

Solving Nonlinear Partial Differential Equations Using Homotopy Perturbation Method

حل المعادلات التفاضلية الجزئية غير الخطية باستخدام طريقة الهوموتوبي المضطربة

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Abstract

In this paper, we attempt to solve samples of nonlinear partial differential equations (PDE's) of the form $f(u, u_x, u_{xx})$ using He's Homotopy Perturbation Method (HPM) with time fractional derivative, which proposed by J. H. He. We present in this paper an algorithm of the new modification of the homotopy perturbation method to be suitable to be applied in nonlinear PDE's.

خلاصة

في هذا البحث نحاول حل نماذج من المعادلات التفاضلية الجزئية غير الخطية من الشكل $f(u, u_x, u_{xx})$ باستخدام طريقة الهوموتوبي المضطربة (Homotopy Perturbation Method) مع الاشتقاق الكسوري الزمني، والذي سبق وان تم اقتراحه من قبل ج.ح.هيز. نقدم في هذا البحث خوارزمية جديدة مع تحويلات على طريقة الهوموتوبي المضطربة (HPM) لتناسب التطبيق على المعادلات التفاضلية غير الخطية.

1. Introduction

In recent years, that many phenomena in fluid mechanics, viscoelasticity, biology, physics, engineering and other areas of science can be success fully modeled by the use of fractional derivatives and integrals.

Several analytical and numerical methods have been proposed to solve fractional ordinary differential equation integral equations and fractional partial differential equations of physical interest.

The most commonly used ones are; A Domain Decomposition Method (ADM) [15-20], [23] Variational Iteration Method (VIM) [3], [6-8], [20] Fractional Difference Method (FDM) [3], Differential Transform Method (DTM) [8], Homotopy Perturbation Method (HPM) [9].

Also there are some classical solution techniques, e.g. Laplace transform, fractional Green's function method, Mellin transform method and method of orthogonal polynomials [3].

The HPM, proposed first by He [6-7], for solving differential and integral equations, linear and nonlinear, has been the subject of extensive analytical and numerical studies. The method, which is a coupling of the traditional perturb method and homotopy in topology, deforms continuously a simple problem which is easily solved.

This method, which does not require a small parameter in an equation.

The HPM is applied to nonlinear oscillators [11], bifurcation of nonlinear problems [13], nonlinear wave equations [4], boundary value problems [5], quadratic Riccati differential equation of fractional order [9], and to other fields [1-2], [6-10], [21-22].

This HPM yields a very rapid convergence of the solution series in most cases.

This paper is to extend the application of the He's homotopy perturbation method proposed by He [12-13] to solve nonlinear partial differential equations with time fractional derivative.

Consider the nonlinear partial differentials equations with time fractional derivative of the form.

$$D_{*t}^{\alpha} u(x,t) = f(u, u_x, u_{xx}), t > 0, \quad \dots(1)$$

where $m-1 < \alpha < m$, f is a nonlinear function and D_{*}^{α} denotes the differential operator in the sense of Caputo [14], defined by:

$$D_{*}^{\alpha} f(x) = J^{m-\alpha} D^m f(x)$$

here D^m is the usual integer differential operator of order m and J^{α} is the Rieman–Liouville integral operator of order $\alpha > 0$.

2. Basic Definitions

We give some basic definitions and properties of the fraction calculus theory which are used further in this paper.

Definition (2.1): Areal function $f(x)$, $x > 0$, is said to be in the space $C_{\mu} f_1(x) \in [0, \infty)$, and it is said to be in the space C_{μ}^m iff $f^{(m)} \in C_{\mu}$, $m \in N$.

