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Bayesian Reciprocal Adaptive Bridge Composite Quantile Regression with Ordinal Data

Rahim Alhamzawi
rahim.alhamzawi@qu.edu.iq
Zahraa Saad
Al-Qadisiyah University, College of Administration and Economics

Corresponding Author : Zahraa Saad

Abstract : Selecting active variables for a QR model is difficult. Selecting the right group of predictors often improves prediction accuracy. To improve scientific understanding, choose a smaller subset. Several methods have been presented to find the active subset. Estimating model parameters aims to find the best estimators for accurate predictions. Estimating all the parameters in the high-dimensional data request yields a weak prediction with large correlations between independent variables, resulting in incorrect findings. Variable selection (V.S) is a key challenge in modelling high-dimensional data. Linear QR selection variables and estimation are studied using the Bayesian hierarchical approach. Regularization bridge and ordinal composite quantile regression are our specialities. This work proposes a Bayesian reciprocal adaptive bridge composite quantile regression for ordinal variable selection and estimation. A new Gibbs sampling approach is developed for comprehensive conditional posterior distributions. We look at how Bayesian reciprocal adaptive bridge composite quantile regression for ordinal data (BrABCQRO) stacks up against other Bayesian and non-Bayesian approaches. The posterior, prior, and conditional distributions are all talked about together. For full conditional posterior distributions, a new Gibbs sampling method is created. A real-world example and many simulation examples show that the suggested methods often work better than standard ones.
Keywords: Reciprocal adaptive Bridge, Composite Quantile Regression, Gibbs sampler, Ordinal data.

Introduction:

Quantile regression (QReg), proposed by Koeker and Bassett (1978), has garnered attention from statisticians, econometricians, and applied researchers. Economics, agriculture, medicine, genetics, sociology, and others have employed it "(Alhamzawi (2013), Koenker(2005), and Yu et al.(2003))."

QReg has various advantages over SReg (Orsini&Bottai, 2011). It detects distinct response variable effects at different quantities. Since it does not require a data distribution, it is possible (Liu, Saat, Qin &Barkan, 2013). The estimators are insensitive to outliers (Koenker, 2005), and most crucially, they can handle data heterogeneity without requirements. "For any τ^{th} quantile, $(0< \tau <1)$, the τ^{th} quantile regression can be defined as

$\mathbf{Q}_{\mathbf{y_i}|\mathbf{x_i}}(\boldsymbol{\tau}) = \boldsymbol{x}_i'\boldsymbol{\beta}_\tau$,

where $\boldsymbol{y_i}$ is the response variable, $\boldsymbol{x}_i'$ is a K-dimensional vector, $\boldsymbol{\beta}_\tau$ is a coefficient vector of QReg. To estimate the coefficient vector (Koenker, and Bassett,1978) proposed this equation".

$$\sum_{i=1}^{n} \rho_\tau(y_i - x_i'\beta_\tau), \quad (1)$$

where $\boldsymbol{\rho_\tau(u) = \tau(1 - I(u < 0))}, \boldsymbol{I(u < 0)}$ is the indicator function. This problem can be minimised by using a linear programming algorithm (Koenker, and D'Orey, 1987).

