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Numerical Calculation of triple integrals with Continuous Integrands

حساب التكاملات الثلاثية ذات المكاملات المستمرة عدديًا

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Abstract

The main aim of this research is to introduce a new rule to evaluate numerically the triple integrals of continuous integrands using Romberg accelerating method depending on correction terms which we found . We determined that the combined method of Romberg accelerating method on resulting values came from applying Mid-point rule on the outer dimension z and Simpson's rule on the middle and inner dimensions y , x where number of subintervals to which the interval of integral on the inner dimension equal to that number of subintervals of interval of middle dimension and equal to the number of subintervals of outer dimension (i.e $h=\bar{h}=\bar{\bar{h}}$) where h , $\bar{h}$ and $\bar{\bar{h}}$ are respectively distances between ordinates of $z-$ axis, $y-$ axis and $x-$ axis, we call them $RMSS$ on which we can depend to reach solution of higher accuracy.

المستخلص

الهدف الرئيس من هذا البحث هو اشتقاق قاعدة جديدة لحساب التكاملات الثلاثية ذات المكاملات المستمرة باستعمال قاعدتي النقطة الوسطى وسمبسون واشتققنا حدود التصحيح (صيغة الخطأ) لها ولتحسين نتائج التكاملات الثلاثية استعملنا طريقة تعجيل رومبرك بالاعتماد على حدود التصحيح التي وجدناها , فتبين لنا إن الطريقة المركبة من طريقة تعجيل رومبرك على القيم الناتجة من تطبيق قاعدة النقطة الوسطى على البعد الخارجي z وقاعدة سمبسون على البعدين الأوسط والداخلي y x , عندما عدد الفترات الجزئية التي تجزأ إليها فترة التكامل على البعد الداخلي مساوية لعدد الفترات الجزئية التي تجزأ إليها فترة التكامل على البعد الأوسط ومساوية لعدد الفترات الجزئية التي تجزأ إليها فترة التكامل على البعد الخارجي أي إن $(h=\bar{h}=\bar{\bar{h}})$ حيث h المسافات بين الإحداثيات على المحور z و $\bar{h}$ المسافات بين الإحداثيات على المحور y و $\bar{\bar{h}}$ المسافات بين الإحداثيات على المحور x وأسميناها $RMSS$ حيث يمكن الاعتماد عليها في حساب التكاملات الثلاثية ذات المكاملات المستمرة إذ أعطت دقة عالية في النتائج بفترات جزئية قليلة نسبياً .

1.Introduction :

The subject of numerical analysis is characterize in a creation of a variety methods to find approximate solutions to a certain mathematical matters with an effective manner. The efficiency of these methods depend on both the accuracy and the simplify .The modern numerical analysis is the numerical inter face to the broad field of the application analysis. Since the triple integrals have an importance to find the volumes, the medium-sized centers, the moment of inertia of the sizes and find the blocks with variable density, for example, size effect $x^2+y^2=4x$ and over $z=0$ and under $x^2+y^2=4z$ and the medium center calculate of the volume that exist with in $x^2+y^2=9$ and above the plane $z=0$ and under the plane $x+z=4$,as well as found a blocks that have the changeful density like a piece of cylinder wire or a thin plate of metal, Frank Ayers [7]. So that a number of researchers worked in a field of the triple integrals.

In 2009 Dayaa [6] methods of a single integration to create methods to calculate the triple integration that it is $RMRM(RS)$, $RMRM(RM)$, $RMRS(RM)$, $RMRS(RS)$, and these methods were resulted from $RM\,(RS)$, $RM\,(RM)$ $RS\,(RM)$, on the middle dimension (y)and the internal dimension (x), and a Romberg accelerating method with the mid-point rule$\left(RM\right)$ on the external dimension (z), thus, she has been found that the composition method $RMRS\,(RS)$ is the best method to calculate the triple integrals which it's integrands are continuous functions by depending on the accuracy and speed of the approach. .

In 2010, Akkar [5] introduced an numerical method to calculate a values of the triple integrals by using $RMMM$ method which resulting from Romberg accelerating with mid-point rule applied to the dimensions x ,y ,z when the number of subintervals that divides the internal dimension interval with an equal to the number of subintervals that divides the middle dimension interval and equal to the number of subintervals that divides the external dimension interval, and obtained a good results with respect to accuracy and speed of the approach sub intervals of relatively scarce.

In this research we will offer a theorem with it's proof to derive a new method to calculate an approximate values to the triple integrals which its integrands are a continuous functions, also to derive the error formula for it. This method resulted from the application of Romberg accelerating on the values that resulting from the use of two rules (the mid-point on the external dimension z and the Simpson on the two dimensions –the middle and internal x, y) when $2m = 2n_1 = 2n_2$ ($2n_2$ a number of the subintervals that divides the internal dimension interval and $2n_1$ a number of the subintervals that divides the middle dimension interval and $2m$ a number of the subintervals that divides the external dimension interval). We choose $2m, 2n_1, 2n_2$ because the Simpson's rule needs to be an even number from the subintervals at and the same time does not affect on our discussion to the mid-point rule. We will symbolize to this method by a symbol $RMSS$ when R is Romberg accelerating method and MSS is the derived rule. We obtained a good results with respect to the accuracy and speed of the approach with subintervals of relatively scarce.