Definition (2.2): The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of function $f(\mu, \mu \geq -1)$, is defined as:

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha > 0, x > 0$$

$$J^0 f(x) = f(x)$$

Properties of the operator J^{α} can be found in [1,2,3], we mention only the following:

For $f \in C_{\mu}$, $\mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma > -1$:

1. $J^{\alpha} J^{\beta} f(x) = J^{\alpha+\beta} f(x)$
2. $J^{\alpha} J^{\beta} f(x) = J^{\beta} J^{\alpha} f(x)$
3. $J^{\alpha} X^{\gamma} = (\Gamma(\gamma+1)/\Gamma(\alpha+\gamma+1)) X^{\alpha+\gamma}$

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations . Therefore , we shall introduce a modified fractional differential operator D_{*}^{α} proposed by Caputo in this work on the theory of viscoelasticity [23].

Definition (2.3): The fractional derivative of $f(x)$ in the Caputo sense is defined as:

$$D_{*}^{\alpha} f(x) = J^{m-\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt \quad \dots(2)$$

For $m-1 < \alpha \leq m$, $m \in N$, $x > 0$, $f \in C_{-1}^m$.

Lemma(2.4): If $m-1 < \alpha \leq m$, $m \in N$, $x > 0$ and $f \in C_{\mu}^m$, $\mu \geq -1$, then

$$D_{*}^{\alpha} J^{\alpha} f(x) = f(x)$$

$$\text{And } J^{\alpha} D_{*}^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0) \frac{x^k}{k!}, \quad x > 0.$$

Consider the first order differential equation:

$$u'(x) = su(x) + Q(x), \quad u(0) = 0$$

Where s is appositve constant. The analytical solution of (2) is given by:

$$u(x) f(x) = \int Q(x) f(x) dx$$

where $f(x) = e^{-sx}$.

The Laplace transformation:

$$F(s)=L\{Q(x)\}=\int_0^{\infty} Q(x)e^{-sx}dx=\{e^{-sx}u(x)\}\Big|_{x=0}^{x=\infty} \quad \dots(3)$$

3. He's homotopy Perturbation Method

To illustrate the homotopy perturbation method (HPM), He [18], considered the following nonlinear differential equation

$$A(u)=f(r), \quad r \in \Omega \quad \dots(4)$$

With boundary conditions:

$$B(u, \partial u / \partial r) = 0, \quad r \in \Gamma \quad \dots(5)$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of the domain. The operator A can be divided into two parts μ and N . Therefore (4) can be rewritten as follows:

$$L(u)+N(u)=f(r), \quad r \in \Omega \quad \dots(6)$$

with boundary conditions.

The He's homotopy perturbation technique[6,7] constructed a homotopy:

$v(r,p): \Omega_x[0,1] \rightarrow H$ which satisfies:

$$H(v,p)=(1-p)[L(v)-L(u_0)]+p[L(v)+N(v)-f(r)]=0 \quad \dots(7)$$

or

$$\begin{aligned} H(v,p) &= L(v)-L(u_0)-pL(v)+pL(u_0)+pL(v)+pN(v)-pf(r) \\ H(p,v) &= L(v)-L(u_0)+pL(u_0)+p[N(v)-f(r)]=0 \end{aligned} \quad \dots(8)$$

Where $r \in \Omega$ and $p \in [0,1]$ is an imbedding parameter, u_0 is an initial approximation which satisfies the boundary conditions. Obviously, from equation (8), we have:

$$\begin{aligned} H(v,0) &= L(v)-L(u_0)+0*L(u_0)+0*pN(v)-0*f(r) \\ H(v,0) &= L(v)-L(u_0)=0 \end{aligned} \quad \dots(a)$$

$$\begin{aligned} H(v,1) &= L(v)-L(u_0)+L(u_0)+N(v)-f(r) \\ H(v,1) &= L(v)+N(v)-f(r)=0 \end{aligned} \quad \dots(b)$$

and the changing process of p from 0 to 1, is just that of $H(v,p)$ from $L(v)-L(u_0)$ to $A(u)-f(r)$ (i.e u_0 to $u(r)$). In topology, this is called deformation, $L(v)-L(u_0)$ and $A(u)-f(r)$ are called homotopic.