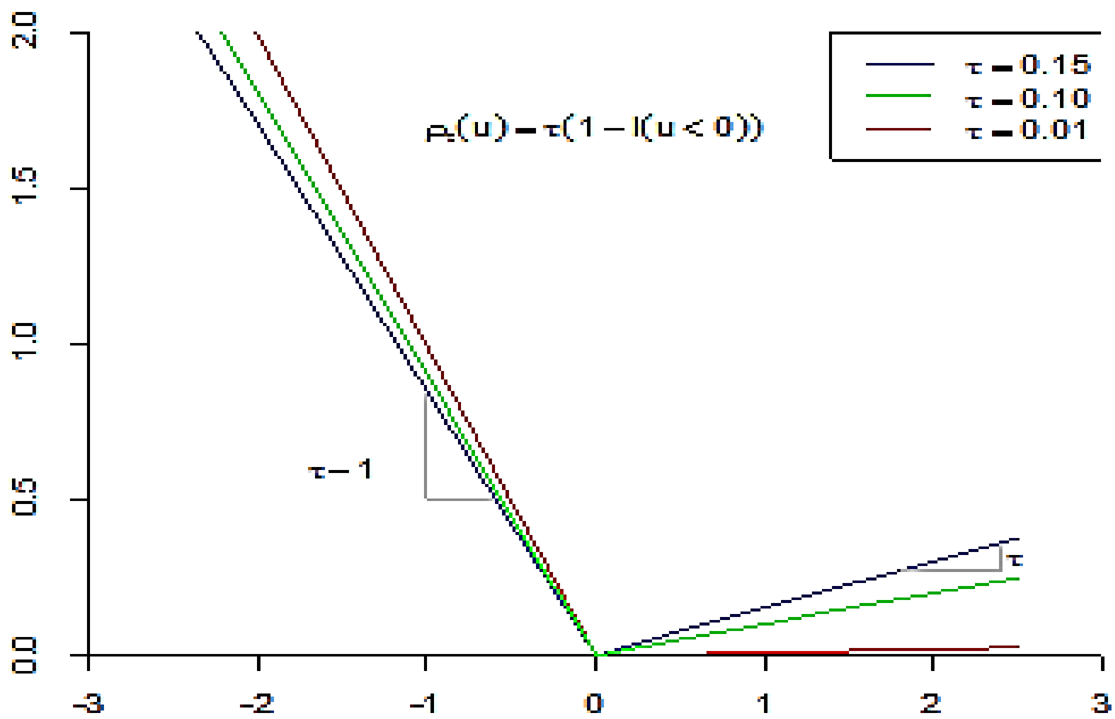

Figure 1 shows the check function at three quantiles 0.01, 0.10, and 0.15 . Since the above check function is not differential at 0 there is no closed-form solution.

Many researchers employed Bayesian approaches to estimate regression coefficients.

QReg parameters are estimated using the Bayesian technique if errors are an asymmetric Laplace distribution (ALD) (Yu and Moyeed, 2001). The Bayesian approach to QReg is accurate even with small sample sizes and acceptable for ordinal replies (Koenker, 2005). Alhamzawi (2013) provides mathematical hints for using ALD for errors.

Active variable selection is difficult in QR models. Many times, selecting the right group of predictors improves prediction accuracy. Scientifically, a smaller selection improves interpretation. Reed et al. (2009) and Ji et al. (2011) present several methods for obtaining the active subset.

Koenker (2004) introduced QReg's first regularization method to eliminate random effects. Wang et al. (2007) verified QReg's low absolute deviation (LAD) oracle characteristic. A posterior technique that creates a Laplace-independent regression parameter hypothesis is a Bayesian lasso. Ordinal response variables complicate things. Many fields use ordinal response models, especially in education where data outputs can be sorted. This study simulates the pth quantile for the latent variable $\boldsymbol{z_i}$ using Rahman's (2016) regression model.

$$\boldsymbol{z_i = x_i'\beta + \epsilon_i}, \qquad \boldsymbol{i = 1, \ldots, n}, \qquad (2)$$

"where $\boldsymbol{x_i}$ is a $\boldsymbol{k \times 1}$ vector of explanatory variables of $\boldsymbol{z_i}$,$\boldsymbol{\beta}$ is a $\boldsymbol{k \times 1}$ vector for the model parameter,$\boldsymbol{\epsilon_i}$ is error follows ALD. Where a description of the ordinal response variable by the latent variable $\boldsymbol{z_i}$ can be written fellow:

$$\boldsymbol{y_i = c \quad if \quad \delta_{c-1} < z_i \le \delta_c \,; \quad c = 1, \ldots, C},$$

where $\boldsymbol{\delta_0}, \ldots, \boldsymbol{\delta_C}$ are cut-points, that fall within the period"

$$-\infty = \boldsymbol{\delta_0 < \delta_1 < \cdots < \delta_{C-1} < \delta_C} = +\infty$$

This study's Bayesian hierarchical approach deals with the selection of variables and estimation of the parameters. Specifically, we propose a regularization bridge method and ordinal composite quantile regression.

Rahman (2016) suggested an ordinal Bayesian model for QReg, assuming ALD error and Gibbs sampling to obtain parameter posteriors.

Recently, Zou and Yuan (2008) came up with a "composite quantile regression (CQReg)" to find parameters that work better than the average regression by more than 70%. The CQReg is more robust, flexible, and efficient than the single QReg because it takes multiple quantities at once. Models using ordinal response variables benefit from CQReg. Ordinal QReg complements the standard ordinal model and has been used for years, see Hong and Zhou (2013), Goffe

et al. (1994), and Hong and He (2010). The nice theoretical characteristics of CQR apply to ordinal outcome models. Medical, ecological, geological, and human and social research use ordinal survey results. Quantiles of ordinal data cannot be obtained by inverting the distribution function, making CQR with ordinal outcomes more challenging.
Koenker (2004) introduced penalty-based parameter estimation using special effects, while Geraci and Bottai (2007) used probability dependency. Yu and Moyeed used Monte Carlo to implement ALD. This study's Bayesian hierarchical approach deals with the selection of variables and estimation of the parameters. Specifically, we propose a regularization bridge method and ordinal composite quantile regression.
In the classical literature, quantile regression estimators have been used with ordinal data, depending on different methods (Kirkpatrick et al., 1983; Goffe et al., 1994). Despite the development of these methods over the years, their use with the Bayesian method has not been addressed, Hong and He (2010).
Rahman (2016)showed a quantile ordinal model that provides a better fit than the classical methods using the Bayesian method (Alhamzawi, R , Bayesian model selection in ordinal quantile regression). we propose a new Bayesian composite quantile regression method with a Bayesian Reciprocal Adaptive Bridge. the attractiveness of this model is shown by suggesting later methods with superior results compared to the existing methods using simulation.
Section 2 introduces composite quantile regression with bridge penalties, preset model parameters, and the MCMC algorithm. Section 3 describes previous assumptions, a Gibbs sampler for model selection, and an ALD-based suitability posterel. Section 4: Simulating the selection and assessment model's proposed procedures. Section 5 uses ordinal data to demonstrate the methods.