2.Evaluation of triple integrals with Continuous Integrands numerically

Theorem:

Let the function $f\,(x\,,y\,,z\,)$ is continuous and differentiable teach point of the region $[a,b]\times[c,d]\times[e,g]$ then approximate value for the integration $I = \int_e^g\int_c^d\int_a^b f\,(x\,,y\,,z\,)$ can be calculate it from the following rule :

$$MSS = \frac{h^3}{9}\sum_{k=1}^{2n}\Bigg[\; f\,(a,c,z_k)+f\,(a,d,z_k)+f\,(b,c,z_k)+f\,(b,d,z_k)+4\sum_{j=1}^{n}\Big(f\,\big(a,y_{(2j-1)},z_k\big)+f\,\big(b,y_{(2j-1)},z_k\big)\Big)$$
$$+2\sum_{j=1}^{n-1}\Big(f\,\big(a,y_{(2j)},z_k\big)+f\,\big(b,y_{(2j)},z_k\big)\Big)+4\sum_{i=1}^{n}\Bigg(f\,(x_{(2i-1)},c,z_k)+f\,(x_{(2i-1)},d,z_k)+4\sum_{j=1}^{n}f\,(x_{(2i-1)},y_{(2j-1)},z_k)$$
$$+2\sum_{j=1}^{n-1}f\,(x_{(2i-1)},y_{(2j)},z_k)\Bigg)+2\sum_{i=1}^{n-1}\Bigg(f\,(x_{(2i)},c,z_k)+f\,(x_{(2i)},d,z_k)+4\sum_{j=1}^{n}f\,(x_{(2i)},y_{(2j-1)},z_k)+2\sum_{j=1}^{n-1}f\,(x_{(2i)},y_{(2j)},z_k)\Bigg)\Bigg]$$

Where as

$x_{(2i-1)} = a+(2i-1)h\;,\; i=1,2,\ldots,n$ and $x_{(2i)} = a+(2i)h\;,\; i=1,2,\ldots,n\text{-}1$

$y_{(2j-1)} = c+(2j-1)h\;,\; j=1,2,\ldots,n$ and $y_{(2j)} = c+(2j)h\;,\; j=1,2,\ldots,n\text{-}1$

$$z_k = e+\frac{(2k-1)}{2}h\;,\; k=1,2,\ldots,2n$$

And the formula of the correction limits(error a formula) are : -

$I - MSS\,(h) = Ah^2 + Bh^4 + Ch^6 + \cdots$

such that $A, B, C, \ldots$ are constants.

Proof We can be written the triple integral I in genera by the following .

$$I=\int_e^g\int_c^d\int_a^b f\,(x,y,z)dxdydz=MSS\,(h)+E\,(h) \qquad ...(1)$$

The $MSS\,(h)$ is the value of integration numerically with using two rules of the mid-point on the dimension z and Simpson on the two dimensions x, y, and $E\,(h)$ is a series of the correction that can beaded to the values $MSS\,(h)$, and $\frac{g-e}{2m}=\frac{d-c}{2n_1}=\frac{b-a}{2n_2}$ such that $2m=2n_1=2n_2$.

The error formula for the single integrals with a continuous integrands by using the mid-point rule is:

$$E_M(h)=\frac{1}{6}h^2(f'_{2n}-f'_0)-\frac{7}{360}h^4(f_{2n}^{(3)}-f_0^{(3)})+\frac{31}{15120}h^6(f_{2n}^{(5)}-f_0^{(5)})-\cdots \qquad ...(2)$$

And by using Simpson's rule is:

$$E_S(h)=-\frac{1}{180}h^4(f_{2n}^{(3)}-f_0^{(3)})+\frac{1}{1512}h^6(f_{2n}^{(5)}-f_0^{(5)})-\cdots \qquad ...(3)$$

and by using the mean-value theorem for derivatives with the formulas (2) and(3) we are obtaining :

$$E_M(h)=\frac{(x_{2n}-x_0)}{6}h^2f^{(2)}(\eta_1)-\frac{7(x_{2n}-x_0)}{360}h^4f^{(4)}(\eta_2)+\frac{31(x_{2n}-x_0)}{15120}h^6f^{(6)}(\eta_3)-... \qquad ...(4)$$

$$E_S(h)=\frac{-(x_{2n}-x_0)}{180}h^4f^{(4)}(\mu_1)+\frac{(x_{2n}-x_0)}{1512}h^6f^{(6)}(\mu_2)+... \qquad ...(5)$$

Such that $i=1,2,3,...$, $\mu_i,\eta_i\in(x_0,x_{2n})$. Akkar[5]

