S-Generalized supplemented modules

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Abstract

Xue introduced the following concept: Let M be an R- module. M is called a generalized supplemented module if for every submodule N of M, there exists a submodule K of M such that M = N +K and $N \cap K \subseteq Rad(K)$.

N. Hamada and B. AL- Hashimi introduced the following concept:
Let S be a property on modules. S is called a quasi – radical property if the following conditions are satisfied:

1. For every epimorphism f: $M \to N$, where M and N are any two R- modules. If the module M has the property S, then the module N has the property S.
2. Every module M contained the submodule S(M).

These observations lead us to introduce S- generalized supplemented modules. Let S be a quasi- radical property. We say that an R-module M is S- generalized supplemented module if for every submodule N of M, there exists a submodule K of M such that M = N + K and $N \cap K \subseteq S(K)$.

The main purpose of this work is to develop the properties of S-generalized supplemented modules. Many interesting and useful results are obtained about this concept. We illustrate the concepts, by examples.

Keywords: quasi-radical property, generalized supplemented module, small submodule.

Introduction:

In this note all rings are commutative with identity and all modules are unitary left R-modules, unless otherwise specified.

An R-module M is called a GS-module if for any submodule N of M, there exists a submodule K of M such that M =N + K and $N \cap K \subseteq Rad(K)$. See [1], [2].

On the other hand, let S be a property on modules an R -module M is called a module of type S (briefly S-module) if M has the property S. A submodule N of M is called S- submodule if N has the property S as an R- module. If there exists a submodule of M has the property S and contained all submodules of M that having the property S, then this submodule is called the radical of M and denoted by S(M).

A property S defined on modules is called a quasi-radical property if the following conditions are satisfied:

1- Epimorphic image of an R-module of type S is an R- module of type S.
2- Every module M contained the submodule S(M).

These observations lead us to introduce the following concept :- Let S be a quasi-radical property and N be a submodule of an R- module M. A submodule K of M is called an S-generalized supplement of N in M, if M= N +K and $N \cap K \subseteq S(K)$.

M is called an S- generalized supplemented module (briefly S- GS

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module), if every submodule of M has S- generalized supplement in M.

In this paper we investigate the properties of S- GS modules. In §1, we recall that the definition of quasi-radical property and list some of their important properties that are relevant to our work.

In §2 of this paper we give the definition of S-GS modules with some examples and basic properties.

In §3, we study the sum of two S-GS module. Also we give a characterization of S- GS rings we prove that a ring R is S- GS ring if and only if every finitely generated R- module is S- GS module. See(3.6)

In §4, we study Soc(Z) -GS modules with some examples and basic properties. Also we give a characterization of Soc(Z)- GS module, we prove that an R- module M is Soc(Z)- GS module if and only if for every submodule N of M, there exists a submodule K of M such that M = N+K and $N\cap K \subseteq$ Soc(K), (respectively, $N\cap K \subseteq$ Z(K)), See (prop.(4.6)).

Also we prove if M is a non – zero projective R- module, where R is an integral domain and not a field, then M is not Soc (respectively Z) -GS module (prop.(4.11)).

1. Quasi- radical Properties

Let S be a property and let M be an R- module. Recall that M is called a module of type S(briefly S- module) if M has the property S.A submodule N of M is called S- module if N has the property S as an R- module(i.e. N is S-module).

If there exists a submodule of M has the property S and contained all submodules of M that having the property S, then this submodule called the radical of M and denoted by S(M). M is called semisimple module of type S if S(M)= 0, See[3].

Let S be a property defined on modules. Recall that S is called a quasi-radical property if the following conditions are satisfied:

1- Epimorphic image of an R-module has the property S is also has the property S.
2- Every R- module M contained the submodule S(M)(radical M) See [3].

Recall that quasi- radical property S is called a hereditary property if every submodule of an R- module has the property S is also has the property S.

(equivalentily) $S(N)= N\cap S(M)$,for every submodule N of a module M.see[3]

Example (1.1): [3]

Define the property Soc as follows:

An R-module M has the property Soc{ Soc- module} if M is semisimple.

One can easily show that the socle property is a quasi - radical property:

Note: The Socle property is a hereditary property, where Soc(N) $=N\cap$ Soc(M), for every submodule N of M [4, p.227].

Example (1.2): [3]

Let M be an R- module. Recall that the singular submodule of M (denoted by Z(M)) is defined by $Z(M)= \{ m\in M ; Ann(m) \subseteq_e R\}$,See [4, p.138].

