On The Implicative Ideal of a BH-Algebra

المثالية الإستنتاجية في جبر- BH

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Abstract

In this paper, we study the implicative ideal of a BH-algebra. We state and prove some theorems which determine the relationship between this notion and the other types of ideals of a BH-algebra, also we give some properties of this ideal and link it with other types of ideals of a BH-algebra.

المستخلص :

في هذا البحث، درسنا المثالية الإستنتاجية في جبر- BH و أعطينا و برهنا بعض المبرهنات التي تحدد العلاقة بين هذا المفهوم و أنواع أخرى من مثاليات جبر – BH و كذلك أعطينا بعض خصائص هذه المثالية وصلتها مع أنواع أخرى من مثاليات جبر-BH.

Introduction:

The notion of BCK-algebras was formulated first in 1966 [14] by Y.Imai and K.Iseki as a generalization of the concept of set-theoretic difference and propositional calculus, where this notion was originated from two different ways: one of the motivations was based on set theory, another motivation was from classical and non classical propositional calculi. In the same year, K.Iseki introduced the notion of a BCI–algebra [6], which was a generalization of a BCK- algebra.

K.Iseki introduced the notion of an ideal of a BCK–algebra[6]. In 1983, Q.P.Hu and X.Li introduced the notion of a BCH-algebra which was a generalization of BCK/BCI-algebras [8]. In 1998, Y.B.Jun et al introduced the notion of BH-algebra, which is a generalization of BCH-algebras[12]. Then, they discussed more properties on BH-algebras [4, 8, 11]. In 2009, A. B. Saeid, A. Namdar and R.A. Borzooei introduced the notions of a p-semisimple BCH-algebra, an associative BCH-algebra, atoms of a BCH-algebra, a BCH-algebra generated by I-atoms, p-ideals, implicative ideals, positive implicative ideals, normal ideals and fantastic ideals in BCH-algebra[2].In the same year, A. B. Saeid and A. Namdar introduced the notions of n-fold p-ideal and n-fold implicative ideal[1].

In this paper, we study the implicative ideal of a BH–algebra and the implicative BH-algebra. We study some properties of this notion and link it with some other types of ideals of a BH-algebra.

1. Preliminaries :

In this section, we give some basic concepts about BCI-algebra, BCK-algebra, BCH-algebra, BH-algebra, subalgebra, ideals of BH-algebra, implicative ideal of BH-algebra and implicative BH-algebra with some theorems, propositions.

Definition (1.1) : [6]

A BCI-algebra is an algebra (X,*,0), where X is a nonempty set, * is a binary operation and 0 is a constant, satisfying the following axioms: $\forall$ x, y, z$\in$ X:

i. (x *y) * (x * z)) * (z * y) = 0,

ii. (x * (x * y)) * y = 0,

iii. x * x = 0,

iv. x * y =0 and y * x = 0 imply x = y.

Definition (1.2) : [14]

A BCK-algebra is a BCI-algebra satisfying the axiom: 0 * x = 0, $\forall$x $\in$ X.

Definition (1.3) : [7]

A BCH-algebra is an algebra (X,*,0), where X is nonempty set, * is a binary operation and 0 is a constant, satisfying the following axioms:

i. x * x = 0, $\forall$ x$\in$ X.

ii. x * y =0 and y * x = 0 imply x = y, $\forall$ x, y$\in$ X.

iii. (x * y) * z = (x * z) * y, $\forall$ x, y, z$\in$ X.

Definition (1.4) : [12]

A BH-algebra is a nonempty set X with a constant 0 and a binary operation * satisfying the following conditions:

i. x * x = 0, $\forall$ x$\in$ X.

ii. x * y = 0 and y * x = 0 imply x = y, $\forall$ x, y$\in$ X.

iii. x *0 = x, $\forall$ x$\in$ X.

Remark (1.5) : [12]

1. Every BCK-algebra is a BCH-algebra.
2. Every BCH-algebra is a BH-algebra.
3. Every BCI-algebra is a BH-algebra.

Theorem(1.6) :[12]

Every BH-algebra satisfying the condition ((x*y)*(x*z))*(z*y)=0; $\forall$ x, y, z $\in$ X is a BCI-algebra.

Theorem (1.7): [12]

Every BCH-algebra is a BH-algebra. Every BH-algebra satisfying the condition:

(x * y)* z = (x * z)*y, $\forall$ x, y, z $\in$ X is a BCH-algebra.

Remark(1.8):

We denote the condition

i. x = x *(y *x), $\forall$ x, y $\in$ X by (a_1).

ii. x*(y*x) $\in$ I imply x$\in$ I, $\forall$ x, y $\in$ X by (a_2).

iii. ((x*y)*(x*z))*(z*y)=0, $\forall$ x, y, z $\in$ X by (a_3).

iv. (x * y)* z = (x * z)*y, $\forall$ x, y, z $\in$ X by (a_4).

Definition (1.9) : [14]

In any BH-algebra X, we can define a **partial order relation $\leq$** by putting x $\leq$ y if and only if x*y=0.

Definition(1.10):[9]

A BH-algebra X is said to be a **normal BH-algebra** if it satisfying the following conditions:

i. 0*(x*y) = (0*x)*(0*y) ,$\forall$x,y $\in$X.

ii. (x*y)*x = 0* y,$\forall$x,y $\in$X.

iii. (x*(x*y))*y = 0,$\forall$x,y $\in$X.