With respect to the single integration $\int_a^b f\,(x,y,z)dx$ we can calculate it numerically by Simpson's rule on the dimension x and (dealing with y and z as constants) and it's valued:

$$\int_a^b f\,(x,y,z)dz=\frac{h}{3}\left(f\,(a,y,z)+f\,(b,y,z)+4\sum_{i=1}^{n}f\,(x_{(2i-1)},y,z)+2\sum_{i=1}^{n-1}f\,(x_{(2i)},y,z)\right)$$
$$-\frac{(b-a)h^4}{180}\frac{\partial^4 f\,(\eta_1,y,z)}{\partial x^4}+\frac{(b-a)h^6}{1512}\frac{\partial^6 f\,(\eta_2,y,z)}{\partial x^6}+... \qquad \cdots(6)$$

Such that

$\eta_1,\eta_2,\ldots\in(a,b)$, $i=1,2,...,n\text{-}1$, $x_{2i}=a+2ih$, $i=1,2,...,n$, $x_{2i\text{-}1}=a+(2i\text{-}1)h$

So by integrated the formula (6) numerically on the interval [c,d] also by using Simpson's rule on the dimension y we are obtaining:

$$\int_c^d\int_a^b f\,(x,y,z)dzdy=\frac{h^2}{9}\Bigg[\;f\,(a,c,z)+f\,(b,c,z)+f\,(a,d,z)+f\,(b,d,z)$$
$$+4\sum_{j=1}^{n}\left(f\,(a,y_{(2j-1)},z)+f\,(b,y_{(2j-1)},z)\right)+2\sum_{j=1}^{n-1}\left(f\,(a,y_{(2j)},z)+f\,(b,y_{(2j)},z)\right)$$
$$+4\sum_{i=1}^{n}\left(f\,(x_{(2i-1)},c,z)+f\,(x_{(2i-1)},d,z)+4\sum_{j=1}^{n}f\,(x_{(2i-1)},y_{(2j-1)},z)+2\sum_{j=1}^{n-1}f\,(x_{(2i-1)},y_{(2j)},z)\right)$$
$$+2\sum_{i=1}^{n-1}\left(\;f\,(x_{(2i)},c,z)+f\,(x_{(2i)},d,z)+4\sum_{j=1}^{n}f\,(x_{(2i)},y_{(2j-1)},z)+2\sum_{j=1}^{n-1}f\,(x_{(2i)},y_{(2j)},z)\right)\Bigg]$$
$$+\int_c^d\left[-\frac{(b-a)h^4}{180}\frac{\partial^4 f\,(\eta_1,y,z)}{\partial x^4}+\frac{(b-a)h^6}{1512}\frac{\partial^6 f\,(\eta_2,y,z)}{\partial x^6}+\cdots\right]dy+\frac{h}{3}\left[\frac{-(d-c)}{180}h^4\frac{\partial^4 f\,(a,\xi_1,z)}{\partial y^4}\right.$$

$$+\frac{(d-c)}{1512}h^{6}\frac{\partial^{6}f(a,\xi_{2},z)}{\partial y^{6}}+\cdots-\frac{(d-c)}{180}h^{4}\frac{\partial^{4}f(b,\xi_{1},z)}{\partial y^{4}}+\frac{(d-c)}{1512}h^{6}\frac{\partial^{6}f(b,\xi_{2},z)}{\partial y^{6}}+\ldots$$

$$+4\sum_{i=1}^{n}\left(\frac{-(d-c)}{180}h^{4}\frac{\partial^{4}f(x_{(2i-1)},\xi_{1i},z)}{\partial y^{4}}+\frac{(d-c)}{1512}h^{6}\frac{\partial^{6}f(x_{(2i-1)},\xi_{2i},z)}{\partial y^{6}}+\ldots\right)$$

$$+2\sum_{i=1}^{n-1}\left(\frac{-(d-c)}{180}h^{4}\frac{\partial^{4}f(x_{2i},\xi_{1i},z)}{\partial y^{4}}+\frac{(d-c)}{1512}h^{6}\frac{\partial^{6}f(x_{2i},\xi_{2i},z)}{\partial y^{6}}+\cdots\right)\Bigg]\qquad\ldots(7)$$

$r=1,2,\ldots,\;\; \xi_{ri}\in(a,b),\quad i=1,2,\ldots n,\; x_{(2i-1)}=a+(2i-1)h,\quad i=1,2,\ldots n-1,\; x_{(2i)}=a+2ih$

$j=1,2,\ldots n\;,\; y_{(2j-1)}=c+(2j-1)h\quad,\quad j=1,2,\ldots n-1\;,\; y_{(2j)}=c+2jh$