The module M is called a singular module if Z(M)= M, the module is called a non singular module if Z(M)=0, See [5]. Define the property Z as follows:

An R-module M has property Z(Z-module) if M is singular(i.e Z(M)=M).

It is easy to see that Z- property is a quasi- radical and hereditary property.

Let R be a ring and let $r \in R$. Recall that r is called a regular element if there is s $\in$ R such that r = rsr. A ring R is called regular ring, if each element of R is regular, See [6]

It is known that a ring R is regular if and only if every cyclic ideal is a direct summand, See [7].

An R- module M is called a regular module if for each x $\in$M and for every r$\in$R, there is s$\in$ R such that r x= rsr x, See[6].

It is known that a projective R-module M is regular if and only if every cyclic submodule is a direct summand, See[7].

Example (1.3): [3]

Let B be an R- module, the Semi Brown-Mecoy radical of B(denoted by M(B)) is defined as follows:

M(B) = {x$\in$ B ; for each r$\in$R, $\exists$ s $\in$ R s.t r x = rsrx}.

Let the regular property M be defined as follows:-

An R-module B has the regular property M, if B is a regular module. It is clear that M is a quasi-radical and hereditary property, See [3, Exa. 3. 54, CH3]

Proposition (1.4) [3, Prop. 3.4, CH3]: Let S be a quasi-radical property and let f: M $\rightarrow$ N be an R-homomorphism, then f (S(M)) $\subseteq$ S(N).

2. S-Generalized supplemented modules.

In this section we introduce the concept of the S- Generalized supplemented modules (or briefly S-GS module) and we illustrate it by some examples we also give some basic properties.

In this section S is a quasi - radical property. Unless otherwise stated.

Definition (2.1):

Let M be an R-module and N be a submodule of M. A submodule K of M is called an S-generalized supplement of N in M, if M = N+K and N $\cap$ K $\subseteq$ S(K).

Let M be an R-module. Recall that if there exist maximal submodules in M, then the intersection of all maximal submodules of M is called the Jacobson radical of M and denoted by Rad(M). If there is no maximal submodule of M, then we define Rad(M) = M, see [4].

Examples (2. 2):

1. Consider the module Z_6 as a Z – module. Let A= $\{\bar{0},\bar{3}\}$ and B= $\{\bar{0},\bar{2},\bar{4}\}$. It is clear that Z_6 = A+ B and A $\cap$ B = 0 $\subseteq$ S(B), for each quasi – radical property S on modules, Thus B is S-generalized supplement of A in Z_6.
2. It is known that the module Q as a Z- module has no maximal submodule and hence Rad (Q) = Q. Let A be any submodule of Q, then Q = A+Q and A $\cap$Q = A $\subseteq$ Rad(Q) = Q. Thus Q is generalized supplement of A.

One can easily show that Soc(Q) =0 and Z(Q)=0.

Now let A be a non- trivial submodule of Q and let B be submodule of Q such that Q = A +B. Since Q is indecomposable as Z- module, then A $\cap$ B $\neq$ 0.

So A$\cap$B$\not\subset$ Soc(B)$\subseteq$ Soc(Q) = 0 and A $\cap$ B $\not\subset$ Z(B) $\subseteq$ Z(Q) = 0. Thus A has no Soc -generalized supplement in Q. Also A has no Z- generalized supplement in Q.

Definition (2.3):

Let M be an R- module. M is called a S- generalized supplemented module (or briefly S-GS module), if every submodule of M has S- generalized supplement in M, where S is a quasi - radical property on modules.

Examples (2.4):

1. The module Z_6 as a Z-module is S-GS module, for each quasi – radical property S on modules.
2. The module Q as a Z- module is GS- module. But the module Q is not Soc-GS module Also Q is not Z - GS module.
3. Let X be an infinite set. Consider the ring (P(X), Δ, $\cap$), where $A \Delta B = (A \cup B)-(A \cap B)$. Since $A^2 = A \cap A = A$, for each A subset of X, then every element A of p(X) is an idempotent. So by[4] every cyclic ideal is a direct summand. So (P(X), Δ, $\cap$) is a regular ring and hence J (P(X)) = 0, See[6]. Thus (P(X), Δ, $\cap$) is M- GS module, But P(X) is not semisimple, See[8, Example 1.2.19], therefore the ring (P(X), Δ, $\cap$) is not GS- module.

Let M be an R- module . Recall that a submodule N of M is called a small submodule of M, (denoted by N <<M), if N+ K $\neq$M, for any proper submodule K of M, see[4].