2. Methods

2.1 Bayesian Ordinal Composite Quantile Regression Model (BOCQReg)

Consider the following model

$$\mathbf{y_i} = \mathbf{b_\tau} + \boldsymbol{x_i'\beta} + \boldsymbol{\epsilon_i}, \qquad \mathbf{i}=1,\ldots\ldots,\mathbf{n}, \qquad (3)$$

where $\boldsymbol{y_i}$ is response variable, $\boldsymbol{b_\tau}$the parameter for the quantile intersection where (0< τ <1), $\boldsymbol{x_i'}$ is the vector of explanatory variables، $\boldsymbol{\beta}$ is a vector for model parameters, $\boldsymbol{\epsilon_i}$ is the error of the quantile regression model and $\boldsymbol{n}$ is several observations. ، the parameters of the composite regression can be estimated by solving the following equation:

$$\left(\widehat{\boldsymbol{b}}_{\tau_1}, \widehat{\boldsymbol{b}}_{\tau_2}, \ldots\ldots, \widehat{\boldsymbol{b}}_{\tau_K}, \widehat{\boldsymbol{\beta}}\right) = \underbrace{\mathbf{arg}\,\boldsymbol{min}}_{\boldsymbol{b_\tau},\boldsymbol{\beta}} \sum_{k=1}^{K}\left\{\sum_{i=1}^{n} \boldsymbol{\rho_{\tau_k}}\left(\boldsymbol{y_i} - \boldsymbol{b_{\tau_k}} - \boldsymbol{x_i'\beta}\right)\right\} \qquad (4)$$

where $\boldsymbol{\rho_{\tau_k}(t) = t\left(\tau_k - I(t<0)\right)}$, is the check function, *I* (.) is indicator function and $\boldsymbol{\tau_k = \frac{k}{K+1}}$, $\boldsymbol{where\ k = 1, 2, \ldots, K}$.

"By assuming the error is asymmetric Laplace distribution ($\boldsymbol{\mu = b_\tau + x_i'\beta, \sigma = 1}$) . The probability function of ALD is given by:"

$$\boldsymbol{p(y|x_i, b_\tau, \beta, \tau) = \tau(1-\tau)exp\left(-\rho_{\tau_k}\left(y_i - b_{\tau_k} - x_i'\beta\right)\right)}, \qquad (5)$$

The check function is not derivable, hence conventional approaches estimate quantile regression utilizing computational and simulation methods with algorithms "(Madsen and Nielsen, 1993; Chen, 2007; Rahman, 2013)." Bayesian approaches minimized the loss function (4) and maximized the probability function likelihood. (5). Kozumi and Kobayashi (2011) used a mixture of the standard exponential distribution with the standard normal of the error term, suppose that $\boldsymbol{u \sim N\,(0,1)}$ and $\boldsymbol{v \sim exp\left(\frac{1}{\tau(1-\tau)}\right)}$. "Therefore, the error term in (2) can be written as $\boldsymbol{\epsilon = \vartheta v + \sqrt{\varphi v}\,u}$ where $\boldsymbol{\vartheta = (1-2\tau)}$ and $\boldsymbol{\varphi = 2}$."

The normal-exponential mixture is useful because it gives us access to the normal distribution's properties, which we will use in this study. After that, this is the conditional distribution of the quantile variable:

$$\boldsymbol{p(y_i|x, b, \beta, v_i) = exp\left(-\sum_{k=1}^{K}\sum_{i=1}^{n}\frac{1}{4v_i}\left(y_i - b_{\tau_k} - x_i'\beta - \vartheta v_i\right)^2\right)\prod_{i=1}^{n}(4\pi v_i)^{\frac{1}{2}}} \qquad (6)$$

where $\boldsymbol{u = (u_1, \ldots, u_n)'}$

Quantile regression has been utilized to treat ordinal response variable models by several researchers (Hong, H. G. and Zhou, J. (2013), Zhou, L. (2010)). Composite quantile regression is more efficient and immune to atypical error distributions than individual quantile regression "(Zou and M. Yuan, 2008)." The response variable $\boldsymbol{y_i}$ can be modelled through the continuous latent variable $\boldsymbol{z_i}$ and cut-off point $\boldsymbol{\delta = \{\delta_0, \ldots, \delta_C\}}$ where we impose $\boldsymbol{y}$ to take C-ordered values $\boldsymbol{\{c_1, c_2, \ldots, c_C\}}$ to be in the following form:

$$\boldsymbol{y_i} = \begin{cases} \boldsymbol{1 \quad if \quad \delta_0 \le z_i < \delta_1} \\ \boldsymbol{c \quad if\ \delta_{c-1} \le z_i < \delta_c;\ c = 2, \ldots, C-1} \\ \boldsymbol{C \quad if \quad \delta_{C-1} \le z_i < \delta_C} \end{cases} \qquad (7)$$