Definition (1.11) : [7]

A BCH-algebra X is called **medial** if x * (x * y) = y, $\forall$ x, y $\in$ X.

We generalize the concept of **medial** to BH-algebra.

Definition (1.12) :

A BH-algebra X is called **medial** if $x * (x * y) = y, \forall\ x, y \in X$.

Definition (1.13) : [3]

A BH-algebra X is called an **associative BH-algebra** if: $(x*y)*z = x*(y*z), \forall\ x, y, z \in X$.

Theorem (1.14): [3]

Let X be an associative BH-algebra. Then the following properties are hold:

i. $0*x=x \quad ; \quad \forall\ x\in X$

ii. $x*y=y*x \quad ; \quad \forall\ x, y\in X$

iii. $x*(x*y)=y \quad ; \quad \forall\ x, y\in X$

iv. $(z*x)*(z*y)=x*y \quad ; \quad \forall\ x, y, z\in X$

v. $x*y=0 \Rightarrow x=y \quad ; \quad \forall\ x, y\in X$

vi. $(x*(x*y))*y=0 \quad ; \quad \forall\ x, y\in X$

vii. $(x*y)*z=(x*z)*y \quad ; \quad \forall\ x, y, z\in X$

viii. $(x*z)*(y*t)=(x*y)*(z*t) \quad ; \quad \forall\ x, y, z, t\in X$

Definition (1.15) :[4]

Let X be a BH-algebra. Then the set $X_+ = \{ x \in X : 0 * x = 0 \}$ is called the **BCA-part of X.**

Definition (1.16) : [3]

Let X be a BH-algebra. Then the elements of the set **$L_K(X)$,** where

$L_K(X) = \{ a\in X_+\backslash\{0\} : x * a = 0 \Rightarrow x = a, \forall\ x \in X\backslash\{0\} \}$ is called a **K-atom of X.**

Definition (1.17) : [12]

A nonempty subset S of a BH-algebra X is called a **Subalgebra** of X if $x * y \in S, \forall\ x, y \in S$.

Definition(1.18) : [6]

An ideal I of a BCH-algebra X satisfies the condition $x \in I$ and $a\in X\backslash I$ imply $x*a\in I$, is called a ***-ideal** of X.

We generalize the concept of a ***- ideal** to a BH-algebra.

Definition(1.19) :

An ideal I of a BH-algebra X satisfies the condition $x \in I$ and $a\in X\backslash I$ imply $x*a\in I$, is called a ***-ideal** of X.

Theorem (1.20) : [2]

In a BCH-algebra X, the following conditions are equivalent:

1. Every nonzero element of X is a K-atom of X, i.e. $X = L_K(X)\cup\{0\}$,
2. $x*y=x, \forall\ x, y \in X$ with $x\neq y$,
3. $x*(x*y) = 0, \forall\ x, y\in X$ with $x\neq y$,
4. every subalgebra of X is a *-ideal of X.

Definition (1.21) : [12]:

Let I be a nonempty subset of a BH-algebra X. Then I is called an **ideal** of X if it satisfies:

i. $0\in I$.

ii. $x*y\in I$ and $y \in I$ imply $x\in I$.

Proposition (1.22) : [3]

Let $\{ I_i , i\in\Gamma\}$ be a family of ideals of a BH-algebra X. Then $\bigcap_{i\in\Gamma} I_i$ is an ideal of X.

Theorem(1.23):[3]

Let $\{ I_i , i\in\Gamma\}$ be a chain ideals of a BH-algebra X. Then $\bigcup_{i\in\Gamma} I_i$ is an ideal of X.

Proposition (1.24) : [3]

Let **$f: X\rightarrow Y$** be a BH- epimorphism, if I is an ideal of X then f(I) is an ideal of Y.

Proposition (1.25) : [3]
Let **f: X→Y** be a BH- homomorphism, if I is an ideal of Y then $f^{-1}(I)$ is an ideal of X.

Definition (1.26):[4]
An ideal I of a BH-algebra X is called a **closed ideal** of X, $0*x\in I$, $\forall x\in I$.

Definition (1.27) :[4]
Let X be a BH-algebra and I be an ideal of X. Then I is called a **closed ideal with respect to an element b∈X** (denoted **b-closed ideal**) if $b*(0*x)\in I$, $\forall$ $x\in I$.

Definition (1.28):[3]
An ideal I of a BH-algebra is called a **completely closed ideal** if $x * y \in I$, $\forall x, y \in I$.

Definition (1.29) : [6]
An ideal I of a BCH-algebra X is called **a normal ideal** if $x*(x*y)\in I$ implies $y*(y*x)\in I$, $\forall$ x , y∈ X.

We generalize the concept of a **normal ideal** to a BH-algebra.

Definition (1.30) :
An ideal I of a BH-algebra X is called **a normal ideal** if $x*(x*y)\in I$ implies $y*(y*x)\in I$, $\forall$ x, y∈X.

Definition(1.31):[3]
Let X be a BH-algebra, a non-empty subset N of X is said to be **normal subset** of X if $(x*a)*(y*b)\in N$ for all $x*y$, $a*b\in N$, $\forall$ x, y, a, b∈X.