Proposition (2.5): Let M be S- GS module, then Rad(M) $\subseteq$ S(M),

Proof:

Assume that M is S-GS module and x$\in$ Rad(M). By[4,coro.9.1.3, p.219] Rx << M. Since M is S - GS module, then there exists a submodule N of M such that M = Rx +N and Rx $\cap$ N $\subseteq$ S(N). But Rx << M, therefore N= M and hence Rx $\cap$ N = Rx $\subseteq$ S(N) $\subseteq$ S(M). Thus Rad (M) $\subseteq$ S(M).

Corollary (2.6): Let M be an R-module such that Rad(M) = M, if S(M) $\neq$ M, then M is not S -GS module.

Proof:

Suppose that M is S - GS module, then by (prop. 2.5) Rad(M) $\subseteq$ S(M). But Rad(M) = M, therefore M $\subseteq$ S(M) which is a proper submodule of M . This is a contradiction. Thus M is not S - GS module.

Remark (2.7): Let M be an R- module such that S(M) = M, then M is S- GS module.

Proof:

Let N be a submodule of M, then M= N+M and N $\cap$ M = N $\subseteq$ M =S(M)

Remark(2.8): Every semisimple R-module M is S- GS module.

Proof:

Let N be a submodule of M since is semisimple, then M = N $\oplus$ K, for some submodule K of M. Thus M= N + K and N $\cap$ K = 0 $\subseteq$ S(K) and hence K is S- generalized supplement of N in M.

Proposition (2.9): Let M be an R-module such that S(M)=0, then M is S-GS module if and only if M is semisimple.

Proof:

Suppose that M is S-GS module. Let N be a submodule of M. So there exists a submodule K of M such that M= N+K and N $\cap$ K $\subseteq$ S(K) $\subseteq$ S(M)=0. Thus M= N $\oplus$ K, and we get so every submodule of M is a direct summand. Therefore M is semisimple. the converse from (remark (2.8)).

Proposition (2.10): Let S be a quasi-radical and hereditary property, then every submodule of S-GS module is S- GS module.

Proof:

Let M be S-GS module and let A, N be a submodules of M, such that $N \subseteq A$. Since M is S- GS module, then there is a submodule K of M such that $M = N + K$ and $N \cap K \subseteq S(K)$, by Modular law $A = A\cap M = A\cap (N+K) = N + (A\cap K)$.

Now $N\cap(A\cap K) = N\cap K \subseteq S(K) \cap (A\cap K) = S(A\cap K)$ by [3, cor. 3.37, CH3].

Thus A is S - GS module.

Proposition (2.11): Let M be S- GS module and let K be a submodule of M such that $K \cap S(M) = 0$. Then K is semisimple submodule of M.

Proof:

Let N be a submodule of K. Since M is S-GS module, then there exists a submodule L of M such that $M = N + L$ and $N \cap L \subseteq S(L)$. By Modular law $K = K \cap M = K \cap (N+L) = N + (K\cap L)$.But $N\cap L \cap K \subseteq S(L) \subseteq S(M) \cap K = 0$, therefore $K = N \oplus (K \cap L)$. Thus K is semisimple.

Proposition (2.12): Let M be S-GS module. Then $M = N \oplus L$, where N is semisimple and $S(M) \oplus N$ is an essential submodule of M.

Proof:

Assume that M is S-GSmodule. By Zorn's lemma S(M) has relative complement N in M, by [5,prop.1.3, p.17]. Then $S(M) \oplus N$ is an essential submodule of M. Since M is S-GS module and $N \cap S(M) = 0$, then by (prop. (2.11)), N is semisimple. Since M is S-GS module, then there exists a submodule L of M such that $M = N+L$ and $N \cap L \subseteq S(L)$.But $S(L) \subseteq S(M)$, so $N \cap L \subseteq S(M) \cap N = 0$.Thus $M = N \oplus L$.

Corollary (2.13): Let M be an indecomposable and not simple R-module. If M is S-GS module, then $S(M) \subseteq_e M$.

Proof:

By (prop.(2.12)), $M = N \oplus L$, where N is a relative complement of S(M) in M. But M is indecomposable, therefore either $N = M$ or $N = \{0\}$. If $N = M$ then $S(M) = 0$ and hence M is semisimple by (ramark 2.9) which is a contradiction, so $N = 0$. But $N \oplus S(M) \subseteq_e M$ [5, prop. 1.3, p.17]. So $S(M) \subseteq_e M$.

