On Near Complex Hadamard Matrices

حول المصفوفات المعقدة المقتربة من مصفوفة هادمرد

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Abstract:

In this paper we have introduced a matrix with special condition, and called it "near complex Hadamard matrix" and we have studied some basic properties of this matrix, also we have found two types of this matrix and conclude some relationships between them.

المستخلص :

في هذا البحث قدمنا مصفوفة جديدة مع شروط خاصة تسمى " المصفوفة المعقدة المقتربة من مصفوفة هادمرد " و درسنا بعض بعض الخواص المهمة لهذه المصفوفة كما و جدنا نوعين منها و استنتجنا علاقات مهمة بينها .

Introduction:

A Hadamared matrices seems such simple matrix structures, they are square matrix with entries equal to ±1 whose rows and columns are orthogonal. In other words a Hadamard matrix of order n is a{1,-1}matrix A satisfying: $AA^T = nI$ where I is the identity matrix [3] these matrices yet have been actively studied for over 146 years and still have more secrets to be discovered [2].

A Hadamared matrices first introduced by Turyn [6] and further discussed by Seberry Wallis [4],this matrix have very important applications in image processing[5][7] .

In 1867 Sylvester[3] proposed a recurrent method for construction of Hadamard matrices of order 2^k. In 1893 Hadamard proved this famous determinantal inequality for a positive semi define matrix A, such that

$\det(A) \leq h(A)$

where h(A) is the product of the diagonal entries of A . it follows from this inequality that if $A=(a_{ij})$ is real matrix of order n Which $|a_{ij}| \leq 1$ then $|\det A| \leq n^{\frac{n}{2}}$,This result gives rise to the term "Hadamard matrix".

A complex Hadamard matrix C, has elements 1,-1,i,-i where $i=\sqrt{-1}$ of order 1 or $2C$ and satisfies $CC^*=2CI_{2c}$, where C^* denotes to the Hermition conjugates (transpose ,complex conjugates) of C.

1.Near Complex Hadamard Matrix

Definition (1.1) :[3]

A square matrix H of size nxn with entries ± 1 is a Hadamard matrix if $HH^T = n.I_n$; where I_n is the identity $n \times n$ matrix Hadamard matrices of size n exist only if n is two or a multiple of four. A Hadamard matrix H is said to be skew if $H + H^T = 2I$, where I denotes the identity matrix.

Definition (1.2):

A *near complex hadamard matrix* is a square complex matrix with entry i or $-i$ satisfying :

1. $A^T.A = -n.I_n$
2. Its rank ; $n = 2^k$;k is any positive integer
3. $\det(A) = n^{n/2}$;$n = 2^k$;k is any positive integer

Notation(1.3):
We will denote to the near complex hadamard matrix by N.C.H. Matrix.
Example (1.4):
For instance the matrices $A=\begin{pmatrix} i & i \\ i & -i \end{pmatrix}$ and $B=\begin{pmatrix} -i & i \\ i & i \end{pmatrix}$ are two near complex hadamard matrices .
Remark (1.5):
We can write the N.C.H. Matrix A of rank 2^k as ablock matrix of tensor +(kronker) product of $A_{2^{k-1}}$ as follows:

$$A_{2^k} = \begin{bmatrix} A_{2^{k-1}} & A_{2^{k-1}} \\ A_{2^{k-1}} & -A_{2^{k-1}} \end{bmatrix} \text{ or } A_{2^k} = \begin{bmatrix} -A_{2^{k-1}} & A_{2^{k-1}} \\ A_{2^{k-1}} & A_{2^{k-1}} \end{bmatrix}$$

Examples (1.6):

let $A_{2^1}=\begin{bmatrix} i & i \\ i & -i \end{bmatrix}$ then

1. $A_4 = A_2{}^2 = \begin{bmatrix} A_{2^1} & \vdots & A_{2^1} \\ \cdots & . & \cdots \\ A_{2^1} & \vdots & -A_{2^1} \end{bmatrix} = \begin{bmatrix} i & i & \vdots & i & i \\ i & -i & \vdots & i & -i \\ \cdots & \cdots & . & \cdots & \cdots \\ i & i & \vdots & -i & -i \\ i & -i & \vdots & -i & i \end{bmatrix}$

det $(A_2{}^2) = (4)^{4/2} = 16$

2. $A_8 = (A_2{}^3) = \begin{bmatrix} A_{2^2} & \vdots & A_{2^2} \\ \cdots & . & \cdots \\ A_{2^2} & \vdots & -A_{2^2} \end{bmatrix} = \begin{bmatrix} i & i & i & i & \vdots & i & i & i & i \\ i & -i & i & -i & \vdots & i & -i & i & -i \\ i & i & -i & -i & \vdots & i & i & -i & -i \\ i & -i & -i & i & \vdots & i & -i & -i & i \\ \cdots & \cdots & \cdots & \cdots & . & \cdots & \cdots & \cdots & \cdots \\ i & i & i & i & \vdots & -i & -i & -i & -i \\ i & -i & i & -i & \vdots & -i & i & -i & i \\ i & i & -i & -i & \vdots & -i & -i & i & i \\ i & -i & -i & i & \vdots & -i & i & i & -i \end{bmatrix}$