Applying the perturbation technique [8], due to the fact that $0 \leq p \leq 1$ can be considered as a small parameter, we can assume that the solution of (7) or (8) can be expressed as a series in p , as follows:

$$V=V_0+pV_1+p^2V_2+p^3V_3+\dots \quad \dots(9)$$

When $p \rightarrow 1$, (7) or (8) corresponds to (a) and (b) becomes the approximate solution of (9), i.e,

$$u = \lim_{p \rightarrow 1} p = v_0 + v_1 + v_3 + \dots \quad \dots(10)$$

The series (10) is convergent for most cases, and also the rate of convergence depends on $A(v)$, [9].

$$L(v)-L(u_0)+pL[N(r)-Q]L(v)-L(u_0)+pL(u_0)+p[N(r)-Q]=0$$

$$N(u)=u'(x)$$

$$L(v)-L(u_0)+pL(u_0)+p[N(v)-Q]=0,$$

or

$$p[v'(x)-Q(x)]-sv_0(x)+su_0(x)-psu_0(x)=0$$

$$p^0: L(v_0) - L(u_0) = 0, p^1: L(v_1) + L(u_0) + N(v_0) - Q = 0, p^2: L(v_2) + N(v_1) = 0, \\ p^3: L(v_3) + N(v_2) = 0, \dots, p^{n+1}: L(v_{n+1}) + N(v_n) = 0$$

$$F(x) = L[Q(x)] = \int_0^\infty Q(x) e^{-sx} dx = \left\{ e^{-sx} \sum_{i=0}^\infty v_i(x) \right\} \Bigg|_{x=0}^{x=\infty}$$

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Example (3.1): Let $Q(x) = x^n$.

By choosing $u_0(x) = 0$, we have

$$v_0(x) = 0, v_1(x) = -\frac{1}{s} x^n, v_2(x) = -\frac{n}{s^2} x^{n-1}, v_3(x) = -\frac{n(n-1)}{s^3} x^{n-2}, \dots,$$

$$v_{n+1}(x) = -\frac{n!}{s^{n+1}}, v_i(x) = 0, i = n+2, n+3, \dots$$

Therefore

$$F(s) = L[x^n] = L[Q(x)] = \left\{ e^{-sx} \sum_{i=0}^\infty v_i(x) \right\} \Bigg|_{x=0}^{x=\infty} = \frac{n!}{s^{n+1}}$$

4. Laplace Transform

We consider He's homotopy perturbation method, suppose:

$L(u) = -su(x)$ and $N(u) = u'(x)$, in other word, we construct the following simple homotopy

$$L(v) - L(u_0) + pL(u_0) + p[N(v) - Q] = 0 \quad \dots(11)$$

or

$$p[v'(x) - Q(x)] - sv(x) + su_0(x) - psu_0(x) = 0$$

substituting (9) into (11), and equating the terms with the identical powers of p , we have:

$$p^0: L(v_0) - L(u_0) = 0, p^1: L(v_1) + L(u_0) + N(v_0) - Q = 0, p^2: L(v_2) + N(v_1) = 0,$$

$$p^3: L(v_3) + N(v_2) = 0, \dots, p^{n+1}: L(v_{n+1}) + N(v_n) = 0$$

therefore according to (3) and (10) we have:

$$F(x) = L[Q(x)] = \int_0^\infty Q(x) e^{-sx} dx = \left\{ e^{-sx} \sum_{i=0}^\infty v_i(x) \right\} \Bigg|_{x=0}^{x=\infty}$$

Example(4.1): Let $Q(x) = x^n$.