Let M be an R-module and let $a \in M$. Recall that the annihilator of a in M is the set:- $Ann(a) = \{r\in R;\ ra = 0\}$. It is clear that Ann(a) is an ideal of R, See[5].

Recall that the annihilator of M is the set $Ann(M) = \{r \in R;\ rM = 0\}$. It is clear that Ann (M) is an ideal of R, See [5].

Also recall that an R-module M is called a prime R-module if $Ann(x) = Ann(y)$, for every nonzero elements x and y in M, See [9]

Proposition (2.14): Let M be a prime R- module. If M is S-GS module, then either M is semisimple or S(M) is an essential submodule of M.

Proof:

Let M be S- GS module, Since M is prime, then either $Soc(M) = 0$ or $Soc(M) = M$ by [10, lemma 3.18, CH1]. Assume that $Soc(M) = 0$. By (prop.(2.12)), $M = N \oplus L$, Where N is semisimple and $N \oplus S(M) \subseteq_e M$. One can easily show that $N = Soc(N) \subseteq Soc(M) = 0$ and hence S(M) is essential submodule of M.

Corollary (2.15): Let R be an integral domain and let M be a torsion free R-module. If M is S- GS module, then either M is semisimple or S(M) is an essential submodule of M.

Proof: It is clear by (prop. (2.14)).

Corollary (2.16): Let R be an integral domain and let M be a flat (or projective) R-module, if M is S- GS module, then either M is semisimple or S(M) is an essential submodule of M.

Proof: Clear by proposition (2.14).

3. Characterizations of S - GSmodules.

At the start of this section, we show that the sum of two S- GS modules is also S- GS module. And we give a characterization of the S - GS rings.

We start this section by the following proposition.

Proposition (3.1): Let f: $M \to N$ be an epimorphism. If M is S- GS module, then N is S-GS module.

Proof:

Let K be a submodule of N. Since M is S- GS module, then there is a submodule L of M such that $M = f^{-1}(K)+L$ and $(f^{-1}(K)) \cap L \subseteq S(L)$. So $N = f(M)=f(f^{-1}(K)+L)=f(f^{-1}(K)) + f(L)$. But f is an epimorphism, therefore $N = K + f(L)$ by [4, lemma 3.1.8, p.44]

We only need to show that $K \cap f(L) \subseteq S(f(L))$.

Now, $f(L) \cap K = f(L) \cap f(f^{-1}(K)) = f(L \cap f^{-1}(K)) \subseteq f(S(L)) \subseteq S(f(L))$

By (prop.(1.4)). Thus f(L) is S - generalized supplement of K in N.

Corollary (3.2): Let M be a S-GS module, then $\frac{M}{N}$ is S-GS module, for every submodule N of M.

Before we give our next result, we need the following.

Lemma (3.3): Let M be an R- module and let M_1, K be submodules of M. If M_1 is S-GS module and M_1+ K has S-generalized supplement in M, then K has S- generalized supplement in M.

Proof:

Assume that M_1 is S- GS module. Since M_1 + K has S-generalized supplement in M, then there exists a submodule N of M such that $M = (M_1 +K) + N$ and $(M_1+ K) \cap N \subseteq S(N)$. Since $(K+ N) \cap M_1 \subseteq M_1$ and M_1 is S- GS module, then there exists a submodule L of M_1 such that $M_1 = L +((K+N) \cap M_1)$ and $((K+N) \cap M_1) \cap L \subseteq S(L)$, implies that $(K+N) \cap L \subseteq S(L)$. So $M =L+ ((K+N) \cap M_1) + K+N = L+ K+N$. Now by [11, lemma 3.2.3,CH3],

$K \cap (L+N) \subseteq (L \cap (K+N)) + (N \cap (L+K)) \subseteq (L \cap (K+N)) + (N \cap (M_1+ K))$

Thus $K \cap (L+N) \subseteq S(L) + S(N)$. But $S(N) \subseteq S(N+L)$ and $S(L) \subseteq S(N+L)$, so $S(N)+ S(L) \subseteq S(N+L)$.Thus $K \cap (N+L) \subseteq S(N+L)$.Thus N+L is S-generalized supplement of N in M.

Proposition (3.4): Let $M = M_1 + M_2$. If M_1 and M_2 are S- GS modules, then M is S-GS module.

Proof:

Assume that M_1 and M_2 are S-GS modules. Let N be a submodule of M. Since $M= M_1+ M_2 + N$ has S-generalized supplement in M, then by (lemma(3.3)) M_2 + N has S - generalized supplement in M. But M_2 is S-GS module, therefore by (lemma (3.3)) again, N has S-generalized supplement in M. Thus M is S- GS module.