A continuous latent random variable $\boldsymbol{z_i}$can be used to show a composite quantile regression for ordinal data as $\boldsymbol{z_i = b_\tau + x_i'\beta + \epsilon_i}$ $\boldsymbol{i}$=1,……,$\boldsymbol{n}$ where $\boldsymbol{x_i}$ is a $\boldsymbol{k \times 1}$ vector of explanatory variables,$\boldsymbol{\beta}$ is a $\boldsymbol{k \times 1}$ vector for model parameters, $\boldsymbol{\epsilon_i}$ follows an ALD with pdf (4) and $\boldsymbol{n}$ is several observations. Equation (6) can be rewritten as a hierarchical Bayesian model using ordinal composite quantile regression

$$p(z_i|x,b,\beta,v_i) = exp\left(-\sum_{k=1}^{K}\sum_{i=1}^{n}\frac{1}{4v_i}\left(z_i - b_{\tau_k} - x_i'\beta - \vartheta v_i\right)^2\right)\prod_{i=1}^{n}(4\pi v_i)^{\frac{1}{2}} \qquad (8)$$

2.2 Bayesian Reciprocal Bridge for Ordinal Composite Quantile Regression (BRBOCQReg)

The reciprocal bridge estimator can be written by making use of formula (23) with quantile regression (Alhamzawi and Mallick, 2020) which solves the following:

$$\underbrace{argmin}_{b_\tau,\beta}\sum_{i=1}^{n}\left\{\sum_{k=1}^{K}\rho_{\tau_k}\left(z_i - b_{\tau_k} - x_i'\beta\right)\right\} + \lambda\sum_{g=1}^{G}\frac{1}{|\beta_g|^{\alpha}}I\{\beta_g \neq 0\} \qquad (9)$$

λ represents the regularization parameter for α. A value of zero corresponds to L_0, one to reciprocal LASSO, and 2 to reciprocal ridge. The Bayesian approach addresses miniaturization in small samples by using the check function instead of the loss function. According to Mallick et al. (2020), the inclusion of the penalty factor in equation (8) results in bridge estimates that serve as posterior mode estimates when the regression parameters follow an Inverse Generalized Gaussian (IGG) pattern.

$$\pi(\beta) = \prod_{g=1}^{G}\frac{\lambda^{\frac{1}{\alpha}}}{2\beta_g^2\Gamma\left(\frac{1}{\alpha}+1\right)}exp\left\{-\frac{\lambda}{|\beta_g|^{\alpha}}\right\}I\{\beta_g \neq 0\}, \qquad (10)$$

Armagan, Dunson, and Lee's (2013); Mallick, Alhamzawi, and Svetnik's (2020) representation of the scale mixture of normal (SMN) is used by the Gibbs sampler for the Bayesian reciprocal bridge. If we assume that $\boldsymbol{\beta \sim N(0,l)I(|\beta| >}$ ƞ$)$, $\boldsymbol{l \sim Exp\left({\xi^2}/{2}\right)}$, and" $\boldsymbol{\xi \sim Exp}$(ƞ), then the inverse double exponential distribution for $\boldsymbol{\beta}$ with scale parameter $\boldsymbol{\lambda > 0}$ arises when ƞ follows Inverse Gamma (2,λ)."

Where $\boldsymbol{u = \frac{1}{\eta}}$, $\boldsymbol{l = (l_1, \dots, l_G)'}$, and $\boldsymbol{\xi = (\xi_1, \dots, \xi_G)'}$. To specify a prior distribution for $\boldsymbol{\delta}$, we follow Alhamzawi (2016), we assign order statistics from uniform $(\boldsymbol{\delta_0, \delta_C})$ distribution for the $\boldsymbol{C-1}$ unknown cut-points :

$$P_\delta = (C-1)!\left(\frac{1}{\delta_{max}-\delta_{min}}\right)^{C-1}I(\delta \in H), \qquad (11)$$

Where $\boldsymbol{\delta = (\delta_0, \delta_1, \dots, \delta_C)}$ and $\boldsymbol{H = \{(\delta_{min}, \delta_1, \dots, \delta_{max})|\delta_{min} < \delta_1 < \dots < \delta_{C-1} < \delta_{max}\}}$.

To summarize, in our Bayesian hierarchical formulation, we consider the following priors for all parameters and latent variables

$$y_i = \begin{cases} 1 \quad if \quad \delta_0 \leq z_i < \delta_1 \\ c \quad if\ \delta_{c-1} \leq z_i < \delta_c;\ \ c = 2, \dots, C-1 \\ C \quad if \quad \delta_{C-1} \leq z_i < \delta_C \end{cases}$$

$$z_i|x \sim N_n\left(z_i + b_{\tau_k} + x_i'\beta + \vartheta v, 2\sigma v\right),$$

$$P(\delta) = (C-1)!\left(\frac{1}{\delta_{max}-\delta_{min}}\right)^{C-1}I(\delta \in H)\text{when } H = \{(\delta_0, \dots, \delta_C)|\delta_0 < \dots < \delta_C\}.$$

