مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
Journal of the College of Basic Education Vol.30 (NO. 126) 2024, pp. 162-173

Variational Iteration Method for Solving First Kind Bessel Differential Equation of Order Zero

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Abstract

The aim of this paper is to study the first kind Bessel differential equation of order zero. The variational iteration method (VIM) is used to obtain the approximate solutions of first kind Bessel differential equation (BDE) of order zero. The approximate solution of this equation is calculated in the form of a series which its components are computed, can be calculated rapidlyon computer systems and the convergence of this method is studied. Also two numerical examples are given . The findings, which illustrate the effectiveness and precision of the suggested strategy, are presented in tables and figures.

Keywords: Variational Iteration Method, Bessel Differential Equation, Analytic solution, Approximate solution.

1. Introduction

Ordinary differential equations (ODEs) are frequently occured as mathematical models in many branches of science, engineering and economy. Many models have no closed form solutions and hence it is a need to find approximate solutions by means of numerical methods. Moreover, Bessel differential equation (BDE) is considered among the most important ODEs due to its wide applications in heat transfer, vibrations, stress analysis, optics, signal processing and fluid mechanics [1-3].

Although the German mathematician and astronomer Friedrich Wilhelm Bessel (1784–1846) is credited with giving this equation its name, Daniel Bernoulli was the first to propose the idea of Bessel functions (BFs) in 1732. Its comes up in many engineering applications such as heat transfer involving the analysis of circular fins, vibration analysis, stress analysis and fluid mechanics. There are various numerical methods to solve such problems, for example, in 2016, Entisar and Magdi are used different methods to found the numerical solution of BDe of order zero [7]. In 2019, Falade Andubakar are found the numerical solution of BDE of order zero by using chebyshev polynomials of first kind [8]. Many researchers studied the variational iteration method. In 2019, Mohamed and Kacem are found the numerical solution of ODEs by using VIM [4]. This method is one of the few numerical methods which can be solved without needing to a computer program unlike other many numerical methods which they need to use the computer programs [10]. In 2022, Ndipmong and Udech are studying the numerical solution of first order non linear ordinary differential equations by using VIM [5]. In 2022, Huseyin Kayabasetal applied VIM to found the numerical solution of second order liner non-homogeneous ODEs with constant coefficients [6].

The continuous BDE is first described in this paper. Next, the VIM form of the problem is discovered. Finally, the approximate solution of the equation is computed as a series with computed components, which can be quickly calculated on computer systems. Finally, the convergence of this method is examined. An algorithm to show the steps of solution is given. Lastly, illustrations examples are presented to solve different problems using the VIM method and the computations have been performed by using Matlab R2010b. The findings, which illustrate the effectiveness and precision of the suggested strategy, are presented in tables and figures.

Definition(1.1): Bessel differential equation of order r is second order ODE of the form:

$$t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}+(t^2-r^2)y=0 \quad ,r\geq 0 \qquad (1)$$

Definition(1.2): Bessel functions $J_r(t)$ of the first kind of order r are the solutions of BDE Eq. (1) which are finite at $t = 0$, for integer or positive r, and diverge when t approaches zero for negative non-integer r.

By using the series expansion of the Bessel functions of the first kind, $J_r(t)$ around $t = 0$, we can define $J_r(t)$ in term of gamma function as :

$$J_r(t)=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!\Gamma(s+r+1)}\left(\frac{t}{2}\right)^{2s+r} \qquad (2)$$

Since $\Gamma(s+r+1) = (s+r)!$ then (2) can be written as:

$$J_r(t)=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(s+r)!}\left(\frac{t}{2}\right)^{2s+r} \qquad (3)$$

