CC_P \|e^c\|^{2r} ||| e^c |||^2$$

$$+ C\|u_h^d\|_{2+2r}^{2+2r} + \|\varepsilon\|_{2+2r}^{2+2r},$$

valid for all $0 \pounds r < 2$ for $d = 2$ and $0 \pounds r \pounds \frac{4}{3}$ for $d = 3$. Substituting this into our earlier inequality (12), this leads to

$$\int_\Omega \|f(u_h) - f(u_h)\| dx$$

$$\leq \alpha(t) \left(\sqrt{1 + 4^r |u_h|_\infty^{2r}} (||| e^c ||| + \|u_h^d\|)\right.$$

$$\left. + \sqrt{2}\|\varepsilon\|_{2+2r}^{1+r}\right)$$

$$+ \sqrt{1 + 4^r |u_h|_\infty^{2r}} \|\varepsilon\| + \sqrt{2}CC_P \|e^c\|^r ||| e^c ||| +$$

$$C\|u_h^d\|_{2+2r}^{1+r}. \tag{13}$$

Now, putting $\beta(t) = \sqrt{1 + 4^r |u_h|_\infty^{2r}}$, this becomes

$$\int_\Omega \|f(u_h) - f(u_h)\|^2 dx$$

$$\leq \alpha^2(t) \left(\beta^2(t) ||| e^c |||^2 + \beta^2(t) \|u_h^d\|^2\right.$$

$$\left. + 2\|\varepsilon\|_{2+2r}^{2+2r}\right)$$

$$\int_\Omega \|f(u_h) - f(u_h)\|^2 dx$$

$$\leq \alpha^2(t) \left(\beta^2(t) ||| e^c |||^2 + \beta^2(t) \|u_h^d\|^2\right.$$

$$\left. + 2\|\varepsilon\|_{2+2r}^{2+2r}\right)$$

$$+ \beta^2(t) \|\varepsilon\|^2 + 2C^2 C_P^2 \|e^c\|^{2r} ||| e^c |||^2 +$$

$$C^2 \|u_h^d\|_{2+2r}^{2+2r}. \tag{14}$$

Refuring (11) and Young's inequality, so that
$$||| \varepsilon^c ||| \, |||e^c |||$$
$$\leq C ||| \varepsilon^c |||^2$$
$$+ \frac{1}{2} |||e^c |||^2. \tag{15}$$

Setting
$$\mathcal{F}(t) = 2C ||| \varepsilon^c |||^2 + 2\alpha^2(t)\beta^2(t)\|\varepsilon\|^2 +$$
$$4\alpha^2(t)\|\varepsilon\|_{2+2r}^{2+2r} + 2\alpha^2(t)\beta^2(t)\|u_h^d\|^2 +$$
$$2C^2 \|u_h^d\|_{2+2r}^{2+2r} + 2\left\|\frac{\partial u_h^d}{\partial t}\right\|^2,$$
$$\mathcal{M}(t) = 2\alpha^2(t)\beta^2(t) + \frac{1}{2},$$ and, by combining these terms together, implies that

$$\frac{d}{dt}\|e^c\|^2 + 2||| e^c |||^2$$
$$\leq \mathcal{F}(t) + \mathcal{M}(t) ||| e^c |||^2$$
$$+ 2\alpha^2(t)C^2 C_P^2 \|e^c\|^{2r} ||| e^c |||^2. \tag{16}$$

Integrating the time variable over $s \in (0, t)$, and set $C_2 = 2\alpha^2(t)C^2 C_P^2$, to give

$$\|e^c\|^2 + 2\int_0^t ||| e^c |||^2 ds$$

$$\leq \int_0^t \mathcal{F}(s) ds + + \mathcal{M}(t) \int_0^t ||| e^c |||^2 ds$$

$$\leq \int_0^t \mathcal{F}(s) ds + \mathcal{M}(t) \int_0^t ||| e^c |||^2 ds$$

$$+ C_2 \sup_{s\in[0,t]} \| e^c(s)\|^{2r} \int_0^t ||| e^c |||^2 ds$$

$$\leq \int_0^t \mathcal{F}(s) ds + \mathcal{M}(t) \int_0^t ||| e^c |||^2 ds$$

$$+ C_2 \left(\sup_{s\in[0,t]} \| e^c\|^2 + \int_0^t ||| e^c |||^2 ds\right)^{1+r}$$

$$\leq \int_0^t \mathcal{F}(s) ds + \mathcal{M}(t) \int_0^t ||| e^c |||^2 ds$$

$$+ C_2 (\| e^c(s)\|_*)^{1+r}$$

Letting $\theta(\varepsilon^c)^2 = \int_0^t \mathcal{F}(s) ds$, and implies that

$$\|e^c\|^2 + 2\int_0^t ||| e^c |||^2 ds \leq \theta(\varepsilon^c)^2$$

$$+ \mathcal{M}(t) \int_0^t ||| e^c |||^2 ds$$
$$+ C_2 (|| e^c(s) ||_*)^{1+r}. \quad (17)$$