Proposition (3.5): Let M be S- GS module, then every finitely M - generated module is S-GS module.

Proof:

Assume that A is a finitely M-generated module, then there exists an epimorphism $f : \overset{n}{\underset{i=1}{\oplus}} M \to A$, for some $n \in N$. By (Prop.(3.4)), $\overset{n}{\underset{i=1}{\oplus}} M$ is S-GSmodule and hence by (prop. (3.1)) A is S- GS module.

The following proposition gives a characterization of S- GS rings.

Proposition (3.6): Let R be a ring and let S be a quasi-radical hereditary property then the following statements are equivalent.

1. R is S-GS ring.
2. R $\oplus$ R is S-GS module.
3. Every finitely generated R - module is S -GS module.
4. Every finitely generated projective R - module is S -GS module.
5. Every finitely generated free R - module is S -GS module.

Proof: (1) $\Rightarrow$(2) Clear by (prop. (3.4)).
(2) $\Rightarrow$ (1) Clear by (prop. (2.10)).
(1) $\Rightarrow$ (3) Clear by (prop. (3.5)).
(3) $\Rightarrow$ (4) $\Rightarrow$ (5) It is clear
(5) $\Rightarrow$ (1)
Since R is isomorphic to a free R-module generated by one element, then R is S-GS ring.

Let us recall that , An R- module M is said to be π-projective if for every two submodules N, K of M with M = N+K, there exists f $\in$ End (M) with Imf $\subseteq$N and Im(I-f) $\subseteq$ K, See [12]

Theorem (3.7):

Let M be a π - projective R-module. If M is S- GS module and M = N+K, then N has S- generalized supplement contained in K.

Proof:

Assume that N and K are submodules of M such that M= N+K, Since M is π-projective, then there is an endomorphism e of M such that e(M) $\subseteq$ N and (I-e)(M) $\subseteq$ K, see [12]. One can easily show that (I-e) (N) $\subseteq$ N. Since M is S -GS module, then there exists a submodule L of M such that M = N+L and N $\cap$ L $\subseteq$ S(L). Now M = e(M) + (I-e) (M),
M= e(M) + (I-e)(N+L) = e(M) + (I-e)(N) + (I-e) (L) $\subseteq$ N + (I-e)(L) $\subseteq$ M.
Thus M = N + (I-e)(L). It is clear that (I-e)(L) $\subseteq$ K. Claim that N $\cap$ (I-e)(L) = (I-e) (N$\cap$ L). To verify this, let y$\in$N $\cap$ (I-e)(L), then y $\in$N and y$\in$(I-e) (L). So there exists x $\in$ L such that y = (I-e)(x) = x –e (x) and hence x = y + e (x) $\in$ N. Thus y $\in$ (I-e) (N$\cap$ L).It is clear that (I-e) (N$\cap$ L) $\subseteq$ N $\cap$ (I-e) (L). Since N$\cap$L $\subseteq$ S(L),
Then N $\cap$ (I-e)(L) = (I-e)(N $\cap$L) $\subseteq$ (I-e) (S(L)) $\subseteq$ S((I-e)(L)) by (prop. (1.4)). Thus (I-e) (L) is S-generalized supplement submodule of N in K.

Corollary (3.8): Let R be a ring and let N, K be two ideals of R. If R is S-GS ring and R = N + K , then N has S- generalized supplement contained in K.

Proof: Clear.

The following theorem gives a characterization of S- generalized supplement submodule.

Theorem (3.9):

Let M be an R- module and U be a submodule of M. The following statements are equivalent.

1. There is a decomposition M = N $\oplus$ K with N $\subseteq$ U and K $\cap$U $\subseteq$ S(K).
2. There is an idempotent e $\in$ End (M) with e (M) $\subseteq$ U and (I-e) (U) $\subseteq$ S((I-e)(M)).
3. There is a direct summand N of M with N $\subseteq$ U and $\frac{U}{N} \subseteq S\left(\frac{M}{N}\right)$.
4. U has S- generalized supplement V in M such that U $\cap$ V is a direct summand of U.