$$\beta|l \sim \prod_{g=1}^{G}N(0,t^2)I\left\{|\beta_g|^{\alpha} > \frac{1}{u_g}\right\},$$

$$l|\xi \sim \prod_{g=1}^{G}Exp\left(\xi_g^2\right),$$

$$\xi|u \sim \prod_{g=1}^{G}Exp\left(\frac{1}{u_g}\right),$$

$$u \sim \prod_{g=1}^{G}Gamma\ (2,\lambda), \qquad (12)$$

$$\sigma \sim \sigma^{-1},$$

$$\lambda \sim \lambda^{-1},$$

Then the condition posteriors are:

$$\beta|z_i \sim N_P\left((X'\Omega^{-1}X + T^{-1})^{-1}X'\Omega^{-1}(z-\vartheta v), (X'\Omega^{-1}X + T^{-1})^{-1}\right)\prod_{g=1}^{G}I\left\{|\beta_g|^{\alpha} > \frac{1}{u_g}\right\},$$

$$v_i^{-1}|z_i \sim Inverse - Gaussian\left(\frac{1}{2}, \frac{1}{|z_i + b_{\tau_k} + x_i'\beta|}, \frac{1}{2\sigma}\right),$$

$$l^{-1}|z_i \sim \prod_{g=1}^{G}Inverse - Gaussian\left(\frac{1}{2}, \sqrt{\frac{\xi_g^2}{\beta_g^2}}, \xi_g^2\right),$$

$$\xi|z_i \sim \prod_{g=1}^{G}Gamma\left(|\beta_g|^{\alpha} + \frac{1}{u_g}\right),$$

$\boldsymbol{u|z_i \sim \prod_{g=1}^{G} Exponential(\lambda) I\left\{u_g > \frac{1}{|\beta_g|^{\alpha}}\right\}}$,

$\boldsymbol{\sigma|z_i \sim Inverse - Gamma\left(a + \frac{3n}{2}, b + \frac{1}{4}(z_i - b_{\tau_k} - x_i'\beta - \vartheta v)' V^{-1}(z_i - b_{\tau_k} - x_i'\beta - \vartheta v)\right)}$,

$\boldsymbol{\lambda|z_i \sim Gamma\left(\gamma + 2p, d + \sum_{g=1}^{G} \frac{1}{|\beta_g|^{\alpha}}\right)}$,

Where $\boldsymbol{L = diag(l_1, \ldots, l_G)}$, $\Omega = \boldsymbol{diag}((\boldsymbol{2\sigma v_1}), \ldots, (\boldsymbol{2\sigma v_n}))$,$\boldsymbol{\gamma, p, and\ d}$ are fixed hyperparameters.

"Algorithm 1. MCMC sampling for the Bayesian reciprocal Bridge composite quantile regression (SMN)"

Input: (z, x)

Initialize: $(\boldsymbol{b_\tau, \beta, \sigma, v, u, \lambda, \alpha})$

For $\boldsymbol{t = 1, \ldots, (t_{max} + t_{burn-in})}$ **do**

1. sample $\boldsymbol{v}|.\sim \prod_{i=1}^{n}$ $\boldsymbol{Inverse\ Gaussian}\left(\frac{1}{2\sigma}, \frac{1}{|z_i - b_{\tau_k} - x_i'\beta|}, \frac{1}{2\sigma}\right)$

2. sample $\boldsymbol{u}|. \sim \prod_{g=1}^{G} \boldsymbol{Exponential\ (\lambda) I}\left\{\boldsymbol{u_g} > \frac{1}{|\beta_g|^{\alpha}}\right\}$

3. sample $\boldsymbol{l}|.\sim \prod_{g=1}^{G} \boldsymbol{Inverse - Gaussin}\left(\frac{1}{2}, \sqrt{\frac{\xi_k^2}{\beta_k^2}}, \xi_k^2\right)$

4. sample$\boldsymbol{\xi}\Big|.\sim \prod_{g=1}^{G} \boldsymbol{Gamma}\left(\boldsymbol{2}, \left(|\boldsymbol{\beta_g}|^{\alpha} + \frac{1}{u_g}\right)\right)$

5. "sample $\boldsymbol{\beta}|.$ From a truncated multivariate normal proportional to

$\boldsymbol{N_P}((\boldsymbol{X'}\Omega^{-1}\boldsymbol{X} + \boldsymbol{T^{-1}})^{-1}\boldsymbol{X'}\Omega^{-1}(\boldsymbol{z} - \boldsymbol{\vartheta v}), (\boldsymbol{X'}\Omega^{-1}\boldsymbol{X} + \boldsymbol{T^{-1}})^{-1}) \prod_{k=1}^{p} \boldsymbol{I}\left\{|\boldsymbol{\beta_k}| > \frac{1}{u_k}\right\}$,"

$\widehat{\boldsymbol{\beta}} = \left(\sum_{i=1}^{n}\sum_{k=1}^{K} \frac{x'x}{2\sigma v_i}\right)$ and $\widehat{\boldsymbol{\beta}} = \widehat{\boldsymbol{B}}\left(\sum_{i=1}^{n}\sum_{k=1}^{K} \frac{\left(x_i\left(z_i - b_{\tau_k} - x_i'\beta - \vartheta v_i\right)\right)}{2\sigma v_i}\right)$