And by integrated the formula (7) numerically on the interval [e,g] by using mid-point rule on the dimension z we are obtaining:

$$\int_{e}^{g}\int_{c}^{d}\int_{a}^{b}f(x,y,z)dxdydz=\frac{h^{3}}{9}\Big[f(a,c,z_{k})+f(b,c,z_{k})+f(a,d,z_{k})+f(b,d,z_{k})$$

$$+4\sum_{j=1}^{n}\left(f(a,y_{(2j-1)},z_{k})+f(b,y_{(2j-1)},z_{k})\right)+2\sum_{j=1}^{n-1}\left(f(a,y_{(2j)},z_{k})+f(b,y_{(2j)},z_{k})\right)$$

$$+4\sum_{i=1}^{n}\left(f(x_{(2i-1)},c,z_{k})+f(x_{(2i-1)},d,z_{k})+4\sum_{j=1}^{n}f(x_{(2i-1)},y_{(2j-1)},z_{k})+2\sum_{j=1}^{n-1}f(x_{(2i-1)},y_{(2j)},z_{k})\right)$$

$$+2\sum_{i=1}^{n-1}\left(f(x_{(2i)},c,z_{k})+f(x_{(2i)},d,z_{k})+4\sum_{j=1}^{n}f(x_{(2i)},y_{(2j-1)},z_{k})+2\sum_{j=1}^{n-1}f(x_{(2i)},y_{(2j)},z_{k})\right)\Bigg]$$

$$+\int_{e}^{g}\int_{c}^{d}\left[-\frac{(b-a)h^{4}}{180}\frac{\partial^{4}f(\eta_{1},y,z)}{\partial x^{4}}+\frac{(b-a)h^{6}}{1512}\frac{\partial^{6}f(\eta_{2},y,z)}{\partial x^{6}}+\cdots\right]dydz+\frac{h}{3}\int_{e}^{g}\Bigg[\frac{-(d-c)}{180}h^{4}\frac{\partial^{4}f(a,\xi_{1},z)}{\partial y^{4}}$$

$$+\frac{(d-c)}{1512}h^{6}\frac{\partial^{6}f(a,\xi_{2},z)}{\partial y^{6}}+\cdots-\frac{(d-c)}{180}h^{4}\frac{\partial^{4}f(b,\xi_{1},z)}{\partial y^{4}}+\frac{(d-c)}{1512}h^{6}\frac{\partial^{6}f(b,\xi_{2},z)}{\partial y^{6}}+\ldots$$

$$+4\sum_{i=1}^{n}\left(\frac{-(d-c)}{180}h^{4}\frac{\partial^{4}f(x_{2i-1},\xi_{1i},z)}{\partial y^{4}}+\frac{(d-c)}{1512}h^{6}\frac{\partial^{6}f(x_{2i-1},\xi_{2i},z)}{\partial y^{6}}+\ldots\right)$$

$$+2\sum_{i=1}^{n-1}\left(\frac{-(d-c)}{180}h^{4}\frac{\partial^{4}f(x_{2i},\xi_{1i},z)}{\partial y^{4}}+\frac{(d-c)}{1512}h^{6}\frac{\partial^{6}f(x_{2i},\xi_{2i},z)}{\partial y^{6}}+\cdots\right)\Bigg]dz$$

$$+\frac{h^{2}}{9}\Bigg[\frac{(g-e)}{6}h^{2}\frac{\partial^{2}f(a,c,\lambda_{1})}{\partial z^{2}}-\frac{7(g-e)}{360}h^{4}\frac{\partial^{4}f(a,c,\lambda_{2})}{\partial z^{4}}+\frac{31(g-e)}{15120}h^{6}\frac{\partial^{6}f(a,c,\lambda_{3})}{\partial z^{6}}-\cdots$$

$$+\frac{(g-e)}{6}h^{2}\frac{\partial^{2}f(a,d,\lambda_{1})}{\partial z^{2}}-\frac{7(g-e)}{360}h^{4}\frac{\partial^{4}f(a,d,\lambda_{2})}{\partial z^{4}}+\frac{31(g-e)}{15120}h^{6}\frac{\partial^{6}f(a,d,\lambda_{3})}{\partial z^{6}}-\cdots$$

$$+\frac{(g-e)}{6}h^{2}\frac{\partial^{2}f(b,c,\lambda_{1})}{\partial z^{2}}-\frac{7(g-e)}{360}h^{4}\frac{\partial^{4}f(b,c,\lambda_{2})}{\partial z^{4}}+\frac{31(g-e)}{15120}h^{6}\frac{\partial^{6}f(b,c,\lambda_{3})}{\partial z^{6}}-\cdots$$

$$+\frac{(g-e)}{6}h^{2}\frac{\partial^{2}f(b,d,\lambda_{1})}{\partial z^{2}}-\frac{7(g-e)}{360}h^{4}\frac{\partial^{4}f(b,d,\lambda_{2})}{\partial z^{4}}+\frac{31(g-e)}{15120}h^{6}\frac{\partial^{6}f(b,d,\lambda_{3})}{\partial z^{6}}-\cdots$$

