

On Semi- α -Connected Subspace

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Abstract:

In this paper, the concept of semi- α -open set will be used to define a new kind of strongly connectedness on a topological subspace namely "semi- α -connectedness". Moreover, we prove that semi- α -connectedness property is a topological property and give an example to show that semi- α -connectedness property is not a hereditary property. Also, we prove that semi- α -irresolute image of a semi- α -connected space is a semi- α -connected space.

Keywords: α -open, semi- α -open, semi- α -irresolute, semi- α -connected subspace, semi- α - τ -separated.

1- Introduction

Njåsted [1] introduced the concept of semi- α -open sets in 1965. This concept depends on the concept of α -open set, see [1], [2], [3] and [4]. Al-Tabatabai [5] introduced semi- α -continuous and semi- α -irresolute functions in 2004. In 2005 Maleki [6] introduced and studied semi- α -connected (resp. α -connected) on a topological space (X, τ) . In this paper a new type of connectedness on a topological subspace, namely semi- α -connected subspace, is introduced and studied. Also, we prove that semi- α -connectedness property is a topological property and give an example to show that semi- α -connectedness property is not a hereditary property. Moreover, we prove that the semi- α -irresolute image of a semi- α -connected space is a semi- α -connected space. Also, we introduce and study a new kind of sets, namely semi- α -separated sets.

2- Preliminaries

2.1 Definition

A subset A of a space (X, τ) is called

(1) an α -open set [1] if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$.

(2) a semi- α -open set [1] if $U \subseteq A \subseteq \text{cl}(U)$, for some α -open set U .

The family of all semi- α -open sets is denoted by $S_{\alpha}O(X)$.

2.2 Remarks [5]

(1) Every α -open set is semi- α -open set, not conversely.

(2) Every open set is α -open set, so it is semi- α -open set, not conversely.

2.3 Definition [6]

A topological space (X, τ) is said to be semi- α -connected (resp. α -connected) space if and only if X can't be expressed as the union of two disjoint semi- α -open (resp. α -open), nonempty subset of X . Otherwise, X is said to be semi- α -disconnected (resp. α -disconnected) space.

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2.4 Proposition, [6]

The continuous image of semi- α -connected space is a connected space.

2.5 Definition

A function $f: (X, \tau) \longrightarrow (Y, \tau')$ is said to be

(1) α -continuous [7] if $f^{-1}(V)$ is an α -open set in (X, τ) for every open set V of (Y, τ') .

(2) α -irresolute [8] if $f^{-1}(V)$ is an α -open set in (X, τ) for every α -open set V of (Y, τ') .

(3) semi- α -continuous [5] if $f^{-1}(V)$ is a semi- α -open set in (X, τ) for every open set V of (Y, τ') .

(4) semi- α -irresolute [5] if $f^{-1}(V)$ is a semi- α -open set in (X, τ) for every semi- α -open set V of (Y, τ') .

2.6 Remarks

(1) Every open and continuous function is α -irresolute, not conversely [9].

(2) Every α -continuous function is a semi- α -continuous, not conversely [5].

(3) Every α -irresolute function is α -continuous and semi- α -continuous, not conversely [5].

2.7 Definition, [10]

Let (X, τ) be a topological space. Two nonempty subset A and B of X are said to be τ -separated sets if and only if $A \cap \text{cl}(B) = \emptyset$ and $\text{cl}(A) \cap B = \emptyset$.

3- Some Properties of Semi- α -Connected Space

In this section, we prove the semi- α -irresolute (resp. α -irresolute, semi- α -continuous) image of a semi- α -connected space is a semi- α -connected (resp. α -connected, connected) space. Also, the relationships between semi- α -irresolute and semi- α -continuous functions have been studied. Moreover,

we prove the semi- α -connectedness property is a topological property.

3.1 Proposition

Every quotient space of a semi- α -connected space is connected.

Proof:

Let f be a function from a topological space X on to quotient space Y . It is known that f is continuous and onto function, so by proposition (2.4), we get Y is connected space.