Definition (1.32):[10]
Let X be a BH-algebra. For a fixed a∈X, we define a map R_a: X→X such that $\mathbf{R_a(x)=x*a}$, $\forall$ x∈X, and call R_a a **right map** on X. The set of all right maps on X is denoted by R(X). A left map L_a is defined by a similar way, we define a map L_a : X→X such that $\mathbf{L_a(x)=a*x}$, $\forall$x∈X, and called L_a a **left map** on X. The set of all left maps on X is denoted by L(X).

Definition (1.33): [4]
A nonempty subset I of a BH-algebra X is called an **implicative ideal** of X if:
i. $0 \in I$.
ii. $(x*(y*x))*z \in I$ and $z \in I$ imply $x\in I$, $\forall$ x, y, z $\in$ X.

Proposition (1.34) :[4]
Every implicative ideal of a BH-algebra X is an ideal of X.

Definition (1.35) : [5]
A BCI-algebra is said to be an implicative if it satisfies $(x*(x*y))*(y*x) =y*(y*x)$, $\forall$ x, y ∈X.

We generalize the concept of an **implicative** BCI-algebra to a **BH-algebra**.

Definition (1.36):
A BH -algebra is said to be an implicative if it satisfies $(x*(x*y))*(y*x) =y*(y*x)$, $\forall$ x, y ∈X.

Example (1.37):
Consider the BH-algebra X= {0, 1, 2} with the binary operation '*' defined by the following table**:**

*	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

Then (X,*,0) is an implicative BH-algebra.

Theorem (1.38) : [15]
A BCI-algebra is implicative if and only if every closed ideal of X is an implicative.

Definition (1.39):[10]

A BH-algebra (X,*, 0) is said to be **a positive implicative** if it satisfies the condition, $\forall$ x, y, z $\in$X, (x*z)*(y*z) = (x*y)*z.

Remark (1.40):[10]

Let X be a positive implicative BH-algebra and $\oplus$ be a binary operation defined on L(X) by $(L_a \oplus L_b)(x) = L_a(x) * L_b(x)$ and $(L_a \oplus L_b)(x) = L_{a*b}(x)$; $\forall L_a, L_b \in L(X)$ and $\forall x \in X$

Theorem (1.41) :[10]

If X is a positive implicative BH-algebra, then $(L(X),\oplus,L_0)$ is a positive implicative BH-algebra.

Remark (1.42):[13]

Let X and Y be BH-algebras. A mapping f: X$\to$Y is called a **homomorphism** if f(x*y)=f(x)*'f(y), $\forall$x, y $\in$ X. A homomorphism f is called a **monomorphism** (resp., **epimorphism**) if it is an injective (resp**.,** surjective). A bijective homomorphism is called an isomorphism. Two BH-algebras X and Y are said to be **isomorphic**, written X $\cong$ Y , if there exists an isomorphism f:X$\to$Y . For any homomorphism f:X $\to$Y, the set {x$\in$X ; f(x)=0'} is called the **kernel** of f, denoted by Ker(f), and the set {f(x):x$\in$X} is called the **image** of f, denoted by Im(f). Notice that f(0)=0', for all homomorphism f.

Definition (1.43):[11]

An ideal A of a BH-algebra X is said to be **a translation ideal** of X if x *y $\in$ A and y* x $\in$ A $\Rightarrow$ (x*z)*(y*z) $\in$A and (z*x)*(z*y) $\in$A, $\forall$ x, y, z $\in$ X.

Remark (1.44):[12]

Let (X,*,0) be a BH-algebra and let A be a translation ideal of X. Define a relation $\sim_A$ on X by $x \sim_A y$ if and only if x*y $\in$ A and y*x $\in$ A, where x , y $\in$ X .Then $\sim_A$ is an equivalence relation on X. Denote the equivalence class containing x by $[x]_A$, i.e., $[x]_A=\{y\in X|x\sim_A y\}$ and $X/A=\{[x]_A|x\in X\}$. And define $[x]_A\oplus [y]_A =[x*y]_A$, then $((X/A),\oplus,[0]_A)$ is a BH-algebra.

Theorem(1.45):[12]

Let f: X → Y be a homomorphism of BH-algebra. Then Ker(f) is a translation ideal of X.

Definition(1.46):[3]

Let X be a BH-algebra, a non-empty subset N of X is said to be **normal subalgebra** of X if

i. (x*a)*(y*b)$\in$N, $\forall$x*y, a*b$\in$ N, $\forall$ x, y, a, b$\in$X.

ii. x * y $\in$ N, $\forall$ x, y $\in$ N.

Remark (1.47):

Let (X,*, 0) be a BH-algebra and let N be a normal subalgebra of X. Define a relation $\sim_N$ on X by $x \sim_N y$ if and only if x*y $\in$N and y*x $\in$N, where x, y$\in$ X .Then$\sim_N$ is an equivalence relation on X. Denote the equivalence class containing x by $[x]_A$, i.e., $[x]_N=\{y\in X|\ x\sim_N y\}$ and $X/N=\{[x]_N \mid x\in X\}$. And define $[x]_N\oplus [y]_N =[x*y]_N$, then $((X/N),\oplus,[0]_N)$ is a BH-algebra.

Remark (1.48):[3]

The BH-algebra X/N is called the quotient BH-algebra of X by N.

Theorem(1.49):[3]

Let N be a normal subalgebra of a BH-algebra X. Then X/N is a BH-algebra.

Definition (1.50) :[4]

Let X be a BH-algebra and a ∈ med(X). B(a) = {x ∈ X : a*x = 0} is called **a branch subset** of X **determined by a.**

2. The Main Results:

Proposition(2.1):

Let X = L_K(X)∪{0} be a BH-algebra. Then every ideal of X is an implicative ideal.