6. sample$\boldsymbol{b_\tau}|.\sim \boldsymbol{N}\left(\frac{\sum_{i=1}^{n}\sum_{k=1}^{K}\left(z_i - b_{\tau_k} - x_i'\beta - \vartheta v_i\right)}{\sum_{i=1}^{n} 1/2\sigma v_i}, \frac{1}{\sum_{i=1}^{n} 1/2\sigma v_i}\right)$

7. sample $\boldsymbol{\sigma}|. \sim \boldsymbol{Inverse\ Gamma}\left(\boldsymbol{a}, \frac{3n}{2}, \boldsymbol{b} + \frac{1}{4}\sum_{i=1}^{n}\sum_{K=1}^{k} \left(\boldsymbol{z_i} - \boldsymbol{b_{\tau_k}} + \boldsymbol{x_i'\beta} - \boldsymbol{\vartheta v_i}\right)' \boldsymbol{V^{-1}} \sum_{i=1}^{n}\sum_{k=1}^{K} \left(\boldsymbol{z_i} - \boldsymbol{b_{\tau_k}} + \boldsymbol{x_i'\beta} - \boldsymbol{\vartheta v_i}\right)\right)$

8. sample $\boldsymbol{\lambda}|. \sim \boldsymbol{Gamma}\left(\boldsymbol{\gamma} + \boldsymbol{2p}, \boldsymbol{d} + \sum_{g=1}^{G} \frac{1}{|\beta_g|^{\alpha}}\right)$

9. sample $\boldsymbol{\delta_c}$, with $\boldsymbol{c}$ from 1 to $\boldsymbol{C - 1}$, from a uniform distribution over the interval

$\begin{pmatrix} \boldsymbol{min}\,\{\boldsymbol{min}(z_i|y_i = c + 1), \boldsymbol{\delta_{c+1}}, \boldsymbol{\delta_C}\}, \\ \boldsymbol{max}\{\boldsymbol{max}(z_i|y_i = c)\}, \boldsymbol{\delta_{c-1}}, \boldsymbol{\delta_0} \end{pmatrix}$.

10. Sample $\boldsymbol{z_i}$, $\boldsymbol{for\ i\ from\ 1\ to\ n}$, from truncated normal (TN) distribution

$\boldsymbol{TN_{(\delta_{c-1},\delta_c)}}(\boldsymbol{z_i} + \boldsymbol{b_{\tau_k}} + \boldsymbol{x_i'\beta} + \boldsymbol{\vartheta v}, \boldsymbol{2\sigma v})$.

end for

2.3 "Bayesian Reciprocal Adaptive Bridge for Ordinal Composite Quantile Regression (BRABOCQReg)"

To demonstrate ordinal composite quantile regression using the adaptive bridge penalty function with a reciprocal parameter, we solve the following equation:

$$\underbrace{\boldsymbol{argmin}}_{b_\tau,\beta} \sum_{i=1}^{n}\left\{\sum_{k=1}^{K} \boldsymbol{\rho_{\tau_j}}\left(\boldsymbol{z_i} - \boldsymbol{b_{\tau_k}} - \boldsymbol{\acute{x}_i\beta}\right)\right\} + \sum_{g=1}^{G} \frac{\lambda_g}{|\beta_g|^{\alpha}} \boldsymbol{I}\{\boldsymbol{\beta_g} \neq \boldsymbol{0}\}, \quad (13)$$

Where $\boldsymbol{\lambda_g} \geq \boldsymbol{0}$, g=1,…,G . By utilizing the scale combination described in (11), it is able to create the Gibbs sampler for the Bayesian reciprocal adaptive Bridge,

$$\frac{\lambda_g^{\frac{1}{\alpha}}}{2\beta_g^2\Gamma\left(\frac{1}{\alpha}+1\right)} \boldsymbol{e}^{-\lambda|\beta_g|^{-\alpha}} = \frac{\lambda_g^{\frac{1}{\alpha}}}{2\beta_g^2\Gamma\left(\frac{1}{\alpha}+1\right)} \int_{u_g > |\beta|^{-\alpha}} \boldsymbol{\lambda_g e^{-\lambda_g u_g}} \quad (14)$$

Under (10), the hierarchical model for the reciprocal adaptive Bridge is the same as (12) with λ replaced with $\boldsymbol{\lambda_g}$'s as follows :

$\boldsymbol{u_g|\lambda_g \sim Gamma(2, \lambda_g)}$,

$\boldsymbol{\lambda_g \sim \lambda_g^{-1}}$,

"Algorithm 2. MCMC sampling for the Bayesian reciprocal adaptive Bridge composite quantile regression (SMN)**"**

Input: (z,x)
Initialize: $(\boldsymbol{b_\tau}, \boldsymbol{\beta}, \boldsymbol{\sigma}, \boldsymbol{v}, \boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{\alpha})$
For $\boldsymbol{t = 1, \dots, (t_{max} + t_{burn-in})}$ ***do***