3.2 Proposition

The semi- α -continuous image of a semi- α -connected space is a connected space.

Proof:

Let $f: (X, \tau) \longrightarrow (Y, \tau')$ be a semi- α -continuous, onto function and X is a semi- α -connected space. We want to prove Y is connected space. Suppose Y is a disconnected space. So, $Y = A \cup B$, where $A \neq \emptyset \neq B$, $A \cap B = \emptyset$ and $A, B \in \tau'$. So, $f^{-1}(Y) = f^{-1}(A \cup B)$, then $X = f^{-1}(A) \cup f^{-1}(B)$, but f is a semi- α -continuous function then $f^{-1}(A), f^{-1}(B)$ are semi- α -open sets in X . Since $A \neq \emptyset \neq B$ and f is onto, then $f^{-1}(A) \neq \emptyset \neq f^{-1}(B)$ and $A \cap B = \emptyset$, so $f^{-1}(A) \cap f^{-1}(B) = \emptyset$. Hence, X is a semi- α -disconnected space, which is a contradiction. Therefore, Y is a connected space.

3.3 Corollary

Every α -continuous image of a semi- α -connected space is a connected space.

Proof:

This following immediately from part (1) of remarks (2.6) and the above proposition.

3.4 Proposition

Every α -irresolute image of a semi- α -connected space is an α -connected space.

Proof:

Let $f : (X, \tau) \longrightarrow (Y, \tau')$ be an α -irresolute, onto function and X is a semi- α -connected space. We want to prove Y is an α -connected space. Suppose Y is an α -disconnected space. So, $Y = A \cup B$, where $A \neq \emptyset \neq B$, $A \cap B = \emptyset$ and A, B are α -open sets in Y . So, $f^{-1}(Y) = f^{-1}(A \cup B)$, then $X = f^{-1}(A) \cup f^{-1}(B)$, but f is an α -irresolute function then $f^{-1}(A)$ and $f^{-1}(B)$ are α -open sets in X . Since $A \neq \emptyset \neq B$ and f is onto, then $f^{-1}(A) \neq \emptyset \neq f^{-1}(B)$ and $A \cap B = \emptyset$, then $f^{-1}(A) \cap f^{-1}(B) = \emptyset$. Since every α -open set is a semi- α -open set by part (1) of remarks (2.2), hence, X is a semi- α -disconnected space, which is a contradiction. Therefore, Y is an α -connected space.

3.5 Proposition

Every semi- α -irresolute function is a semi- α -continuous function.

Proof:

Let $f : X \longrightarrow Y$ be a semi- α -irresolute function, to prove f is a semi- α -continuous function. Let O be an open set in Y , then O is semi- α -open set in Y by part (2) of remarks (2.2). Therefore, $f^{-1}(O) \in S_{\alpha}O(X)$ since f is a semi- α -irresolute function. Then f is a semi- α -continuous function.

3.6 Remark

The converse of the above proposition is not true in general. Let $X = \{1, 2, 3\}$, $\tau = \{X, \emptyset, \{1, 2\}\}$. Then $S_{\alpha}O(X) = \tau$. Let $Y = \{a, b, c\}$, $\tau' = \{Y, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then $S_{\alpha}O(Y) = \tau' \cup \{\{a, c\}, \{b, c\}\}$. Define $f : (X, \tau) \longrightarrow (Y, \tau')$ by $f(1) = f(2) = a, f(3) = c$. It is clear that f is a semi- α -continuous function but it is not semi- α -irresolute function, since $\{b, c\} \in S_{\alpha}O(Y)$ but $f^{-1}(\{b, c\}) = \{3\} \notin S_{\alpha}O(X)$.