3. $\det(A_8) = (8)^{8/2} = 8^4 = 4096$

Lemma(1.7):
If A is near complex hadamard matrix then A is symmetric matrix .
Proof : its clear from Remark (1.5) that N.C.H matrix is must be symmetric to be N.C.H matrix .

Lemma(1.8):
If A is near complex hadamard matrix and I_n is identity matrix of rank n then:
1. $A^2 = -n\, I_n$
2. $A^{2s} = (-n)^s\, I_n$
3. $A^{2s+1} = (-n)^s\, A$

Where s is any positive integer
Proof:
1. From definition of near complex hadamard matrix, we get that
$AA^T = -n\, I_n$ and since $A = A^T$ (from lemma 1.7)

$$\begin{aligned} A^2 &= A\,A \\ &= A\,A^T \\ &= -n\, I_n \end{aligned}$$

So, $A^2 = -n\, I_n$

2. Its generalized (extension) to the hold (1) in this lemma

$$A^4 = A^2\ A^2$$
$$= -n\ I_n - n.I_n$$
$$= (-n)^2\ I_n$$

So by using the same way can find ,

$A^6 = (-n)^3\ I_n$,…., A^8, A^{10}, ….. etc.

Thus we conclude that $A^{2s} = (-n)^s\ I_n$

3. $A^{2s+1} = A^{2s}\ A$ (from $A^{2s} = (-n)^s\ I_n$)

$$= (-n)^s\ \mathbf{I_n}\ \mathbf{A}$$
$$= (-n)^s\ \mathbf{A}$$

Lemma(1.9):

If A and B are two near complex hadamard matrices ,then :

1. trace (A)= trace (B)=0
2. $\bar{\mathbf{A}} = -\mathbf{A}$ where $\overline{\mathbf{A}}$ is the conjugate of the matrix A
3. AB= -BA

proof :

1. trace(A) = $\sum_{i=j}^{2^k} A_{ij} = A_{2^{k-1}} + (-A_{2^{k-1}}) = 0$.

4. Then by using remark (1.5) we get that trace (A)= trace (B)=0

2. since A is N.C.H. Matrix and all it's entry are complex number with zero real parts so it's obviously to see that the conjugate of this matrix is equal to the minus of the matrix A , i.e. $\bar{\mathbf{A}} = -\mathbf{A}$

3. form Remark (1.5) we can conclude that there is two equivalent kind of N.C.H. Matrices are :

$$A_{2^k} = \begin{bmatrix} A_{2^{k-1}} & A_{2^{k-1}} \\ A_{2^{k-1}} & -A_{2^{k-1}} \end{bmatrix} \text{ and } B_{2^k} = \begin{bmatrix} -B_{2^{k-1}} & B_{2^{k-1}} \\ B_{2^{k-1}} & B_{2^{k-1}} \end{bmatrix}$$

And by multiplying them we obtained on the following block matrix :

$$A.B_{(2^k \times 2^k)} = \begin{bmatrix} 0 & 2A_{2^{k-1}}B_{2^{k-1}} \\ -2A_{2^{k-1}}B_{2^{k-1}} & 0 \end{bmatrix} \quad \text{where 0 is zero matrix}$$

And

$$B.A_{(2^k \times 2^k)} = \begin{bmatrix} 0 & -2A_{2^{k-1}}B_{2^{k-1}} \\ 2A_{2^{k-1}}B_{2^{k-1}} & 0 \end{bmatrix}$$

It clear that A.B = - B.A .

2.The Inverse of Near complex hadamard matrix

Theorem(2.1):

If A is near complex hadamard matrix and if B = $\frac{-1}{2^k} A_{2^k}$ then B is the inverse of A.