1. Sample $v|.\sim \prod_{i=1}^{n}$ $\textbf{Inverse Gaussian}\left(\frac{1}{2\sigma}, \frac{1}{\left|z_i - b_{\tau_k} - x_i'\beta\right|}, \frac{1}{2\sigma}\right)$
2. Sample $u|. \sim \prod_{g=1}^{G} \textbf{Exponential}\left(\lambda_g\right) I\left\{u_g > \frac{1}{\left|\beta_g\right|^{\alpha}}\right\}$
3. Sample $l|.\sim \prod_{g=1}^{G} \textbf{Inverse} - \textbf{Gaussin}\left(\frac{1}{2}, \sqrt{\frac{\xi_k^2}{\beta_k^2}}, \xi_k^2\right)$
4. Sample$\xi\Bigg|. \sim \prod_{g=1}^{G} \textbf{Gamma}\left(2, \left(\left|\beta_g\right|^{\alpha} + \frac{1}{u_g}\right)\right)$
5. Sample $\boldsymbol{\beta}|.$ From a truncated multivariate normal proportional to
   $N_P((X'\Omega^{-1}X + T^{-1})^{-1}X'\Omega^{-1}(z - \vartheta v), (X'\Omega^{-1}X + T^{-1})^{-1}) \prod_{k=1}^{p} I\left\{|\beta_k| > \frac{1}{u_k}\right\}$,
   $\widehat{\beta} = \left(\sum_{i=1}^{n}\sum_{k=1}^{K}\frac{x'x}{2\sigma v_i}\right)$ and $\widehat{\beta} = \widehat{B}\left(\sum_{i=1}^{n}\sum_{k=1}^{K}\frac{\left(x_i\left(z_i - b_{\tau_k} - x_i'\beta - \vartheta v_i\right)\right)}{2\sigma v_i}\right)$
6. Sample$b_\tau|.\sim N\left(\frac{\sum_{i=1}^{n}\sum_{k=1}^{K}\left(z_i - b_{\tau_k} - x_i'\beta - \vartheta v_i\right)}{\sum_{i=1}^{n} 1/2\sigma v_i}, \frac{1}{\sum_{i=1}^{n} 1/2\sigma v_i}\right)$
7. Sample $\sigma|. \sim \textbf{Inverse Gamma}\left(a, \frac{3n}{2}, b + \frac{1}{4}\sum_{i=1}^{n}\sum_{K=1}^{k}\left(z_i - b_{\tau_k} + x_i'\beta - \vartheta v_i\right)' V^{-1}\sum_{i=1}^{n}\sum_{K=1}^{K}\left(z_i - b_{\tau_k} + x_i'\beta - \vartheta v_i\right)\right)$
8. Sample $\lambda|. \sim \textbf{Gamma}\left(\gamma + p, d + \frac{1}{\left|\beta_g\right|^{\alpha}}\right)$
9. Sample δ_c, with c from 1 to $C - 1$, from a uniform distribution over the interval $\begin{pmatrix} min\{min(z_i|y_i = c+1), \delta_{c+1}, \delta_C\}, \\ max\{max(z_i|y_i = c)\}, \delta_{c-1}, \delta_0 \end{pmatrix}$.
10. Sample z_i , $\boldsymbol{for\ i\ from\ 1\ to\ n}$, from truncated normal (TN) distribution
   $TN_{(\delta_{c-1}, \delta_c)}\left(z_i + b_{\tau_k} + x_i'\beta + \vartheta v, 2\sigma v\right)$.

end for

3 Simulation Studies

Here, we conduct simulation simulations to evaluate our method, "Bayesian reciprocal adaptive bridge composite quantile regression for ordinal data," or "BrABCQRO," in contrast to other Bayesian and non-Bayesian methods. The following methods are compared here:

- Bayesian QReg for ordinal models
- Bayesian model selection Ordinal QR .
- Akaike Information Criterion AIC
- Bayesian Information Criterion BIC

2 Simulation Studies

Our reciprocal adaptive Bridge ordinal composite quantile regression (rABOCQR) technique was tested in three simulation simulations. Compare the proposed method to Bayeian ordinal quantile regression (Rahman, 2016) and Bayeian model selection in ordinal quantile regression (Alhamzawi, 2016).

2.1 Simulation 1

Consider data generated from the ordinal regression model,

$$z_i = x_i'\beta + \varepsilon_i \quad i = 1, \dots, 100, \qquad (15)$$

" where $x_i = (1, x_{1i})'$and $\beta = (0, 4)'$, including the intercept. The variable x1i is produced using the conventional normal distribution. We included eleven noise variables in the model. N10(0, Σx) was used as a model independently for these variables, using (Σx)gh = 0.75|g−h|, where gh is the (g, h)th element of Σx. εi ≈ N(0, 1) in this simulation investigation. Based on the cut-point vector δ = (0.5, 2, 3.5)J, the outcome of interest y was produced, resulting in four categories. There are 150 data produced, with n = 100 observations in each data set. For our suggested approach, we use K = 3. We use the median to test alternative approaches. Additionally, rABOCQR's performance is contrasted with the AIC (Akaike, 1998) and BIC (Schwarz et al.)"