See [11] the formula

$$J_{-r}(t)=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!\Gamma(s-r+1)}\left(\frac{t}{2}\right)^{2s-r} . \qquad (4)$$

is obtained by replacing r in Eq.(2) by –r. If r is positive integer, for s=0,1,2,3,….,r-1 the value s-r+1≤ 0 and then all the coefficients in Eq.(4) are

not defined since gamma function is not defined at zero and for negative integer, $J_r(t)$ and $J_r(t)$ are linearly independent only when r is not an integer.
The BDEs of the second kind, $Y_r(t)$ are solutions of the BDE that have a singularity at t=0, and are multivalued.
When r is not an integer $Y_r(t)$ defined in terms of $J_r(t)$ and $J_{-r}(t)$ as

$$Y_r(t) = \frac{J_r(t)\cos(r\pi) - J_{-r}}{\sin(r\pi)} \quad . \tag{5}$$

2.General Solution of BDE of order r

If r is not an integer , the general solution of BDE of order r in Eq.(1) is of the form:
$y = a_1 J_r(t) + a_2 Y_r(t)$ where a_1 and a_2 are given constants.
The values for the Bessel functions can be found in most collections of mathematical tables. .
Definition(2.1):BDE of order zero is the second order ODE of the form

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + t^2y = 0 \quad . \tag{6}$$

3.The Variation iteration method for solving Bessel differential equation

We consider the following linear differential operator (LDO)
$Ly = 0$ where L is a linear operator (7)

The following correction functional according to the VIM is constructed

$$y_{n+1}(t) = y_n(t) + \int_0^t \mu(v)\big(Ly_n(v)\big)\,dv \tag{8}$$

Where μ is the Lagrange multiplier (LM) which can be determined optimally.

4. Formulation of Approximate LM

Applying restriction on both sides of Eq. (8) to get

$$\xi\, y_{n+1}(t) = \xi\, y_n(t) + \xi \int_0^t \mu(v)\big(Ly_n(v)\big)\,dv \tag{9}$$

Now, the linear is solved by using integration by parts to fine out the value of μ

$$\xi\, y_{n+1}(t) = \xi\, y_n(t) + \xi \int_0^t \mu(v)\, Ly_n(v)\,dv$$

$$\xi\, y_{n+1}(t) = \xi\, y_n(t) + \xi \int_0^t \mu(v)\, y_n'(v)\, dv$$

$$\xi\, y_{n+1}(t) = \xi\, y_n(t) + \xi \left[\mu(v)\, y_n(v) - \int_0^t y_n(v)\, \mu'(v)\right] dv$$

$$\xi\, y_{n+1}(t) = \xi y_n(t) + \xi\, \mu(v)\, y_n(v) - \xi \int_0^t y_n(v)\, \mu'(v)\, dv$$

$$0 = \xi y_n(t) + \xi\, \mu(t)\, y_n(t) - \xi \int_0^t y_n(v)\, \mu'(v)\, dv$$

$$0 = \xi y_n(t)\big(1 + \mu(t)\big) - \xi \int_0^t y_n(v)\, \mu'(v)\, dv \qquad (10)$$

Equating the terms on both sides
$\xi y_n(t);$ $1 + \mu(t) = 0$

$\xi y_n(t);$ $\mu'(v) = 0$

Eq. (10) is the stationary conditions. By using first stationary condition
$1 + \mu(v) = 0,$ $\mu(v) = 1$

By using second stationary condition
$\mu(v) = -1$

So, the value of $\mu(v)$ for first order DE is always -1, for second order

$\mu_1(v) = v - t$, for third order equation $\mu_2(v) = \frac{1}{2!}(v - t)^2$.

Generally, for the n^{th}-order ODE $\mu_n(v) = \frac{(-1)^n}{(n-1)!}(v - t)^{n-1}$.

5.The Algorithm for solving(BDE)

Input: a ,b, n, Bessel j(0,t),initial condition, LM.
Output: The Numerical solution of the BDE.
Step1:Calculate $t_i = a + i\frac{(b-a)}{n}, i = 0,1,\dots,n.$

Step2:Calculate the numerical solution $y_{n+1}(t)$.

Step3:Calculate absolute error is the comparison between the exact and the numerical solutions.
Step4:End.