Proof: (1) $\Rightarrow$(2)

Assume that M = N $\oplus$ K with N $\subseteq$ U and K $\cap$U $\subseteq$ S(K)
Let e: M $\rightarrow$ M be a map defined as follows: e(a+b)=a, where a $\in$ N and b$\in$K . One can easily show that e$\in$End (M) and e^2=e. Let x$\in$M. Since M=N$\oplus$K. Then x=a+b, where a$\in$N and b$\in$K. Now, e(x)=e(a+b)=a. Thus e(M)

$\subseteq$ N $\subseteq$U. Now (I-e)(M)= {(I-e)(x), x$\in$M}

= {(I-e)(a+b),x = a+b, a$\in$N, b$\in$K}

= {a+b-a, a$\in$N, b$\in$K}={b, b$\in$K}=K. Thus (I-e)(M)=K.

Claim that (I-e) (U) = U $\cap$ (I-e) (M). To show that, let x $\in$ U$\cap$ (I-e) (M). So x $\in$ U and x $\in$ (I-e) (M) and hence x = (I-e) (y), for some y$\in$ M. Now x=(I-e)(y) = y-e(y), y=x +e(y)$\in$U So x $\in$ (I-e) (U) and hence U$\cap$ (I-e) (M) $\subseteq$ (I-e)(U).Now, let z$\in$(I-e)(U), so there is y$\in$U such that z=(I-e)(y) = y-e (y)$\in$U. Thus z $\in$ U $\cap$ (I-e)(M),

(I-e)(U) =U $\cap$ (I-e)(M)= U $\cap$ K $\subseteq$ S(K) = S((I-e)(M))

(2)$\Rightarrow$(3)

Since e is an idempotent element , then by [4, cor.7.2.4, p. 176]

M = e(M) $\oplus$ (I-e)(M). Let N = e(M) and K = (I-e)(M). Since (I-e)(U) =U$\cap$ K and S((1-e)(M)) = S(K), then U$\cap$K$\subseteq$S(K), but by the second isomorphism theorem, $\frac{M}{N}=\frac{N\oplus K}{N}\cong K$ and $\frac{U}{N}=\frac{U\cap(N\oplus K)}{N}=\frac{N\oplus(K\cap U)}{N}\cong$ K$\cap$U. Claim that $\frac{U}{N}\subseteq S\left(\frac{M}{N}\right)$. Let $\phi:K\to\frac{M}{N}$ be an isomorphism. Since U$\cap$K$\subseteq$S(K), then ϕ(U$\cap$K) $\subseteq$ ϕ (S(K)). But S is a quasi-radical property, therefore ϕ(S (K)) $\subseteq$ S (ϕ(K)) = $S\left(\frac{M}{N}\right)$. Thus $\frac{U}{N}=\phi(U\cap K)\subseteq S\left(\frac{M}{N}\right)$.

(3)$\Rightarrow$(1)

Assume that M=N$\oplus$K, where N $\subseteq$ U and $\frac{U}{N}\subseteq S\left(\frac{M}{N}\right)$. Thus $\frac{M}{N}\cong K$ and $\frac{U}{N}\cong U\cap K$, by the second isomorphism theorem. By the same argument of the proof of (2)$\Rightarrow$(3), we get $K\cap U\subseteq S(K)$

(1)$\Rightarrow$(4) Let M= N$\oplus$ K with N $\subseteq$ U and K $\cap$U $\subseteq$ S(K), then M= U+K and hence K is S-generalized supplement of U. By Modular law, U = U $\cap$ M = U $\cap$ (N$\oplus$K) thus U = N $\oplus$ (U$\cap$K) and hence U $\cap$ K is a direct summand of U.

(4)$\Rightarrow$(1)

By our assumption, there exists a submodule V of M such that M=U+V and U$\cap$V $\subseteq$ S(V) and U=(U$\cap$V)$\oplus$L, for some submodule L of U. But M = U+V= (U$\cap$V) $\oplus$ L + V =L+V and L$\cap$V=(U$\cap$L) $\cap$V=L$\cap$(U$\cap$V)=0, therefore M=L$\oplus$V, where L $\subseteq$ U and V$\cap$U$\subseteq$S(V).

Corollary (3.10): Let M be an R-module. Then the following statements are equivalent:

1- For every submodule U of M, there exists a decomposition M = N$\oplus$ K with N $\subseteq$ U and K $\cap$ U$\subseteq$S(K).
2- For every submodule U of M, there is an idempotent e$\in$End (M) with e(M) $\subseteq$ U and (I-e)(U) $\subseteq$ S((I-e)(M)).
3- For every submodule U of M, there is a direct summand N of M with N$\subseteq$U and $\frac{U}{N}\subseteq S\left(\frac{M}{N}\right)$.
4- Every submodule U has S-generalized supplement V in M with U $\cap$V is a summand of U.
5- M is S-GS module.