$$+4\sum_{j=1}^{n}\Bigg[\frac{(g-e)}{6}h^{2}\frac{\partial^{2}f(a,y_{2j-1},\lambda_{1j})}{\partial z^{2}}-\frac{7(g-e)}{360}h^{4}\frac{\partial^{4}f(a,y_{2j-1},\lambda_{2j})}{\partial z^{4}}+\frac{31(g-e)}{15120}h^{6}\frac{\partial^{6}f(a,y_{2j-1},\lambda_{3j})}{\partial z^{6}}-\cdots$$

$$+\frac{(g-e)}{6}h^{2}\frac{\partial^{2}f(b,y_{2j-1},\lambda_{1j})}{\partial z^{2}}-\frac{7(g-e)}{360}h^{4}\frac{\partial^{4}f(b,y_{2j-1},\lambda_{2j})}{\partial z^{4}}+\frac{31(g-e)}{15120}h^{6}\frac{\partial^{6}f(b,y_{2j-1},\lambda_{3j})}{\partial z^{6}}-\cdots\Bigg]$$

$$+2\sum_{j=1}^{n}\Bigg[\frac{(g-e)}{6}h^{2}\frac{\partial^{2}f(a,y_{2j},\lambda_{1j})}{\partial z^{2}}-\frac{7(g-e)}{360}h^{4}\frac{\partial^{4}f(a,y_{2j},\lambda_{2j})}{\partial z^{4}}+\frac{31(g-e)}{15120}h^{6}\frac{\partial^{6}f(a,y_{2j},\lambda_{3j})}{\partial z^{6}}-\cdots$$

$$+\frac{(g-e)}{6}h^2\frac{\partial^2 f(b,y_{2j},\lambda_{1j})}{\partial z^2}-\frac{7(g-e)}{360}h^4\frac{\partial^4 f(b,y_{2j},\lambda_{2j})}{\partial z^4}+\frac{31(g-e)}{15120}h^6\frac{\partial^6 f(b,y_{2j},\lambda_{3j})}{\partial z^6}-\cdots\Bigg]$$

$$+4\sum_{i=1}^{n}\Bigg[\frac{(g-e)}{6}h^2\frac{\partial^2 f(x_{2i-1},c,\lambda_{1i})}{\partial z^2}-\frac{7(g-e)}{360}h^4\frac{\partial^4 f(x_{2i-1},c,\lambda_{2i})}{\partial z^4}+\frac{31(g-e)}{15120}h^6\frac{\partial^6 f(x_{2i-1},c,\lambda_{3i})}{\partial z^6}-\cdots$$

$$+\frac{(g-e)}{6}h^2\frac{\partial^2 f(x_{2i-1},d,\lambda_{1i})}{\partial z^2}-\frac{7(g-e)}{360}h^4\frac{\partial^4 f(x_{2i-1},d,\lambda_{2i})}{\partial z^4}+\frac{31(g-e)}{15120}h^6\frac{\partial^6 f(x_{2i-1},d,\lambda_{3i})}{\partial z^6}-\cdots$$

$$+4\sum_{j=1}^{n}\left(\frac{(g-e)}{6}h^2\frac{\partial^2 f(x_{(2i-1)},y_{(2j-1)},\lambda_{1ij})}{\partial z^2}-\frac{7(g-e)}{360}h^4\frac{\partial^4 f(x_{(2i-1)},y_{(2j-1)},\lambda_{2ij})}{\partial z^4}+\frac{31(g-e)}{15120}h^6\frac{\partial^6 f(x_{(2i-1)},y_{(2j-1)},\lambda_{3ij})}{\partial z^6}-\cdots\right)$$

$$+2\sum_{j=1}^{n-1}\left(\frac{(g-e)}{6}h^2\frac{\partial^2 f(x_{(2i-1)},y_{(2j)},\lambda_{1ij})}{\partial z^2}-\frac{7(g-e)}{360}h^4\frac{\partial^4 f(x_{(2i-1)},y_{(2j)},\lambda_{2ij})}{\partial z^4}+\frac{31(g-e)}{15120}h^6\frac{\partial^6 f(x_{(2i-1)},y_{(2j)},\lambda_{3ij})}{\partial z^6}-\cdots\right)\Bigg]$$

$$+2\sum_{i=1}^{n-1}\Bigg[\frac{(g-e)}{6}h^2\frac{\partial^2 f(x_{2i},c,\lambda_{1i})}{\partial z^2}-\frac{7(g-e)}{360}h^4\frac{\partial^4 f(x_{2i},c,\lambda_{2i})}{\partial z^4}+\frac{31(g-e)}{15120}h^6\frac{\partial^6 f(x_{2i},c,\lambda_{3i})}{\partial z^6}-\cdots$$

$$+\frac{(g-e)}{6}h^2\frac{\partial^2 f(x_{2i},d,\lambda_{1i})}{\partial z^2}-\frac{7(g-e)}{360}h^4\frac{\partial^4 f(x_{2i},d,\lambda_{2i})}{\partial z^4}+\frac{31(g-e)}{15120}h^6\frac{\partial^6 f(x_{2i},d,\lambda_{3i})}{\partial z^6}-\cdots$$