Proof:

Let $\mathbf{B} = \frac{-1}{2^k} \mathbf{A_{2^k}}$

Then, $\mathbf{AB} = \mathbf{A_{2^k}}(\frac{-1}{2^k}\mathbf{A_{2^k}})$

$$= \frac{-1}{2^k}(\mathbf{A_{2^k}})^2 \quad ……. (1)$$

And
since $\mathbf{A}_{2^k}^2$ = -2^k I (from lemma(1.8) number (2)) so, by substitute this in equation (1) , we get that

$$AB = \frac{-1}{2^k}(-2^k\,\mathbf{I})$$
$$= I \quad \text{, where I is the identity matrix}$$

This implies B is the inverse matrix of A (i.e. $\mathbf{A}_{2^k}^{-1}$ =B)
so,

$$\mathbf{A}_{2^k}^{-1} = \frac{-1}{2^k}\mathbf{A}_{2^k}$$

Examples(2.2):
1.If **A** is **N. C. H** complex matrix of order 2 then:

$$\mathbf{A}^{-1} = \frac{-1}{2}\mathbf{A}_2 = \frac{1}{2i^2}\begin{bmatrix} i & i \\ i & -i \end{bmatrix} = \begin{bmatrix} \frac{1}{2i} & \frac{1}{2i} \\ \frac{1}{2i} & -\frac{1}{2i} \end{bmatrix}$$

2.If **A** is **N. C. H** complex 4×4 matrix then:

$$\mathbf{A}^{-1} = \frac{-1}{(2)^2}\mathbf{A}_{2^2} = \frac{1}{4i^2}\begin{bmatrix} i & i & i & i \\ i & -i & i & -i \\ i & i & -i & -i \\ i & -i & -i & i \end{bmatrix} = \begin{bmatrix} \frac{1}{4i} & \frac{1}{4i} & \frac{1}{4i} & \frac{1}{4i} \\ \frac{1}{4i} & -\frac{1}{4i} & \frac{1}{4i} & -\frac{1}{4i} \\ \frac{1}{4i} & \frac{1}{4i} & -\frac{1}{4i} & -\frac{1}{4i} \\ \frac{1}{4i} & -\frac{1}{4i} & -\frac{1}{4i} & \frac{1}{4i} \end{bmatrix}$$

Theorem(2.3):
If **A** is **nxn** matrix ,then **A** is nonsingular if and only if det(**A**)$\neq \mathbf{0}$
Proof: see [1]

Theorem(2.4): [1]
If **A** is **nxn** nonsingular matrix ,then det($\mathbf{A}^{-1}$) = $\frac{1}{\det(\mathbf{A})}$

Proof: see [1]
Corollary(2.5):
If A is Near complex hadamard matrices then:

$$\det(A^{-1}) = (2^k)^{-\frac{2^k}{2}}$$

Proof:

By definition of Near complex hadamard matrices det(A)= $(2^k)^{\frac{2^k}{2}}$
So, det(A)≠0 ; that is A is non singular (Theorem(2.3))
Thus, det(A^{-1}) = $\frac{1}{\det(A)}$ (Theorem(2.4))

So,

$$\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{(2^k)^{\frac{2^k}{2}}} = (2^k)^{-\frac{2^k}{2}}$$

Examples(2.6):

1.the determine of the matrix in example (2.2) is ,

$$\det(A_2^{-1}) = (2^1)^{\frac{-2^1}{2}} = 2^{-1} = \frac{1}{2}$$

2.the determine of the matrix o in example (2.2) is ,

$$\det(A_{2^2}^{-1}) = (2^2)^{\frac{-2^2}{2}} = (4)^{\frac{-4}{2}} = 4^{-2} = \frac{1}{4^2} = \frac{1}{16} \ .$$

3.The Relation Between Two N.C.H Matrices

Theorem(3.1):

If $\mathbf{A}, \mathbf{B}$ are two equivalent near complex hadamard matrix then $\mathbf{AB}$ is skew symmetric matrix .

Proof:

$(\mathbf{A}\boldsymbol{B})^{\mathbf{T}} = \mathbf{B}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}$ (since $\mathbf{A}, \mathbf{B}$ are symmetric)

$= -BA$ (since $AB = -BA$ from lemma 1.9)

This implies , $(AB)^T = -AB$

So, AB is skew symmetric matrix

Theorem(3.2):

If A, B are two equivalent near complex hadamard matrix then, $AB^T = -BA^T$.

Proof:

$AB^T = AB$ (B is symmetric matrix)

$= -BA$

$= -BA^T$ (A is symmetric matrix)

Theorem(3.3):

If A, B are two equivalent Near complex hadamard matrices then:

1. $A\overline{B} = -B\overline{A}$.
2. $\overline{A.B} = A.B$

Proof :

1. Since $-B = \overline{B}$ then ,

$A.\overline{B} = A(-B)$

$= -AB$

$= BA$

$= B(-\overline{A})$ ($-A = \overline{A}$ then $A = -\overline{A}$)