Then (1)$\Leftrightarrow$(2) $\Leftrightarrow$(3) $\Leftrightarrow$(4) $\Rightarrow$(5).

Proof: Clear.

4. Soc (Z)-GS modules:

In this section we study Soc(Z) - GS modules .We give some their basic properties. Also we give a characterization of Soc (Z)-GSmodule, when M is a prime R-module.

Let M be an R-module. Recall that the socle of M = Soc(M) = $\sum_{\substack{A\text{simple}\\ \text{submodule}}} A = \cap$ {B ; B $\subseteq_e$ M}, and M is called semisimple module if Soc(M)=M, See[5].

The singular of M = Z(M) ={ x $\in$M; Ann x $\subseteq_e$ R }, See[5]

If Z (M) =M then M is called singular module.

If Z (M) =0 then M is called non singular module, See[5] .

It is know that each of the socle (Soc) and the singular (Z) is a hereditary and

quasi-radical property. See (example(1.1)) and (example (1.2)).

Definition (4.1):

An R - module M is called Soc-GS module if for each submodule N of M, there exists a submodule K of M such that M = N+K and N $\cap$ K$\subseteq$ Soc(K).

Definition (4.2):

An R –module M is called Z- GS module if for each submodule N of M, there exists a submodule K of M such that M = N+K and N $\cap$ K $\subseteq$ Z(K).

We start this section by the following examples

Examples (4.3):

1. It is clear that the module Z_n as Z - module is Z-GS module, for each n $\in$ Z. Also Z_n as a Z-module is Soc-GS module, for each square free n$\in$Z, see [4].

   For example Z_6 as Z - module is Soc (Z)-GS module.
2. Consider the module Z_8 as Z-module. Z (Z_8) = Z_8, and hance by (remark (2.7)). Z_8 is Z- GS module. One can easily show that Soc (Z_8) = $\{\bar{0},\bar{4}\}$. Note that A = $\{\bar{0},\bar{2},\bar{4},\bar{6}\}$, B= Z_8 the only submodule of Z_8 such that Z_8=A+B. But A $\cap$ B = A = $\{\bar{0},\bar{2},\bar{4},\bar{6}\}$ $\not\subset$ Soc(B)= $\{\bar{0},\bar{4}\}$, so A has no Soc - generalized supplement in Z_8. Thus Z_8 is not Soc - GS module.
3. Consider the module Z as Z-module. It is easy to see that Soc(Z) = 0 and Z(Z) = 0. Let nZ be a non- trivial submodule of Z and let mZ be a submodule of Z such that Z= nZ + mZ. It is clear that mZ $\neq$0. Since Z is indecomposable, then nZ$\cap$mZ $\neq$0.Thus nZ$\cap$mZ $\not\subset$ Soc(mZ) $\subseteq$ Soc(Z) =0 and hence nZ has no Soc - generalized supplement in Z.Thus Z is not Soc-GS module.

By the same way we can show that Z is not Z-GS module.

Remark (4.4): Every singular R-module is Z- GS module.

Proof: Clear by (remark (2.7)).

Remark (4.5): Let M be an R -module and let N is essential submodule of M, then $\frac{M}{N}$ is Z- GS module.

Proof:

Since N is essential submodule of M, then by [5, prop.1.20,p.31] $\frac{M}{N}$ is singular. Thus by (prop. (4.2)) $\frac{M}{N}$ is Z -GS module.

Proposition (4.6): Let M be an R-module, then M is Soc (Z)-GS module if and only if for every submodule N of M, there exists a submodule K of M such that M = N+ K and

$N\cap K \subseteq Soc(M)$ (respectively, $N\cap K \subseteq Z(M)$).

Proof:

Assume that M is Soc- GS module. Let N be a submodule of M, then there is a submodule K of M such that M= N+K and $N\cap K \subseteq Soc(K)$. But $Soc(K) \subseteq Soc(M)$, therefore $N\cap K \subseteq Soc(M)$.

For the converse, let N be a submodule of M, then there exists a submodule K of M such that M = N+K and $N\cap K \subseteq Soc(M)$. So $N\cap K\subseteq(Soc(M)) \cap K$. But $Soc(K) = Soc(M) \cap K$, by [4,Th. 9.7.3, p.226], therefore $N \cap K \subseteq Soc(K)$. Thus M is Soc -GS module.

By the same argument we can prove the proposition(4.6) for Z -GS module.

Remark (4.7): Let M be Soc (respectively Z) - GS module. Then every submodule of M is Soc (respectively Z)-GS module.

Proof: clear by (prop. (2.10)).