By choosing $u_0(x) = 0$, we have

$$v_0(x) = 0,$$

$$v_1(x) = -\frac{1}{s} x^n, v_2(x) = -\frac{n}{s^2} x^{n-1}, v_3(x) = -\frac{n(n-1)}{s^3} x^{n-2}, \dots, v_{n+1}(x) = -\frac{n!}{s^{n+1}}, v_i(x) = 0, i = n+2, n+3, \dots$$

Therefore

$$F(s) = L[x^n] = L[Q(x)] = \left\{ e^{-sx} \sum_{i=0}^\infty v_i(x) \right\} \Bigg|_{x=0}^{x=\infty} = \frac{n!}{s^{n+1}}$$

Example(4.2): Let $Q(x) = \sin(ax)$. By choosing $u_0(x) = 0$, we have:

$$v_0(x) = 0, v_1(x) = -\frac{1}{s} \sin(ax), v_2(x) = -\frac{a}{s^2} \cos(ax), v_3(x) = -\frac{a^2}{s^3} \sin(ax),$$

$$v_4(x) = -\frac{a^3}{s^4} \cos(ax), \dots$$

Therefore

$$F(s)=L[\sin(ax)]=\left\{e^{-sx}\sum_{i=0}^{\infty}v_i(x)\right\}\Bigg]_{x=0}^{x=\infty}=\frac{a}{s^2}+\frac{a^3}{s^4}+\frac{a^5}{s^6}+\cdots=\frac{a}{s^2+a^2}$$

$$u=\sin(ax), du=a\cos(ax)$$

$$dv=e^{-sx}dx, v=\frac{e^{-sx}}{-s}$$

$$\begin{aligned} L[\sin(ax)] &= \int_0^{\infty} \sin(ax)e^{-sx}dx \\ &= \left[\sin(ax)\frac{e^{-sx}}{-s} \right]_{x=0}^{x=\infty} - \int_0^{\infty} a\cos(ax)\frac{e^{-sx}}{-s}dx \\ &= 0 + \frac{a}{s} \left[\cos(ax)\frac{e^{-sx}}{-s} \right]_{x=0}^{x=\infty} - \frac{a}{s} \int_0^{\infty} -a\sin(ax)\frac{e^{-sx}}{-s}dx \\ &= -\frac{a}{s^2} [\cos(ax)(0) - (1)(1)] - \frac{a^2}{s^2} \int_0^{\infty} \sin(ax)e^{-sx}dx \\ &= \frac{a}{s^2} - \frac{a^2}{s^2} L[\sin(ax)] \end{aligned}$$

$$L[\sin(ax)] \left(1 + \frac{a^2}{s^2} \right) = \frac{a}{s^2}$$

$$L[\sin(ax)] \left(\frac{s^2 + a^2}{s^2} \right) = \frac{a}{s^2}$$

$$L[\sin(ax)] = \frac{a}{s^2} \left(\frac{s^2}{s^2 + a^2} \right) = \left(\frac{a}{s^2 + a^2} \right)$$

The convergence of the series (10), has been proved in [6,7].

5. New Interpretation

Anew interpretation of the concept of constant expansion in the Homotopy perturbation method is given in [20]. Toillustrate the new interpretation of the parameter-expansion, He [8] considered the nonlinear oscillator.

$$u''+\varepsilon u^3=0, u(0)=A, u'(0)=0, \quad \dots(12)$$

Where the parameter ε is not required to be small, $0<\varepsilon<\infty$.

For the first-order approximate solution, we construct a homotopy in the form:

$$u''+(\omega^2+pc_1)u+p\varepsilon u^3=0, u(0)=A, u'(0)=0, p\in[0,1], \quad \dots(13)$$

where

$$\omega^2+pc_1=0 \quad \dots(14)$$

The homotopy parameter p always changes from zero to unity.