6. Numerical Results

In this section two numerical examples are solved by using the presented method on the collocation points $t_i = a + i\frac{(b-a)}{n};\ \ i = 0,1,\dots,n$. Two algorithms were written in MATLAB and numerical results were obtained using a computer.

Example 1: Consider the BDE [9]

$$ty'' + y' + ty = 0$$

with initial conditions $y(0) = 1,\ y'(0) = 0$

The exact solution is given by $y(t) = Besselj(0,t)$.

Table 1 Compare with Exact solution and Approximate solution when y_{15}.

t_i	Exact Sol.	$ApproximateSol.y_{15}$	absolute error
0	1.000000000000000	1.000000000000000	0
0.1	0.997501562066040	0.997004681922960	0.4969 * 1.0e-03
0.2	0.990024972239576	0.990074951774010	0.0500 * 1.0e-03
0.3	0.977626246538296	0.977379790406744	0.2465 * 1.0e-03
0.4	0.960398226659563	0.960201867852461	0.1964 * 1.0e-03
0.5	0.938469807240813	0.938939119320296	0.4693 * 1.0e-03
0.6	0.912004863497211	0.912106997914742	0.1021 * 1.0e-03
0.7	0.881200888607405	0.881341450723277	0.1406 * 1.0e-03
0.8	0.846287352750480	0.846402678740088	0.1153 * 1.0e-03
0.9	0.807523798122545	0.807179755276250	0.3440 * 1.0e-03
1.0	0.765197686557967	0.765696192153943	0.4985 * 1.0e-03

Table 2 Compare with Exact solution and Approximate solution when y_{20}.

t_i	Exact Sol.	$Approximate\ Sol.\ y_{20}$	absolute error
0	1.000000000000000	1.000000000000000	0
0.1	0.997501562066040	0.997504687652186	0.0313 * 1.0e-04
0.2	0.990024972239576	0.990075044098262	0.5007 * 1.0e-04
0.3	0.977626246538296	0.977680263376144	0.5402 * 1.0e-04
0.4	0.960398226659563	0.960303387600053	0.9484 * 1.0e-04
0.5	0.938469807240813	0.938442908954705	0.2690 * 1.0e-04
0.6	0.912004863497211	0.912015060496650	0.1020 * 1.0e-04
0.7	0.881200888607405	0.881256844181434	0.5596 * 1.0e-04
0.8	0.846287352750480	0.846229858960755	0.5749 * 1.0e-04
0.9	0.807523798122545	0.807525006675355	0.0121 * 1.0e-04
1.0	0.765197686557967	0.765168168922211	0.2952 * 1.0e-04

Table 3 Compare with Exact solution and Approximate solution when y_{30}.

t_i	Exact Sol.	$ApproximateSol.\ y_{30}$	absolute error
0	1.000000000000000	1.000000000000000	0
0.1	0.997501562066040	0.997501435790212	0.1263 * 1.0e-06
0.2	0.990024972239576	0.990024889264376	0.0830 * 1.0e-06
0.3	0.977626246538296	0.977626172729572	0.0738 * 1.0e-06
0.4	0.960398226659563	0.960398882560987	0.6559 * 1.0e-06
0.5	0.938469807240813	0.938469248485680	0.5588 * 1.0e-06
0.6	0.912004863497211	0.912004693290204	0.1702 * 1.0e-06
0.7	0.881200888607405	0.881200073037386	0.8156 * 1.0e-06
0.8	0.846287352750480	0.846287558432143	0.2057 * 1.0e-06
0.9	0.807523798122545	0.807523107847523	0.6903 * 1.0e-06
1.0	0.765197686557967	0.765197471547620	0.2150 * 1.0e-06