Proof:

i. Since I is an ideal of X, so 0∈ I

ii. Let I be an ideal of X and x, y, z∈X such that (x*(y*x))*z ∈ I and z∈I.

⇒ x*(y*x) ∈ I [Since I is an ideal]

We have two cases:

Case1: if x=y, we will have x*(y*x) = x*(x*x) =x*0=x

[Since X is a BH-algebra; x*x=0 and x*0=x]

⇒ x ∈ I [Since x*(y*x) ∈ I] . Then I is an implicative ideal of X.

Case2 : if x≠y, then x*(y*x) = x*y = x

[Since X = L_K(X)∪{0},then y*x=y, ∀x, y ∈ X with x≠y ; by Theorem (1.20,2)]

⇒ x ∈ I [Since x*(y*x) ∈ I].

Then I is an implicative ideal of X.

Proposition(2.2):

If X is a BH-algebra satisfies the condition, ∀ x , y∈X ; x = x *(y *x) (a_1) , then every ideal is an implicative ideal of X.

Proof :

Let I be an ideal of X and x, y, z ∈X such that (x*(y*x))*z ∈I and z∈I

⇒ x*(y*x)∈I. [Since I is an ideal of X.]

⇒ x ∈ I. [By (a_1)]

Then I is an implicative ideal of X. ■

Remark (2.3) :

In any BH-algebra, the set I=X is an implicative ideal of X, but the set I={0} may not be an implicative ideal of X, as in the following example ,

Example (2.4):

Consider the BH-algebra X= {0, 1, 2, 3} with the binary operation '*'defined by the following table**:**

*	0	1	2	3
0	**0**	**0**	**2**	**3**
1	**1**	**0**	**2**	**2**
2	**2**	**1**	**0**	**1**
3	**3**	**2**	**3**	**0**

Then (X,*,0) is a BH-algebra. The subset **I={0}** is not an implicative ideal of X. Since if x=2 , y=0 ,z=0, then (2*(0*2))*0 =0*0=0∈I and 0∈I but x=2∉I.

Theorem (2.5):

Let X be BH-algebra and let I be an ideal of X. Then I is an implicative ideal of X if and only if $x*(y*x) \in I$ imply $x \in I$ (a_2).

Proof:

Let I be an implicative ideal of X and x, y $\in$X such that $x*(y*x) \in I$. Then $(x*(y*x))*0 \in I$.

[Since X is a BH-algebra; $x*(y*x) = (x*(y*x))*0$]

Now, we have $(x*(y*x))*0 \in I$ and $0 \in I$. Then $x \in I$. [Since I is an implicative ideal of X]

Conversely,

Let I be an ideal of X and x, y, z $\in$X such that $(x*(y*x))*z \in I$ and $z \in I$.

$\Rightarrow$ $x*(y*x) \in I$. [Since I is an ideal of X.]

$\Rightarrow$ $x \in I$. [By (a2)]

Then I is an implicative ideal of X.■

Proposition(2.6):

Let X be BH-algebra. If {0} is an implicative ideal of X, then $0*x \neq x$, $\forall$ $x \in X/\{0\}$.

Proof:

Suppose I = {0} be an implicative ideal of X and $x \in X/\{0\}$ such that $0*x=x$.

Now,

$\Rightarrow$ $x*(0*x) = x*x = 0$ [Since X is an associative BH-algebra; $x*x=0$ and $0*x=x$].

We have $(x*(0*x))*0=0 \in I$ and $0 \in I$

$\Rightarrow$ $x \in I$ [Since I is an implicative ideal]

$\Rightarrow$ $x=0$ [Since I = {0}],

we get a contradiction . [Since $x \in X/\{0\}$]

Then $0*x \neq x$. ■

Remark (2.7):

The converse of proposition (2.6) is not correct in general, as in the following example:

Example (2.8):

Consider the BH-algebra X= {0, 1, 2, 3, 4} with the binary operation '*' defined by the following table:

*	0	1	2	3	4
0	0	2	1	0	3
1	1	0	2	1	1
2	2	1	0	2	2
3	3	2	3	0	3
4	4	4	4	4	0

$0*x \neq x$, $\forall$ $x \in X/\{0\}$, but the set I={0} is not an implicative ideal of X. Since

If x=1 , y=2 , z=0, then $(1*(2*1))*0 = 1*1 = 0 \in I$, but $x=1 \notin I$.

Theorem(2.9) :

Every associative BH- algebra is an implicative BH-algebra.

Proof :

Let X be an associative BH- algebra. Then

$(x*(x*y))*(y*x)=((x*x)*y)*(y*x)$ [Since X is an associative BH-algebra]

=(0*y)*(y*x) [Since X is a BH-algebra; x*x=0]

= y*(y*x) [Since X is an associative BH-algebra; 0*y=y, by Theorem (1.14,i)]

Then X is an implicative BH-algebra.

Theorem(2.10) :

Let X be a BH-algebra and satisfies the condition, $((x*y)*(x*z))*(z*y)=0, \forall x, y, z \in X$ (a_3).
Then X is an implicative if and only if every closed ideal of X is an implicative ideal of X.

Proof: Directly from Theorem (1.6) and (1.38).

Lemma (2.11): Every medial BH- algebra is an implicative BH-algebra.