Table 1: Comparing average numbers of correct and wrong zeros for different methods in Simulation example 1, averaged over 150 replications. The standard deviations are listed in parentheses.

			Methods		
	rABOCQR	BOQR	BMOQR	AIC	BIC
correct	9.22 (0.09)	6.42 (0.14)	6.19 (0.39)	6.81 (0.42)	6.99 (0.08)
wrong	0.04 (0.11)	0.48 (0.45)	0.37 (0.42)	0.18 (0.38)	0.12 (0.13)

1978). Here, AIC and BIC are respectively given by
AIC = 2k − 2 ln(L),
and
BIC = k ln(n) − 2 ln(L),

where L is the subset-specific maximum of the likelihood function. The models with the smallest AIC or BIC are favoured when multiple options are provided. Based on a sample of 150 synthetic data sets, Table 1 compares the proportion of correct to incorrect zero regression parameters for the best model. The average number of right and wrong zeros shows that the proposed strategy performs quite well.

2.2 Simulation 2

This Simulation example is similar to Simulation 1 except that we set

$$\boldsymbol{z_i = x_i'\beta + \varepsilon_i \quad i = 1, \ldots, 100}, \qquad (16)$$

where $\boldsymbol{x_i} = (\boldsymbol{1}, \boldsymbol{x_{1i}}, \boldsymbol{x_{2i}}, \boldsymbol{x_{3i}})'$ and$\boldsymbol{\beta} = (\boldsymbol{1}, \boldsymbol{4}, \boldsymbol{2}, -\boldsymbol{2})'$, including the intercept.The normal distribution standard is used to create the variables x1i, x2i, and x3i. We included eleven noise variables in the model. The independent simulation of these variables was done using N10(0, Σx) and (Σx)gh = 0.75|g−h|, where gh is the (g, h)th element of Σx. Based on 150 created datasets, the number of true and false zero regression coefficients is compared in Table 2. Once more, the outcomes demonstrate how well the suggested strategy performs in terms of the average numbers of accurate and incorrect zeros.

Table 2: In Simulation Example 2, compare the average numbers of genuine zeros and false zeros for various approaches, averaged across 150 replications. The parenthesis include a list of the standard deviations.

			Methods		
	rABOCQR	BOQR	BMOQR	AIC	BIC
correct	8.93 (0.14)	6.19 (0.23)	6.01 (0.42)	6.33 (0.51)	6.02 (0.15)
wrong	0.09 (0.16)	0.35 (0.42)	0.29 (0.33)	0.17 (0.41)	0.29 (0.53)

4 A real data example

"Work of the rABOCQR method is shown here. The national research (NLSY79) gave it BOQR and BMOQR on academic accomplishment, as modelled by Alhamzawi (2016) and Rahman (2016). Over 12,000 youth were interviewed annually by the NLSY on demographic topics starting in 1979. This dataset was subsampled by Alhamzawi (2016). This subsample has 11 independent factors and one dependent variable, education. The square root of family income (x1), mother's education (x2), father's education (x3), mother's working position (x4), gender (x5), race (x6), and whether the youngster resided in an urban area (x7) or the South at 14 (x8) are regressors To account for age cohort effects, three dummy variables are used to reflect an individual's 1979 age (age cohort 2 (x9), 3 (x10), and 4 (x11)). Interest outcomes include four categories: (1) less than high school, (2) high school, (3) some college, and (4) graduate degree (Jeliazkov et al., 2008). The outcome variable categories have 897, 1392, 876, and 758 observations, respectively. As in simulation research, we choose K = 3 and compare it with different media techniques. The results are in Table 3. The DIC-based model selection study found rABOCQR, BOQR, and BMOQR to be 9337.19, 9781.02, and 9568.31. These data show that the recommended strategy works well. Thus, simulations and real data analysis support the proposed approach."

Table 3: estimates for the model parameters used in the application for educational achievement.

Covariate	rABOCQR	BOQR	BMOQR
	DIC=9337.19	DIC=9781.02	DIC=9568.31
Intercept	-3.27	-3.12	-2.01
x_1	0.31	0.30	0.35
x_2	0.05	0.15	0.27
x_3	0.09	0.13	0.00
x_4	0.00	0.10	0.00
x_5	0.52	0.33	0.51
x_6	0.48	0.41	0.22
x_7	0.00	-0.10	-0.26
x_8	0.00	0.11	0.00
x_9	0.00	-0.04	0.00
x_{10}	-0.10	-0.05	0.00
x_{11}	0.61	0.38	0.33

To address the need for concurrent estimation and variable selection in ordinal models, we present the Bayesian reciprocal adaptive bridge composite quantile regression. This approach yields a sparse solution and takes advantage of the computational benefits of the reciprocal bridge. To draw samples from the whole conditional posterior distributions, a novel Gibbs sampling procedure is developed. Extensive illustrative examples from both simulation and real data show that the proposed methods routinely outperform the state-of-the-art alternatives.

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