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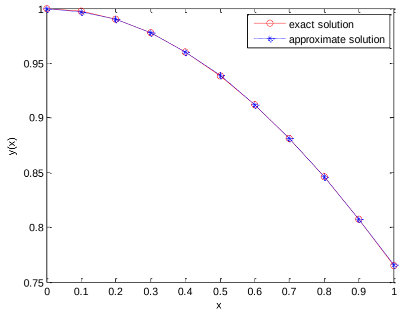
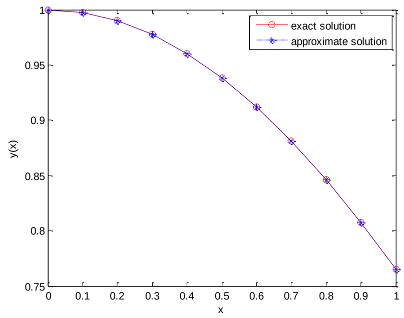


(a)



(b)

Fig. 1 The exact solution and the approximate solution $y(t)$ for (a) $n = 15$

(b) $n = 20$

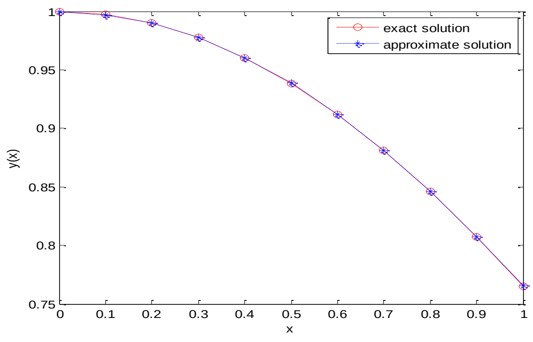


Fig. 2 The exact solution and the approximate solution $y(t)$ for $n = 30$

Example 2: Consider the BDE [7]
$ty'' + y' + 4ty = 0$

with initial conditions $y(0) = 3,\ y'(0) = 0$

The exact solution is given by $y(t) = 3Besselj(0,2t)$.

Table 4 Compare with Exact solution and Approximate solution when y_{15}.

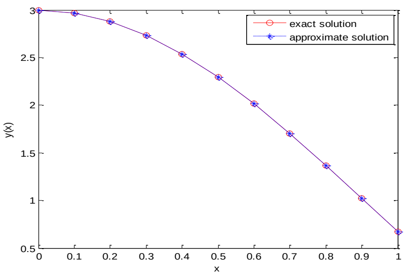
t_i	Exact Sol.	$Approximate\ Sol.\,y_{15}$	absolute error
0	3.000000000000000	3.000000000000000	0
0.1	2.970074916718729	2.970027809228142	0.0471 * 1.0e-03
0.2	2.881194679978690	2.881391416815031	0.1967 * 1.0e-03
0.3	2.736014590491632	2.736434218640069	0.4196 * 1.0e-03
0.4	2.538862058251441	2.538337676235436	0.5244 * 1.0e-03
0.5	2.295593059673900	2.295131896722949	0.4612 * 1.0e-03
0.6	2.013398232793088	2.013386786010991	0.0114 * 1.0e-03
0.7	1.700565361122867	1.700549126330960	0.0162 * 1.0e-03
0.8	1.366206502918142	1.366723337812310	0.5168 * 1.0e-03
0.9	1.019959233127675	1.019970304562114	0.0111 * 1.0e-03
1.0	0.671672337423707	0.671642269797690	0.0301 * 1.0e-03

Table 5 Compare with Exact solution and Approximate solution when y_{20}.

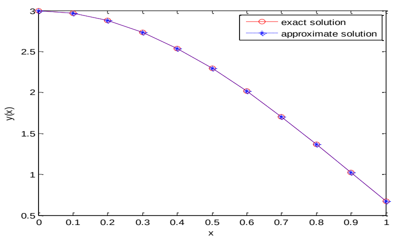
t_i	Exact Sol.	$ApproximateSol.\,y_{20}$	absolute error
0	3.000000000000000	3.000000000000000	0
0.1	2.970074916718729	2.970072440568144	0.0248 * 1.0e-04
0.2	2.881194679978690	2.881190443709395	0.0424 * 1.0e-04
0.3	2.736014590491632	2.736019773255926	-0.0518 * 1.0e-04
0.4	2.538862058251441	2.538866710449065	-0.0465 * 1.0e-04
0.5	2.295593059673900	2.295590556360016	0.0250 * 1.0e-04
0.6	2.013398232793088	2.013355656395818	0.4258 * 1.0e-04
0.7	1.700565361122867	1.700568724196384	-0.0336 * 1.0e-04
0.8	1.366206502918142	1.366207391602961	-0.0089 * 1.0e-04
0.9	1.019959233127675	1.019952322005531	0.0691 * 1.0e-04
1.0	0.671672337423707	0.671672773469909	-0.0044 * 1.0e-04