$$+4\sum_{j=1}^{n}\left(\frac{(g-e)}{6}h^2\frac{\partial^2 f(x_{(2i)},y_{(2j-1)},\lambda_{1ij})}{\partial z^2}-\frac{7(g-e)}{360}h^4\frac{\partial^4 f(x_{(2i)},y_{(2j-1)},\lambda_{2ij})}{\partial z^4}+\frac{31(g-e)}{15120}h^6\frac{\partial^6 f(x_{(2i)},y_{(2j-1)},\lambda_{3ij})}{\partial z^6}-\cdots\right)$$

$$+2\sum_{j=1}^{n-1}\left(\frac{(g-e)}{6}h^2\frac{\partial^2 f(x_{(2i)},y_{(2j)},\lambda_{1ij})}{\partial z^2}-\frac{7(g-e)}{360}h^4\frac{\partial^4 f(x_{(2i)},y_{(2j)},\lambda_{2ij})}{\partial z^4}+\frac{31(g-e)}{15120}h^6\frac{\partial^6 f(x_{(2i)},y_{(2j)},\lambda_{3ij})}{\partial z^6}-\cdots\right)\Bigg]\Bigg]$$

Where as

$x_{(2i-1)}=a+(2i-1)h\ ,\ i=1,2,\ldots,n$ and $x_{(2i)}=a+(2i)h\ ,\ i=1,2,\ldots,n\text{-}1$

$y_{(2j-1)}=c+(2j-1)h\ ,\ j=1,2,\ldots,n$ and $y_{(2j)}=c+(2j)h\ ,\ j=1,2,\ldots,n\text{-}1$

$z_{(k)}=e+\frac{(2k-1)}{2}h\ ,\ k=1,2,\ldots,2n$

Since $\frac{\partial^4 f}{\partial x^4},\frac{\partial^6 f}{\partial x^6},\ldots$ and $\frac{\partial^4 f}{\partial y^4},\frac{\partial^6 f}{\partial y^6},\ldots$ and $\frac{\partial^2 f}{\partial z^2},\frac{\partial^4 f}{\partial z^4},\ldots$ are continuous in each point from the region $[a,b]\times[c,d]\times[e,g]$.

So, a formula of the correction term to triple integrals I by MSS rule becomes:-

$$E_{MSS}(h)=(g-e)(d-c)(b-a)\left(\frac{-h^4}{180}\frac{\partial^4 f\left(\overline{\overline{\overline{n_1}}},\overline{\overline{\overline{\mu_1}}},\overline{\overline{\overline{\kappa_1}}}\right)}{\partial x^4}+\frac{h^6}{1512}\frac{\partial^6 f\left(\overline{\overline{\overline{n_2}}},\overline{\overline{\overline{\mu_2}}},\overline{\overline{\overline{\kappa_2}}}\right)}{\partial x^6}-\cdots\right)+(g-e)(d-c)(b-a)\left(\frac{-h^4}{180}\frac{\partial^4 f\left(\overline{\overline{n_1}},\overline{\overline{\mu_1}},\overline{\overline{\kappa_1}}\right)}{\partial y^4}\right.$$

$$\left.+\frac{h^6}{1512}\frac{\partial^6 f\left(\overline{\overline{n_2}},\overline{\overline{\mu_2}},\overline{\overline{\kappa_2}}\right)}{\partial y^6}-\cdots\right)+(g-e)(d-c)(b-a)\left(\frac{h^2}{6}\frac{\partial^2 f\left(\overline{n_1},\overline{\mu_1},\overline{\kappa_1}\right)}{\partial z^2}-\frac{7h^4}{360}\frac{\partial^4 f\left(\overline{n_2},\overline{\mu_2},\overline{\kappa_2}\right)}{\partial z^4}+\frac{31h^6}{15120}\frac{\partial^6 f\left(\overline{n_3},\overline{\mu_3},\overline{\kappa_3}\right)}{\partial z^6}-\cdots\right)$$

$$E_{MSS}(h)=(g-e)(d-c)(b-a)\frac{h^2}{6}\frac{\partial^2 f\left(\overline{n_1},\overline{\mu_1},\overline{\kappa_1}\right)}{\partial z^2}-(g-e)(d-c)(b-a)\frac{h^4}{180}\left(\frac{\partial^4 f\left(\overline{\overline{\overline{n_1}}},\overline{\overline{\overline{\mu_1}}},\overline{\overline{\overline{\kappa_1}}}\right)}{\partial x^4}+\frac{\partial^4 f\left(\overline{\overline{n_1}},\overline{\overline{\mu_1}},\overline{\overline{\kappa_1}}\right)}{\partial y^4}+\frac{7}{2}\frac{\partial^4 f\left(\overline{n_2},\overline{\mu_2},\overline{\kappa_2}\right)}{\partial z^4}\right)$$