$= -B\overline{A}$

2. $\overline{A\,B} = \overline{A}\,\overline{B}$

$= -A - B$

$= AB$

Corollary (3.4):
If A,B are two equivalent Near complex hadamard matrices then A&B are skew normal i.e. $A(B^*) = -B(A^*)$ where A^* & B^* are the transpose conjugate of the matrices A and B
proof:

$$A(B^*) = A\overline{(B^T)}$$

Since $B=B^T$

$$A\overline{(B^T)} = A\overline{B}$$

Since $(A\overline{B} = -B\overline{A})$

$$= -B\overline{A} = -B(A^*)$$

4. The eigen Value of N.C.H Matrix

Theorem(4.1) :
If A is N.C.H matrix of rank 2^k and if $\lambda = \pm(\sqrt{2})^k i$ then λ is the eigen value of A. Proof : we will use the mathematical induction for the proof
first we must proof that $\lambda = \pm(\sqrt{2})^k i$ is true when k=1
second we let $\lambda = \pm(\sqrt{2})^k i$ is true when k = n
then we must proof that $\lambda = \pm(\sqrt{2})^k i$ is true when k = n+1

a) first if k=1 then the N.C.H Matrix will be $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ where
a_{12}, a_{21} are equal to i and if $a_{11} = i$ *then* $a_{22} = -i$ conversely
if $a_{11} = -i$ *then* $a_{22} = i$

Thus to be λ *eigen value to A* we must proof that $\det(\lambda I - A) = 0$
so $\begin{vmatrix} \lambda - a_{11} & a_{12} \\ a_{21} & \lambda - a_{22} \end{vmatrix} = 0$ then $(\lambda - a_{11})(\lambda - a_{22}) - a_{12}a_{21} = 0$ *then*

1. if $a_{11} = i$ & $a_{22} = -i$

then $(\lambda - i)(\lambda + i) - (i)(i) = 0$ so $\lambda^2 + 1 + 1 = 0$ *then* $\lambda^2 = -2$ *thus* $\lambda = \pm\sqrt{2}i$

2. by the same way if $a_{11} = -i$ & $a_{22} = i$

then $(\lambda + i)(\lambda - i) - (i)(i) = 0$ so
$\lambda^2 + 1 + 1 = 0$ *then* $\lambda^2 = -2$ *thus* $\lambda = \pm\sqrt{2}i$

b) we assume that $\lambda = \pm(\sqrt{2})^k i$ is true when k = n
finally, we proof that $\lambda = \pm(\sqrt{2})^k i$ is true when k = n+1 as follows

c) $\lambda = \pm(\sqrt{2})^k i = \left(\sqrt{2}\right)^{n+1} i = \pm(\sqrt{2})^n (\sqrt{2})\, i$
$= \{\pm(\sqrt{2})^n i\}\{\pm(\sqrt{2})\, i\}$ {where i^2= -1}
from (a)&(b) we get that $\lambda = \pm(\sqrt{2})^k i$ is true when k = n+1

Examples (4.2):

1. If A is N.C.H matrix of rank 2 then the eigen value be $\lambda = \pm\sqrt{2}\, i$
2. If A is N.C.H matrix of rank 4 then the eigen value be $\lambda = \pm 2i$
3. If A is N.C.H matrix of rank 8 then the eigen value be $\lambda = \pm 2\sqrt{2}\, i$

Theorem(4.3):
If A and B are two N.C.H matrices of rank 2^k and $\lambda = \pm(2)^k i$
then λ is the eigen value of AB .

first we must proof that $\lambda = \pm(2)^k i$ is true when n=1

a) first if k=1 then the AB will be AB= $\begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}$ where a_{12}, a_{21} are equal to 2 or -2 we must proof that

$\det(\lambda I - AB) = 0$

so $\begin{vmatrix} \lambda & a_{12} \\ a_{21} & \lambda \end{vmatrix} = 0$ then

in two cases if $a_{12} = 2$ & $a_{21} = -2$ or if $a_{12} = 2$ & $a_{21} = -2$

then $\lambda^2 + 4 = 0$ *then* $\lambda^2 = -4$ *thus* $\lambda = \pm 2i$

second we let $\lambda = \pm(2)^k i$ is true when k = n
then we must proof that $\lambda = \pm(2)^k i$ is true when k = n+1

b) we assume that $\lambda = \pm(2)^k i$ is true when k = n
finally, we proof that $\lambda = \pm(2)^k i$ is true when k = n+1 as follows

c) $\lambda = \pm(2)^k i = \pm(2)^{n+1} i = \pm(2)^n (2) i$
$= \{\pm(2)^n i\}\{\pm(2) i\}$ {where $i^2 = -1$}

from (a)&(b) we get that $\lambda = \pm(2)^k i$ is true when k = n+1

Examples(4.4):
1. If A and B are two N.C.H matrices of rank 2 then the eigen value be $\lambda = \pm 2i$
2. If A and B are two N.C.H matrices of rank 4 then the eigen value be $\lambda = \pm 4i$
3. If A and B are two N.C.H matrices of rank 8 then the eigen value be $\lambda = \pm 8i$

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