Table 6 Compare with Exact solution and Approximate solution when y_{30}.

t_i	Exact Sol.	$ApproximateSol.y_{30}$	absolute error
0	3.000000000000000	3.000000000000000	0
0.1	2.970074916718729	2.970074243226963	0.6735 * 1.0e-06
0.2	2.881194679978690	2.881194065620945	0.6144 * 1.0e-06
0.3	2.736014590491632	2.736014089629845	0.5009 * 1.0e-06
0.4	2.538862058251441	2.538862460910995	0.4027 * 1.0e-06
0.5	2.295593059673900	2.295593003430744	0.0562 * 1.0e-06
0.6	2.013398232793088	2.013398088198442	0.1446 * 1.0e-06
0.7	1.700565361122867	1.700565273410107	0.0877 * 1.0e-06
0.8	1.366206502918142	1.366206171507051	0.3314 * 1.0e-06
0.9	1.019959233127675	1.019959974114486	0.7410 * 1.0e-06
1.0	0.671672337423707	0.671672998378258	0.6610 * 1.0e-06



(a)



(b)

Fig. 3 The exact solution and the Approximate solution $y(t)$ for (a) $n = 15$ (b) $n = 20$

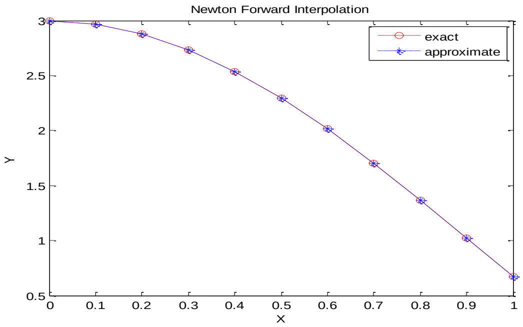


Fig. 4 The exact solution and the approximate solution $y(t)$ for $n = 30$

7. Conclusion

The results obtained from the two given examples have shown that VIM is a powerful and efficient technique for finding an approximate solution of the (BDE) of the first kind and order zero. In Example 1, Table 1 shows small errors, while Table 2 and Table 3 show that the error is the smallest compared to the exact solution. In Example 2, Table 4 shows small errors, while Table 5 and Table 6 show that the error is the smallest compared to the exact solution. Therefore, increasing the number of terms in VIM makes the approximate solution tend towards the exact solution.

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Journal of the College of Basic Education Vol.30 (NO. 126) 2024, pp. 162-173

طريقة التغاير التكرارية لحل معادلة بيسل التفاضلية من النوع الاول ومن الرتبة صفر.

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مستخلص البحث:
الهدف من هذا البحث هو دراسة معادلة بيسل التفاضلية من الرتبة(zero).طريقة التغاير التكرارية أستخدمت لايجاد الحل التقريبي لمعادلة بيسل التفاضلية من النوع الاول ومن الرتبة (zero).تم حساب الحل التقريبي لهذه المعادلة على شكل سلسلة يتم حساب مكوناتها، ويمكن حسابها على أنظمة الكمبيوتر وتمت دراسة التقارب لهذه الطريقة.ايضا تم أعطاء مثالين،النتائج أعطيت بالاشكال والجداول حيث بينت كفاءة ودقة الطريقة المقترحة.
الكلمات المفتاحية: طريقة التغاير التكرارية، معادلة بيسل التفاضلية ،الحل التحليلي ، الحل التقريبي.