Proof : Let X be a medial BH- algebra. Then

(x*(x*y))*(y*x) = y*(y*x) [Since X is medial ; x*(x*y)=y].

Then X is an implicative BH-algebra.

Theorem (2.12) :

Let X be an implicative BH-algebra satisfies (a_3) and let I be an ideal of X. Then

i. If $I \subseteq X_+$, then I is an implicative ideal of X.
ii. If $L_0(I) \subseteq I$, then I is an implicative ideal of X.
iii. If X is equal to a branch subset of X determined by " 0 ", then I is an implicative ideal of X.

Proof :

i. Let $I \subseteq X_+$ and I be an ideal of X.

$\Rightarrow$ $0 * x = 0 \in I, \forall x \in X$.

$\Rightarrow$ $0 * x = 0 \in I, \forall x \in I$. [Since $I \subseteq X_+$]

$\Rightarrow$ every ideal of X is a closed ideal of X. [by Definition (1.26)]

$\Rightarrow$ X is a BCI-algebra. [Since X is BH-algebra and satisfies (a3), By Theorem(1.6)]

$\Rightarrow$ I is an implicative ideal of X.

[Since every closed ideal of X is an implicative ideal of X. By Theorem (1.38)]. ■

ii. Let $x \in I$. Then $L_0(x) \in I$. [Since $L_0(I) \subseteq I$]

$\Rightarrow$ $0*x \in I$ [Since $L_0(x) = 0*x$]

$\Rightarrow$ I is a closed ideal of X. [By Definition (1.26)]

$\Rightarrow$ X is a BCI-algebra.[Since X is BH-algebra and satisfies (a3), By Theorem (1.6)]

$\Rightarrow$ I is an implicative ideal of X.

[Since every closed ideal of X is an implicative ideal of X. By Theorem (1.38)]. ■

iii. Let X is equal to a branch subset of X determined by " 0 " and let I be an ideal of X.

$\Rightarrow$ X=B(0)

$\Rightarrow$ $0 * x = 0 \in I, \forall x \in X$. [Since X=B(0)]

$\Rightarrow$ $0 * x = 0 \in I, \forall x \in I$. [Since $I \subseteq X$]

$\Rightarrow$ I is a closed ideal of X. [By Definition (1.26)]

$\Rightarrow$ I is an implicative ideal of X.

[Since every closed ideal of X is an implicative ideal of X. By Theorem (2.10)].■

Theorem (2.13) :

Let X be an associative BH-algebra. Then

i. every proper subset of X is not an implicative ideal of X.

ii. X_+ is not an implicative ideal of X.

iii. a branch subset of X determined by " 0 " is not an implicative ideal of X.

Proof :

i. Suppose I is an implicative ideal of X and I is a proper subset of X. Then

There exist $x \in X$ such that $x \notin I$ [Since $I \subset X$]

Now, Since X is a BH-algebra, we have $x*0=x$. So $(x*(0*x))*0 = x*(0*x)$

$= x*x$ [Since $0*x=x$; by Theorem (1.14,i)]

$= 0 \in I$ [since X is a BH- algebra ; $x*x=0$]

We have

$(x*(0*x))*0 \in I$ and $0 \in I$.

$\Rightarrow x \in I$ [since I is an implicative ideal of X]

We get a contradiction (By assumption $I \subset X$, $x \notin I$]

$\Rightarrow$ I is not an implicative ideal of X.

Then every proper subset of X is not an implicative ideal of X. ■

ii. To prove X_+ is not an implicative ideal of X.

$X_+=\{x \in X ; 0*x=0\} =\{0\}$ [since X is an associative ; $0*x=x$; by Theorem (1.14,i)]

Now,

Since $X_+ \subset X$

Then X_+ is not an implicative ideal of X [by (i)]. ■

iii. To prove a branch subset of X determined by " 0 " is not an implicative ideal of X.

$B(0)=\{x \in X ;0*x=0\} =\{0\}$ [since X is an associative ; $0*x=x$; by Theorem (1.14,i)]

Now,

Since $B(0) = X_+$.

$\Rightarrow$ B(0) is not an implicative ideal of X [by (ii)]. ■

Then a branch subset of X determined by " 0 " is not an implicative ideal of X.

Corrolary (2.14): Let X be an associative BH-algebra. Then X is a unique implicative ideal of X.

Proof : Directly by Theorem (2.13 ,i) and Remark (2.3). ■

Theorem (2.15) :

Let X be a medial BH-algebra and satisfies (a_3). Then every normal ideal of X is an implicative ideal of X.

Proof :

Let I be a normal ideal of X and let $x \in X$. Then

$(0*x)*((0*x)*0)=(0*x)*(0*x)=0 \in I$ [Since X is s BH-algebra ; $x*0=x$ and $x*x=0$]

$\Rightarrow 0*(0*(0*x)) \in I$ [Since I is a normal ideal]

$\Rightarrow 0*x \in I$; $\forall x \in X$ [Since X is a medial ; $x*(x*y) =y$]

$\Rightarrow 0*x \in I$; $\forall x \in I$

$\Rightarrow$ I is a closed ideal of X. [By Definition (1.26)]

$\Rightarrow$ I is an implicative ideal of X. [Since every closed ideal of X is an implicative ideal of X. By Theorem(2.10)]].■

Theorem (2.16):

Let X be an implicative BH-algebra and satisfies (a_3).Then every completely closed ideal of X is an implicative ideal of X.