To deal with final term on the right-hand side of (17), applying the inequality $z_1^r z_2 \le (z_1^r + z_2)^{1+\frac{1}{r}}$ and assume that the mesh-size h_{max} is small enough that

$$\theta(\varepsilon^c) \le \frac{1}{(C_2)^r} \left(4 e^{\mathcal{M}(t)T} \right)^{-\left(1+\frac{1}{r}\right)} \Rightarrow \theta(\varepsilon^c)^r$$
$$\le \frac{1}{C_2 (4 e^{\mathcal{M}(t)T})^{(1+r)}}$$
$$\Rightarrow C_2 \theta(\varepsilon^c)^r \left(4 e^{\mathcal{M}(t)T} \right)^{1+r} \le 1$$

$$\Rightarrow C_2 \left(4 \theta(\varepsilon^c)^2 e^{\mathcal{M}(t)T} \right)^{1+r} \le \theta(\varepsilon^c)^2.$$

Since the left-hand side of (17) depends continuously on t, which implies that the set

$\mathcal{A} = \left\{ t \in [0,T] : || e^c(s) ||_*^2 \le 4 \theta(\varepsilon^c)^2 e^{\mathcal{M}(t)T} \right\}$ is non-empty and closed. Therefore, setting $t^* = \max \mathcal{A}$ and assuming that $t^* < T$, so that

$$|| e^c(t) ||^2 + 2 \int_0^t ||| e^c |||^2 ds \le 2\theta(\varepsilon^c)^2 +$$
$$\mathcal{M}(t) \int_0^t ||| e^c |||^2 ds, \quad (18)$$

and using Gronwall's inequality, such that

$$|| e^c(t) ||^2 + 2 \int_0^t ||| e^c |||^2 ds \le 2\theta(\varepsilon^c)^2 e^{\mathcal{M}(t)t}$$
$$< 4\theta(\varepsilon^c)^2 e^{\mathcal{M}(t)T}. \quad (19)$$

This leads to contradiction with hypothesis $t^* < T$ because of the continuity of the left-hand side of (19). Hence, $t^* = T$. Setting $t^* = t$, and $|| e^c(t^*) || = || e^c ||_{L_\infty(0,t;L(\Omega))}$ due to the continuity with respect to t, (19) implies

$$|| e^c(t) ||_{L_\infty(0,t;L_2(\Omega))}^2 \le || e^c(t^*) ||^2 + \int_0^{t^*} ||| e^c |||^2 ds$$
$$\le 2\theta(\varepsilon^c)^2 e^{\mathcal{M}(t^*)T}$$
$$\le 2\theta(\varepsilon^c)^2 e^{\mathcal{M}(t)T}.$$

With T being the final time. Combining this with (19) for $t = T$, to arrive

$$|| e^c ||_*^2 = || e^c ||_{L_\infty(0,t;L(\Omega))}^2 + \int_0^T ||| e^c |||^2 ds$$
$$\le 2\theta(\varepsilon^c)^2 e^{\mathcal{M}(t)T}$$

The triangle inequality along with Lemma 3.1, now implies

$$|| e ||_*^2 \le 2 || e^c ||_*^2 + 2 || u_h^d ||_*^2 \le 4\theta(\varepsilon^c)^2 e^{\mathcal{M}(t)T} + 2 || u_h^d ||_*^2,$$

where

$$|| u_h^d ||_*^2 = || u_h^d ||_{L_\infty(0,t;L(\Omega))}^2 + \int_0^T ||| u_h^d |||^2 ds$$

The result therefore, follows from the last three inequalities.

Conclusion:

This paper aims to derive an optimal order a posteriori error estimates in the $L_\infty(L_2) + L_2(H^1)$-norm for semdiscrete semilinear parabolic interface problems. An important factor in our analysis to derive this estimator is to use the elliptic reconstruction framework of Makridakis and Nochetto (12) although, crucially, a number of challenges have to be overcome due to non-linearity on the forcing term depending on Gronwall's Lemma and Sobolev embedding through continuation argument. The main use for these bounds is in designing an efficient adaptive scheme, and consequently leading to a reduction in the computational cost of the scheme. In the future, This work can be extended to the fully discrete case for semilinear parabolic interface problems in $L_¥(L_2)$ and $L_¥(H^1)$ norms.

Author's declaration:

- Conflicts of Interest: None.
- I hereby confirm that all the Figures and Tables in the manuscript are mine. Besides, the Figures and images, which are not mine, have been given the permission for re-publication attached with the manuscript.
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تقدير خطأ البعدي في كالكرين غير مستمرة لمسائل ذات الوسط البيني لشبه متكافئة خطية لشبه متقطعة

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الخلاصة:

ان الهدف من هذا البحث هو اشتقاق الخطأ البعدي لمسائل ذات الوسط البيني لشبه خطية متكافئة شبه متقطعة. وبشكل أكثر تحديدًا، تم تحليل الخطأ البعدي الأمثل $L_\infty(L_2) + L_2(H^1)$ لمسائل ذات الوسط البيني لشبه خطية متكافئة شبه متقطعة باستخدام تقنية إعادة الإهليلجية المقدمة من مارك داكس ونوكيتو (2003). الفكرة الأساسية لهذه التقنية هي استخدام مقدرات الأخطاء المشتقة من مشاكل الواجهة الإهليلجية للحصول على مقدرات مكافئ ذات ترتيب أمثل في المجال والزمان.

الكلمات المفتاحية: تخمين الخطأ البعدي، طرائق كالركين غير المستمرة، مسائل ذات الوسط البيني شبه خطية متكافئة شبه متقطعة.