In case $p=0$, equation (13) becomes the linearized equation:

$$u''+\omega^2u=0, u(0)=A, u'(0)=0, \quad \dots(15)$$

put $p=1$,

$$u''+(\omega^2+c_1)u+\varepsilon u^3=0,$$

$$u(0)=A, u'(0)=0, p\in[0,1],$$

$$u=u_0+pu_1+p^2u_2 \text{ (since } u'=u'_0+pu'_1+p^2u'_2, u'(0)=0 \text{ this condition above).}$$

6. Modified Homotopy Perturbation Method

We present the algorithm of the new modification of the homotopy perturbation method. To illustrate the basic ideas of the new modification, we consider the following nonlinear differential equation of fraction order.

$$D^{\alpha} * u(t) + L(u(t)) + N(u(t)) = f(t), t > 0, m-1 < \alpha < m, \quad \dots(16)$$

or

$$D^{\alpha} * u(x,t) + L(u, u_x, u_{xx}) + N(u, u_x, u_{xx}) = f(x,t), t > 0,$$

Where L is a linear operator which might include other fractional derivatives of order less than α , N is a nonlinear operator which also might include other fractional derivatives of order less than α , f is a known analytic function and $D^{\alpha} *$ the Caputo fractional derivative of order α , subject to the initial conditions:

$$u^k(0) = c_k, k=0,1,2,\dots,m-1, \text{ or } u^k(x,0) = g_k(x), \quad \dots(17)$$

In view of the homotopy technique, we can construct the following homotopy:

$$u^{(m)} + L(u) - f(t) = p[u^{(m)} - N(u) - D^{\alpha} * u], p \in [0,1], \quad \dots(18)$$

or

$$u^{(m)} - f(t) = p[u^{(m)} - L(u) - N(u) - D^{\alpha} * u], p \in [0,1], \quad \dots(19)$$

The homotopy parameter p always changes from zero to unity.

In case $p=0$, equation (18) becomes the linearized equation:

$$\frac{d^m u}{dt^m} + L(u) = f(t) \quad \dots(20)$$

and equation (20) becomes the linearized equation

$$\frac{d^m u}{dt^m} = f(t) \quad \dots(21)$$

And when it is one, equation (18) or (19) turns out to be the original fractional differential equation(16). The basic assumption is that the solution of equation (18) or (19) can be written as a power series in p

$$u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \dots \quad \dots(22)$$

Substituting equation(22) into (18) or (19), and equating the terms with identical powers of p , we can obtain a series of linear equation of the form:

$$\left. \begin{aligned} p^0 : \frac{d^m u_0}{dt^m} + L_0(u_0) &= f(t), u^k(0) = c_k, \\ p^1 : \frac{d^m u_1}{dt^m} + L_1(u_0, u_1) &= \frac{d^m u_0}{dt^m} - N_1(u_0) - D^{\alpha} * u_0, u^k(0) = 0, \\ p^2 : \frac{d^m u_2}{dt^m} + L_2(u_0, u_1, u_2) &= \frac{d^m u_1}{dt^m} - N_2(u_0, u_1) - D^{\alpha} * u_1, u^k(0) = 0, \\ p^3 : \frac{d^m u_3}{dt^m} + L_3(u_0, u_1, u_2, u_3) &= \frac{d^m u_2}{dt^m} - N_3(u_0, u_1, u_2) - D^{\alpha} * u_2, u^k(0) = 0, \end{aligned} \right\} \quad \dots(23)$$

or the form:

$$\left. \begin{aligned} p^0 : \frac{d^m u_0}{dt^m} &= f(t), u^k(0)=0, \\ p^1 : \frac{d^m u_1}{dt^m} &= \frac{d^m u_0}{dt^m} - L_0(u_0) - N_1(u_0) - D_*^\alpha u_0, u^k(0)=0, \\ p^2 : \frac{d^m u_2}{dt^m} &= \frac{d^m u_1}{dt^m} - L_1(u_0, u_1) - N_1(u_0, u_1) - D_*^\alpha u_1, u^k(0)=0, \\ p^3 : \frac{d^m u_3}{dt^m} &= \frac{d^m u_2}{dt^m} - L_2(u_0, u_1, u_2) - N_2(u_0, u_1, u_2) - D_*^\alpha u_2, u^k(0)=0, \end{aligned} \right\} \dots(24)$$