Proof :

Let I be a completely closed ideal of X .Then I is an ideal of X. [By definition (1.28)]

Let y ∈X, if x=0

$\Rightarrow$ 0*y ∈ I, ∀ y ∈ X.

$\Rightarrow$ 0*y ∈ I, ∀ y ∈ I.

Then I is a closed ideal of X.

$\Rightarrow$ I is an implicative ideal of X. [Since every closed ideal of X is an implicative ideal of X. By Theorem (2.10)].■

Proposition (2.17):

Let X be a normal BH-algebra such that $X=X_+$ and let I be an implicative ideal of X. Then I is a completely closed ideal of X.

Proof:

Let I be a an implicative ideal of X. Then I is an ideal of X. [By proposition(1.34)]

Let x , y ∈ I. Then

((x*y)*(0*(x*y)))*x= ((x*y)*0)*x [Since 0*(x*y)=0 ; $X=X_+$. By Definition (1.15)]

=(x*y)*x [Since X is a BH-algebra . x*0=x]

= 0*y [Since X is a normal, By Definition (1. 10, ii)]

=0 ∈ I [Since $X=X_+$. By Definition(1.15)]

$\Rightarrow$ ((x*y)*(0*(x*y)))*x ∈ I and x ∈ I $\Rightarrow$ x*y ∈ I. [Since I is an implicative ideal of X]

Therefore, I is a completely closed ideal of X.■

Theorem (2.18):

Let { I_i , i∈Γ} be a family of implicative ideals of a BH-algebra X. Then $\bigcap_{i\in\Gamma} I_i$ is an implicative ideal of X.

Proof:

To prove that $\bigcap_{i\in\Gamma} I_i$ is an implicative ideal of X.

i. $0\in I_i$,∀i∈Γ [Since each I_i is an implicative ideal of X,∀i∈Γ.By Definition(1.33)]

$\Rightarrow 0\in \bigcap_{i\in\Gamma} I_i$

ii. Let $(x*(y*x))*z \in \bigcap_{i\in\Gamma} I_i$ and $z \in \bigcap_{i\in\Gamma} I_i$

$\Rightarrow$ $(x*(y*x))*z \in I_i$ and $z \in I_i$,∀i ∈Γ

$\Rightarrow$ $x\in I_i$,∀i ∈Γ [Since each I_i is Implicative ideal of X, ∀ i ∈Γ. By Definition(1.33)]

$\Rightarrow$ $x \in \bigcap_{i\in\Gamma} I_i$. Therefore, $\bigcap_{i\in\Gamma} I_i$ is an implicative ideal of X. ■

Corollary (2.19): Let $X=L_K(X)\cup\{0\}$ and let { I_i , i∈Γ} be a family of ideals of a BH-algebra X. Then $\bigcap_{i\in\Gamma} I_i$ is an implicative ideal of X.

Proof: Let $\{I_i , i\in\Gamma\}$ be a family of ideals of X. Then $\bigcap_{i\in\Gamma} I_i$ is an ideal of X. [By Theorem(1.22)].

Therefore, $\bigcap_{i\in\Gamma} I_i$ is an implicative ideal of X. [Since $X=L_K(X)\cup\{0\}$, by Proposition (2.1)].■

Theorem (2.20):

Let $\{ I_i , i\in\Gamma\}$ be a chain implicative ideals of a BH-algebra X. Then $\bigcup_{i\in\Gamma} I_i$ is an implicative ideal of X.

Proof : To prove that $\bigcup_{i\in\Gamma} I_i$ is an implicative ideal of X.

i. $0\in I_i , \forall i\in\Gamma$

[Since each I_i is an implicative ideal of X, $\forall i\in\Gamma$. By Definition(1.33)]

$\Rightarrow 0\in \bigcup_{i\in\Gamma} I_i$

ii. Let $(x*(y*x))*z \in \bigcup_{i\in\Gamma} I_i$ and $z \in \bigcup_{i\in\Gamma} I_i$

$\exists I_j , I_k \in \{ I_i \}_{i\in\Gamma}$, such that $(x*(y*x))*z \in I_j$ and $z \in I_k$,

$\Rightarrow$ either $I_j \subseteq I_k$ or $I_k \subseteq I_j$ [Since $\{I_i\}_{i\in\Gamma}$ is a chain]

$\Rightarrow$ either $(x*(y*x))*z \in I_j$ and $z\in I_j$ or $(x*(y*x))*z \in I_k$ and $z\in I_k$

$\Rightarrow$ either $x \in I_j$ or $x \in I_k$

[Since I_j and I_k are implicative ideals of X. By Definition(1.33)]

$\Rightarrow x \in \bigcup_{i\in\Gamma} I_i$. Therefore $\bigcup_{i\in\Gamma} I_i$ is an implicative ideal of X. ■

Corollary (2.21): Let $X=L_K(X)\cup\{0\}$ and let $\{ I_i , i\in\Gamma\}$ be a Chain of ideals of a BH-algebra X. Then $\bigcup_{i\in\Gamma} I_i$ is an implicative ideal of X.

Proof: Let $\{ I_i , i\in\Gamma\}$ be a chain of ideals of X. Then $\bigcup_{i\in\Gamma} I_i$ is an ideal of X. [by Theorem(1.23)]

Therefore, $\bigcup_{i\in\Gamma} I_i$ is an implicative ideal of X.[Since $X=L_K(X)\cup\{0\}$, by Proposition (2.1)] .■

Proposition (2.22) :

Let **f: (X,*,0)→(Y,*',0')** be a BH- epimorphism. If I is an implicative ideal of X, then f(I) is an implicative ideal of Y.