$$+(g-e)(d-c)(b-a)\frac{h^6}{1512}\left(\frac{\partial^6 f\left(\overline{\overline{\overline{n_2}}},\overline{\overline{\overline{\mu_2}}},\overline{\overline{\overline{\kappa_2}}}\right)}{\partial x^6}+\frac{\partial^6 f\left(\overline{\overline{n_2}},\overline{\overline{\mu_2}},\overline{\overline{\kappa_2}}\right)}{\partial y^6}+\frac{31}{10}\frac{\partial^6 f\left(\overline{n_3},\overline{\mu_3},\overline{\kappa_3}\right)}{\partial z^6}\right)+\cdots \qquad \cdots (7)$$

$$\left(\overline{n_1},\overline{\mu_1},\overline{\kappa_1}\right),\left(\overline{n_2},\overline{\mu_2},\overline{\kappa_2}\right),\ldots\in[a,b]\times[c,d]\times[e,g] \quad , \quad \left(\overline{\overline{n_1}},\overline{\overline{\mu_1}},\overline{\overline{\kappa_1}}\right),\left(\overline{\overline{n_2}},\overline{\overline{\mu_2}},\overline{\overline{\kappa_2}}\right),\ldots\in[a,b]\times[c,d]\times[e,g]$$

$$\left(\overline{\overline{\overline{n_1}}},\overline{\overline{\overline{\mu_1}}},\overline{\overline{\overline{\kappa_1}}}\right),\left(\overline{\overline{\overline{n_2}}},\overline{\overline{\overline{\mu_2}}},\overline{\overline{\overline{\kappa_2}}}\right),\ldots\in[a,b]\times[c,d]\times[e,g]$$

So, if the integrand was continuous function and it's partial derivatives exist in each point from the integral region $[a,b]\times[c,d]\times[e,g]$ we can write the error formula to the rule stated as : -

$$I-MSS(h)=Ah^2+Bh^4+Ch^6+\ldots \qquad \ldots(8)$$

Such that $A, B, \ldots$ are constants depend on the partial derivatives for the function in the integral region, thus the proof is completed.

Moreover using $RMSS$ method region calculate the triple integrals setting $2m=2n_1=2n_2=2$ in above formula then calculate the approximate value for the triple integration which is equal :-

$$\int_e^g\int_c^d\int_a^b f(x,y,z)dxdydz=\frac{h^3}{9}\sum_{k=1}^{2}\Big[f(a,c,z_k)+f(a,d,z_k)+f(b,c,z_k)+f(b,d,z_k)$$

$$+4\left(f(a,c+h,z_k)+f(a+h,c,z_k)+f(a+h,d,z_k)+f(b,c+h,z)+4f(a+h,c+h,z_k)\right)\Big]$$

And we put in our tables this approximate value when $2m=2n_1=2n_2=2$ then puts $2m=2n_1=2n_2=4$ and calculate MSS where it is equal :-

$$\int_e^g\int_c^d\int_a^b f(x,y,z)dxdydz=\frac{h^3}{9}\sum_{k=1}^{4}\Big[f(a,c,z_k)+f(a,d,z_k)+f(b,c,z_k)+f(b,d,z_k)+2\Big(f(a,y_2,z_k)+f(b,y_2,z_k)$$

$$f(x_2,c,z_k)+f(x_2,d,z_k)+2f(x_2,y_2,z_k)+4\sum_{j=1}^{2}f(x_2,y_{(2j-1)},z_k)\Big)+4\sum_{j=1}^{2}\left(f(a,y_{(2j-1)},z_k)+f(b,y_{(2j-1)},z_k)\right)$$

$$+4\sum_{i=1}^{2}\left(f(x_{(2i-1)},c,z_k)+f(x_{(2i-1)},d,z_k)+2f(x_{(2i-1)},y_2,z_k)+4\sum_{j=1}^{2}f(x_{(2i-1)},y_{(2j-1)},z_k)\right)\Big]$$

Also, we can put this value in our tables as the approximate value for the triple integration *I*. And we can improve the two approximate values which were obtained by applied a Romberg accelerating method on it's, as result we obtain on the approximate value for the triple integration by a Romberg accelerating method with MSS rule. And so we continue by application MSS rule for the other values $2m=2n_1=2n_2>4$

then applies on it a Romberg accelerating method to accelerating a values approach to the real value for integral to get high accuracy which we are choosing.