Proposition (4.8): Let M be a non - zero R - module such that Soc(M) =0, then M is not Soc - GS module.

Proof:

Let $M \neq 0$ be an R- module with Soc(M) =0. Suppose that M is Soc -GS module and let N be a submodule of M, then there exists a submodule K of M such that M = N+K and $N\cap K \subseteq Soc(K)\subseteq Soc(M) =0$. So $M = N\oplus K$ and hence M is semisimple (i.e $Soc(M) =M\neq 0$) which is a contradiction.

Before we give our next result we need the following results which they appeared in[8].

Lemma (4.9): [8, prop. 2.3.4]: Let M be a prime R- module, then either Z (M) =0 or Z(M)=M.

Proposition (4.10): [8, prop. 1.2.4, CH1]: Let R be an integral domain such that R is not a field. Then every non - zero torsion free R- module is not regular.

Proposition (4.11): Let R be an integral domain such that R is not a field and let M be a non - zero projective module, then M is not Soc (respectively Z) – GS module.

Proof:

Assume that M is Soc- GS module. Since M is projective then by [13]. M is torsion free and hence M is prime module [9]. By (remark (4.7)) and (prop.(4.6)) M is semisimple. But M is projective, therefore by [7] M is a regular module which is a contracliction with (4.10).

By the same argument we can prove the proposition(4.11) for Z-GS module.

Proposition (4.12): Let M be a non singular R- module and let N, K be submodules of M. If K is a Soc- generalized supplement submodule of N in M and N is an essential submodule of M, then $N \cap K = Soc(K)$.

Proof:

Assume that K is a Soc- generalized supplement of N, then M = N+K and $N\cap K \subseteq Soc(K)$. By the second isomorphism theorem $\frac{M}{N} = \frac{N+K}{N} \cong \frac{K}{N\cap K}$. Since $N \subseteq_e M$, then by [5, prop. 1.21, p.32] $\frac{M}{N}$ is singular and hence $\frac{K}{N\cap K}$ is singular. But M is non singular, therefore by [5, prop. 1.21,p.32], $N \cap K \subseteq_e K$ and hence $Soc(K) \subseteq N\cap K$. Thus $Soc(K) = N\cap K$.

Proposition (4.13): Let R be a ring and N, K be ideals of R such that K is a Soc - generalized supplement ideal of N in R. If N is an essential ideal of R, then $N\cap K = Soc(K)$.

Proof:

Let M = N+K and $N\cap K \subseteq Soc(K)$. By the second isomorphism theorem, $\frac{R}{N}=\frac{N+K}{N}\cong\frac{K}{N\cap K}$, Since $N \subseteq_e R$, then by [5, prop. 1.20, p.31] $\frac{R}{N}$ is singular and hence $\frac{K}{N\cap K}$ is singular. By [5, prop.1.20, p.31], $N\cap K \subseteq_e K$. So $Soc(K) \subseteq N\cap K$. Thus $N\cap K = Soc(K)$.

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المقاسات المكملة المعممة من النمط S

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الكلمات المفتاحية: خاصية شبه جذرية، المقاس المكمل المعمم، المقاسات الجزئية الصغيرة.

الخلاصة:

أكسيو قدم المفهوم الأتي، يقال للمقاس M بأنه مكمل معمم إذا كان لكل مقاس جزئي N من M، يوجد مقاس جزئي K من M بحيث أن M=N+ K و $N\cap K \subseteq Rad(K)$.

نها حمادة والهاشمي قدما المفهوم الأتي، يقال للخاصية S المعرفة على المقاسات بأنها خاصية شبه جذرية أذا تحقق الأتي:

1. ليكن $f: M \rightarrow N$ تشاكلاً شاملاً. أذا كانت M مقاساً يملك الخاصية S فأن N مقاساً يملك الخاصية S.
2. كل مقاس M يحوي على المقاس الجزئي S(M).

هذه الملاحظات قادتنا إلى اقتراح تعريف المقاسات المكملة المعممة من النمط S. لتكن S خاصية شبه جذرية، يقال للمقاس M المعرف على الحلقة R بأنه مقاس مكمل معمم من النمط S. إذا كان لكل مقاس جزئي N من M، يوجد مقاس جزئي K من M بحيث أن M=N+K و $N\cap K \subseteq S(K)$.

الغرض الرئيسي من هذا البحث هو تطوير خواص المقاسات المكملة المعممة من النمط S. لقد أعطينا مجموعة من القضايا الجديدة وأوضحنا المفاهيم بأمثلة.