Proof :

Let I be an implicative ideal of X. Then

i. $f(0) = 0'$, [Since f is an epimorphism, by Remark(1.42)]

$\Rightarrow 0'\in f(I)$

ii. Let $(x*'(y*'x))*'z \in f(I)$ and $z \in f(I)$

$\Rightarrow \exists a ,b\in I$ and $c \in I$ such that $f(a)=x$, $f(b)=y$ and $f(c)=z$

$\Rightarrow (x*'(y*'x))*'z =[f(a)*'(f(b)*'f(a))]*'f(c)=f((a*(b*a))*c) \in f(I)$ [Since f is an epimorphism]

⇒ $(a*(b*a))*c \in I$ and $c\in I$ [Since $f(I)=\{f(x) ; x\in I\}$]
⇒ $a \in I$ [Since I is an implicative ideal of X]
⇒ $f(a) \in f(I)$.
Then f(I) is an implicative ideal of Y. ■

Proposition (2.23) :

Let **f: (X,*,0)→(Y,*',0')** be a BH- homomorphism and I is an implicative ideal of Y. Then $f^{-1}(I)$ is an implicative ideal of X.

Proof :

Let I be an implicative ideal of Y. Then

i. $f(0) = 0'$ [Since f is a homomorphism, by Remark(1.42)]
⇒ $0=f^{-1}(0')\in f^{-1}(I)$

ii. Let x, y, z $\in$X such that $(x*(y*x))*z \in f^{-1}(I)$ and $z \in f^{-1}(I)$
⇒ $f((x*(y*x))*z) \in I$ and $f(z) \in I$
⇒$f((x*(y*x))*z)=(f(x)*'(f(y)*'f(x)))*'f(z)\in I$ and $f(z) \in I$ [Since f is a homomorphism, by Remark(1.42)]
⇒ $f(x) \in I$ [Since I is an implicative ideal of Y]
⇒ $x \in f^{-1}(I)$.
Then $f^{-1}(I)$ is an implicative ideal of X. ■

Theorem (2.24):

Let X be a BH-algebra and N be a normal subalgebra. If I is an ideal of X, then I/N is an ideal of X/N.

Proof :

Let I be an ideal of X. Then

i. Since $0\in I \Rightarrow [0]_N \in I/N$.
ii. Let $[x]_N,[y]_N\in X/N$.
⇒ $[x]_N *[y]_N \in I/N$ and $[y]_N\in I/N$ [Since$[x]_N*[y]_N=[x*y]_N$, By remark(1.47)].
⇒$[x*y]_N \in I/N$ and $[y]_N\in I/N$
⇒ $x*y \in I$ and $y\in I$ [Since $I/N=\{[x]_N | x\in I\}$, By remark(1.47)]
⇒ $x\in I$ [Since I is an ideal of X].
⇒ $[x]_N\in I/N$. Then I/N is an ideal of X/N.■

Theorem (2.25):

Let X be a BH-algebra and N be a normal subalgebra. If I is an implicative ideal of X, then I/N is an implicative of X/N.

Proof:

Let I be an implicative ideal of X. To prove I/N is an implicative ideal of X/N.
⇒I is an ideal of X. [By proposition(1.34)]
⇒I/N is an ideal of X/N. [By proposition(2.24)]
i. Since $0\in I\Rightarrow[0]_N \in I/N$.
ii. Let $[x]_N, [y]_N, [z]_N\in X/N$.
⇒ $([x]_N*([y]_N * [x]_N)) *[z]_N \in I/N$ and $[z]_N\in I/N$
⇒ $([x]_N*[y*x]_N) *[z]_N \in I/N$ and $[z]_N\in I/N$ [Since$[x]_N*[y]_N=[x*y]_N$, By remark(1.47)]
⇒ $[x*(y*x)]_N *[z]_N \in I/N$ and $[z]_N\in I/N$

$\Rightarrow [(x*(y*x))*z]_N \in I/N$ and $[z]_N \in I/N$

$\Rightarrow (x*(y*x))*z \in I$ and $z \in I$ [Since $I/N=\{[x]_N | x\in I\}$, By remark(1.47)]

$\Rightarrow x \in I$ [Since I is an implicative ideal of X]

$\Rightarrow [x]_N \in I/N$.

Then I/N is an implicative ideal of X/N.■

Theorem (2.26):

Let X be a BH-algebra and A be a translation ideal of X. If I is an ideal of X, then I/A is an ideal of X/A.

Proof:

Let I be an ideal of X. To prove I/A is an ideal of X/A.

i. Since $0\in I \Rightarrow [0] \in I/A$.

ii. Let $[x]_A,[y]_A \in X/A$.

$\Rightarrow [x]_A \oplus [y]_A \in I/A$ and $[y]_A \in I/A$ [Since $[x]_A \oplus [y]_A = [x*y]_A$. By remark(1.44)]

$\Rightarrow [x*y]_A \in I/A$ and $[y]_A \in I/A$

$\Rightarrow x*y \in I$ and $y \in I$ [Since $I/A=\{[x]_A | x\in I\}$. By Remark(1.44)]

$\Rightarrow x \in I$ [Since I is an ideal of X]

$\Rightarrow [x]_A \in I/A$

Then I/A is an ideal of X/A.■

Proposition(2.27):

Let X be a BH-algebra and A be a translation ideal. If I is an implicative ideal of X, then I/A is an implicative of X/A.