3. Examples

1. $I=\int_1^2\int_1^2\int_1^2 \ln(x+y+z)dxdydz$ its real value is 1.4978022885754 [approach to thirteen decimals]

2. $I=\int_2^3\int_1^2\int_0^1 xe^{-(x+y+z)}dxdydz$ its real value is 0.005256743455 [approach to twelve decimals]

3. $I=\int_0^1\int_0^1\int_0^1 \sin\left(\frac{\pi}{2}(x+y+z)\right)dxdydz$ its real value is0.5160245509312[approach to twelve decimals]

4 .The results :

The integral $I=\int_1^2\int_1^2\int_1^2 \ln(x+y+z)dxdydz$ has defined integrand for all $(x,y,z)\in[1,2]\times[1,2]\times[1,2]$

So that a formula of the correction terms for the integration be similar to formula (8) and by using $RMSS$ method we got the written results in a table (1)as following:-

$2m=2n_1=2n_2$	Values of MSS rule	k=2	k=4	k=6	k=8
2	1.4983244961941				
4	1.4979351308343	1.4978053423810			
8	1.4978356476677	1.4978024866122	1.4978022962276		
16	1.4978106377219	1.4978023010733	1.4978022887041	1.4978022885847	
32	1.4978043764493	1.4978022893584	1.4978022885774	1.4978022885754	1.4978022885754

Table (1) calculate of the triple integration $I=\int_1^2\int_1^2\int_1^2 \ln(x+y+z)dxdydz=1.4978022885754$

We note from the table when $2m=2n_1=2n_2=32$ the integral value by using MSS rule is correct to five decimals and by using Romberg accelerating method with the rule stated we obtain on Matching value of the real value[approach to three decimals] and with(2^{15} subinterval).

Also, the integral $I=\int_2^3\int_1^2\int_0^1 xe^{-(x+y+z)}dxdydz$ has defined integrand for all $(x,y,z)\in[0,1]\times[1,2]\times[2,3]$ .So that a formula of the correction terms for the integration be similar to formula (8) and the table (2) shows a calculation for above integral numerically by using $RMSS$ method.

$2m=2n_1=2n_2$	Values of MSS rule	k=2	k=4	k=6	k=8
2	0.0051893458660				
4	0.0052422280766	0.0052598554801			
8	0.0052532688882	0.0052569491587	0.0052567554040		
16	0.0052558845930	0.0052567564946	0.0052567436503	0.0052567434638	
32	0.0052565293529	0.0052567442729	0.0052567434581	0.0052567434551	0.0052567434550

Table (2) calculate of the triple integration $I=\int_2^3\int_1^2\int_0^1 xe^{-(x+y+z)}dxdydz=0.005256743455$

We conclude from the table when $2m = 2n_1 = 2n_2 = 32$ the integral value by using MSS rule is correct to six decimals and by using Romberg accelerating method with the rule stated we obtains on Matching value of the real value [approach to twelve decimals] and with(2^{15} subinterval).

Also, the integral $I = \int_0^1\int_0^1\int_0^1 \sin\left(\frac{\pi}{2}(x+y+z)\right)dxdydz$ has defined integrand for all $(x,y,z) \in [0,1]\times[0,1]\times[0,1]$ .So that a formula of the correction terms for the integration be similar to formula (8) and by Applying $RMSS$ method we got the written results in a table (3) as following:-

$2m = 2n_1 = 2n_2$	Values of MSS rule	K=2	k=4	k=6	k=8	k=10
2	0.531947303999					
4	0.519495057715	0.515344308953				
8	0.516862991173	0.515985635659	0.516028390773			
16	0.516232375660	0.516022170489	0.516024606144	0.516024546071		
32	0.516076396125	0.516024402946	0.516024551777	0.516024550914	0.516024550933	
64	0.516037505302	0.516024541695	0.516024550944	0.516024550931	0.516024550931	0.516024550931

Table (2) calculate of the triple integration $I = \int_0^1\int_0^1\int_0^1 \sin\left(\frac{\pi}{2}(x+y+z)\right)dxdydz = 0.516024550931$

We note from the table when $2m = 2n_1 = 2n_2 = 32$ the integral value by using MSS rule is correct to four decimals and by using Romberg accelerating method we obtain on the real value[approach to eleven decimals] with (2^{15} subinterval) as well as we obtain on Matching value of the real value [approach to twelve decimals] with (2^{18} subinterval).

5. The discussion

It is clear according to tables results of this research that when we calculate the approximate value for the triple integration with the continuous integrands by the composition rule from the two rules- Mid-point rule on the dimension z and Simpson's rule on the two dimensions x and y- when the number of subintervals of interval of interior dimension equal to the number of subintervals of interval of middle dimension and equal to the number of subintervals of exterior dimension. This rule–namely MSS rule- gives a good values (to many decimals) comparative with the real values for the integrations and by using a number subintervals without using an operation of the external modification on its, e.g. in the two integrations-first and second- we got on correct value for five and six decimals respectively with (2^{15} subinterval), and in the third integration the correct value was four decimals and with (2^{18} subinterval).

But when we are using Romberg accelerating method with MSS rule we can giving a best results with respect to speed of the approach with a few number from subintervals comparatively to a values of the real integrations because it was Matching value of the real value in the two integrations - first and second – when $2m = 2n_1 = 2n_2 = 32$,And in the third integration when $2m = 2n_1 = 2n_2 = 64$. Thus, we can depend on $RMSS$ method in a calculation the triple integrals with continuous integrands.

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