Respectively, where the terms L_0, L_1, L_2, \dots , and N_0, N_1, N_2, \dots satisfy the following equations:

$$\begin{aligned} L(u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \dots) &= L_0(u_0) + pL_1(u_0, u_1) + p^2 L_2(u_0, u_1, u_2) + \dots \\ N(u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \dots) &= N_0(u_0) + pN_1(u_0, u_1) + p^2 N_2(u_0, u_1, u_2) + \dots \end{aligned}$$

Setting $p=1$ in equation(22) yields the solution of equation(16). It obvious that the linear equation in (23) or equation (24) are easy to solve, and the components determined $u_n, n \geq 0$ of the homotopy perturbation method can be completely determined, and the series solution are thus entirely determined.

Finally, we approximate the solution $u(t) = \sum_{n=0}^{\infty} u_n(t)$ by the truncated series:

$$\varphi_N(t) = \sum_{n=0}^{N-1} u_n(t) \quad \dots(25)$$

7. Numerical Experiments

We shall illustrate the new algorithm of HPM by several examples.

These examples are somewhat artificial in the sense that the exact answer, for the special cases $\alpha=1$ or 2, is known in advance and the initial and boundary condition are directly taken from this answer.

Nonetheless, such an approach is needed to evaluate the accuracy of the analytical technique and examine the effect of varying the order of the time–fractional derivative on the behavior of the solution.

Also, a comparison is made with the results obtained in [3].

Using variational iteration method and A domain decomposition method. All the resulted are calculated by using the symbolic calculus software mathematica.

Example(7.1):

Consider the nonlinear time-fractional Fisher's equation [16]:

$$D_*^\alpha u_{xx}(x, t) + 6u(x, t)(1 - u(x, t)) \quad \dots(26)$$

$t > 0, x \in R, 0 \leq \alpha \leq 1$.

Subject to the initial condition:

$$u(x, 0) = \frac{1}{(1 + e^x)^2} \quad \dots(27)$$

In view of equation (19), the homotopy for equation (26) can be constructed as:

$$\frac{\partial u}{\partial t} = p \left[\frac{\partial u}{\partial t} + u_{xx} + 6u(1 - u) - D_*^\alpha u \right] \quad \dots(28)$$

Substituting (22) and the initial condition (27) into (28), we obtain the following set of linear partial differential equation:

$$\frac{\partial u_0}{\partial t} = 0, \quad u(x,0) = \frac{1}{(1+e^x)^2}, \quad \frac{\partial u_1}{\partial t} = p \left[\frac{\partial u_0}{\partial t} + (u_0)_{xx} + 6u_0(1-u_0) - D_{*t}^\alpha u_0 \right], \quad u_1(x,0)=0.$$

$$\frac{\partial u_2}{\partial t} = p \left[\frac{\partial u_1}{\partial t} + (u_1)_{xx} + 6u_1(1-u_1) - D_{*t}^\alpha u_1 \right]$$

$$u_2(x,0)=0.$$

Consequently the first few components of the homotopy perturbation solution for equation (26) we derived as follows:

$$u_0(x,t) = \frac{1}{(1+e^x)^2}, \quad u_1(x,t) = 10 \frac{e^x}{(1+e^x)^3} t,$$

$$u_2(x,t) = -10 \frac{e^x}{(1+e^x)^3} \frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{1}{(1+e^x)^6} [-200e^{2x}t^3 + (50e^{4x} + 75e^{3x} - 25e^x) \\ + (10e^{4x} + 30e^{3x} + 30e^{2x} + 10e^x)t]$$

8. Concluding Remarks

The modified homotopy perturbation method suggested is an efficient method for calculating approximate solution for nonlinear partial differential equation of fractional order.

The solution obtained using the suggested method has a very high accuracy comparing with variation method and the A domain decomposition method. The method produces the same solution as the variational iteration method with the proper choice of the initial approximation.

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