Proof:

Let I be an implicative ideal of X. To prove I/A is an implicative ideal of X/A.

i. Since $0\in I \Rightarrow [0] \in I/A$.

ii. Let $[x]_A,[y]_A, [z]_A \in X/A$.

$\Rightarrow ([x]_A \oplus ([y]_A \oplus [x]_A)) \oplus [z]_A \in I/A$ and $[z]_A \in I/A$

$\Rightarrow ([x]_A \oplus [y*x]_A) \oplus [z]_A \in I/A$ and $[z]_A \in I/A$ [Since $[x]_A \oplus [y]_A = [x*y]_A$ .By remark(1.44)]

$\Rightarrow [x*(y*x)]_A \oplus [z]_A \in I/A$ and $[z]_A \in I/A$

$\Rightarrow [(x*(y*x))*z]_A \in I/A$ and $[z]_A \in I/A$

$\Rightarrow (x*(y*x))*z \in I$ and $z \in I$ [Since $I/A=\{[x]_A | x\in I\}$. By Remark(1.44)]

$\Rightarrow x \in I$ [Since I is an ideal of X]

$\Rightarrow [x]_A \in I/A$ .Then I/A is an implicative ideal of X/A.■

Corollary (2.28):

Let X be a BH-algebra. If I is an implicative ideal of X,then I/Ker(f) is an implicative of X/ Ker(f).

Proof:

Let I be an implicative ideal of X. To prove I/Ker(f) is an implicative ideal of X/Ker(f).

Since Ker(f) is translation ideal. [By Theorem(1.45)]

$\Rightarrow$ I/ Ker(f) is an implicative ideal of X/ Ker(f). [By Theorem(2.27)].■

Remark (2.29) :

Let X be a BH-algebra and let I be a subset of X. we will define to the set $\{ L_a \in L(X) ; a \in I \}$ by L(I).

Theorem (2.30) :

Let X be a positive implicative BH-algebra. If I is an ideal of X. Then L(I) is an ideal of $(L(X),\oplus,L_0)$.

Proof:

Let I be an ideal of X. To prove L(I) is an ideal of $(L(X),\oplus,L_0)$.

i. $0 \in I \Rightarrow L_0 \in L(I)$ [By Remark (2.29)]

ii. Let $L_a \oplus L_b , L_b \in L(I)$.

We have $L_a \oplus L_b = L_{a*b}$, where a ,b $\in I$

$\Rightarrow a*b \in I$ and $b \in I$

$\Rightarrow a \in I$ [Since I is an ideal of X]

$\Rightarrow L_a \in L(I)$ .Then L(I) is an ideal of $(L(X),\oplus,L_0)$. ■

Corollary (2.31):

Let X be a positive implicative BH-algebra. If I is an implicative ideal of X. Then L(I) is an implicative ideal of $(L(X),\oplus,L_0)$.

Proof:

Let I be an implicative ideal of X. Then I is an ideal of X.

$\Rightarrow$ L(I) is an ideal of L(X). [By Theorem(2.30)]

i. $0 \in I \Rightarrow L_0 \in L(I)$ [Since I is an ideal of X]

ii. Let $(L_a \oplus (L_b \oplus L_a)) \oplus L_c \in L(I)$ and $L_c \in L(I)$

$\Rightarrow (a * (b * a)) * c \in I$ and $c \in I$ [Since $(L_a \oplus (L_b \oplus L_a)) \oplus L_c = L_{(a*(b*a))*c} \in L(I)$]

$\Rightarrow a \in I$ [Since I is an implicative ideal of X]

$\Rightarrow L_a \in L(I)$.Then L(I) is an implicative ideal of $(L(X),\oplus,L_0)$. ■

Theorem (2.31):

If $X= L_K(X) \cup \{0\}$ be a BH-algebra satisfies (a_4) and S be a subalgebra of X, then S is an implicative ideal of X.

Proof:

Since X be a BH-algebra satisfies (a_4), then X is a BCH-algebra. [by Theorem(1.7)]

Let S is a subalgebra of X. Then S is a *-ideal. [By Theorem(1.20,4)]

$\Rightarrow$ S is an ideal. [every *-ideal is an ideal. By Definition (1.19)]

To prove S is an implicative ideal of X.

i) $0 \in S$ [Since S is an ideal]

ii) Let x, y, $z \in X$ such that $(x*(y*x))*z \in S$ and $z \in S$.

$\Rightarrow x*(y*x) \in S$. [Since S is an ideal of X]

We have two cases:

Case 1: if x=y, then $x*(x*x) \in S$

$\Rightarrow x*0 \in S$ [Since X is a BH-algebra ; x*x=0]

$\Rightarrow x \in S$ [Since X is a BH-algebra ; x*0=x]

Then S is an implicative ideal of X.

Case 2: if $x \neq y$, then $x*(y*x) = x*y = x$

$\Rightarrow x*y \in S$ [Since $X = L_K(X) \cup \{0\}$, then $y*x=y$; $\forall$ x, y $\in X$ with $x \neq y$, by Theorem (1.20, 2)]

$\Rightarrow x \in S$ [Since $x*y=x$]

Then S is an implicative ideal of X.■

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