3.7 Proposition

If $f : (X, \tau) \longrightarrow (Y, \tau')$ is continuous and open function, then it is semi- α -irresolute function

Proof:

Let $V \in S_{\alpha}O(Y)$, then there exists an α -open set U of Y such that $U \subseteq V \subseteq \text{cl}(U)$. By taking the inverse image, we get $f^{-1}(U) \subseteq f^{-1}(V) \subseteq f^{-1}(\text{cl}(U))$. Since f is open and continuous function, then by part (1) of remarks (2.6) f is α -irresolute function, so $f^{-1}(U)$ is an α -open subset of X . Since f is an open function, we get $f^{-1}(\text{cl}(U)) \subseteq \text{cl}(f^{-1}(U))$. Now, we get $f^{-1}(U) \subseteq f^{-1}(V) \subseteq \text{cl}(f^{-1}(U))$ where $f^{-1}(U)$ is an α -open subset of X . Hence $f^{-1}(V)$ is a semi- α -open subset of X . Thus f is a semi- α -irresolute function.

3.8 Proposition

Every homomorphism is a semi- α -irresolute function.

Proof:

Since every homomorphism is open and continuous function, proposition (3.7) is applicable.

3.9 Proposition

The semi- α -irresolute image of a semi- α -connected space is a semi- α -connected space.

Proof:

Let $f : (X, \tau) \longrightarrow (Y, \tau')$ be semi- α -irresolute and onto function and X be a semi- α -connected space. We want to prove that Y is a semi- α -connected space. Suppose Y is a semi- α -disconnected space. So, $Y = A \cup B$, such that $A \neq \emptyset \neq B$ and $A \cap B = \emptyset$ and $A, B \in S_{\alpha}O(Y)$. So, $f^{-1}(Y) = f^{-1}(A \cup B)$, then $X = f^{-1}(A) \cup f^{-1}(B)$, but f is a semi- α -irresolute function then $f^{-1}(A)$ and $f^{-1}(B)$ are semi- α -open sets in X . Since $A \neq \emptyset \neq B$, then $f^{-1}(A) \neq \emptyset \neq f^{-1}(B)$ and $A \cap B = \emptyset$, then $f^{-1}(A) \cap f^{-1}(B) = \emptyset$. Hence, X is a semi- α -disconnected

space, which is a contradiction. Therefore, Y is a semi- α -connected space.

3.10 Corollary

Let $X \times Y$ be a semi- α -connected space, then X and Y are semi- α -connected space.

Proof:

Follows from proposition (3.7) and proposition (3.9). Since the projection functions ΠX and ΠY are continuous and open functions.

3.11 Corollary

A semi- α -connectedness property is a topological property.

Proof:

Necessity follows from proposition (3.8) and proposition (3.9).

4- On Semi- α -Connected

Subspace

4.1 Definition

Let (X, τ) be a topological space and (Y, τ_Y) be a subspace of X . A subset A of Y is called

- (1) α -open set in Y if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$ in Y .
- (2) Semi- α -open set in Y if $U \subseteq A \subseteq \text{cl}(U)$ whenever U is α -open set in (Y, τ_Y) . The complement of semi- α -open set in Y is called semi- α -closed set in Y .

4.2 Definition

Let (X, τ) be a topological space and (Y, τ_Y) be a subspace of X . A subset A of Y is called Semi- α -closure set of A in Y if $A =$ the intersection of all semi- α -closed sets in Y containing A , denoted by semi- α -cl(A) in Y .

4.3 Definition

Let (X, τ) be a topological space and (Y, τ_Y) be a subspace of X . Y is said to be semi- α -connected subspace

if and only if Y can't be expressed as the union of two disjoint semi- α -open, nonempty subset of Y . Otherwise, Y is said to be semi- α -disconnected subspace.

4.4 Remark

The semi- α -connectedness property is not a hereditary property.

4.5 Example

Let $X = \{1, 2, 3, 4\}$, $\tau = \{X, \emptyset, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$, $S_{\alpha}O(X) = \tau \cup \{\{2, 4\}, \{1, 2, 4\}, \{2, 3, 4\}\}$. Let $Y = \{1, 3\}$, $\tau_Y = \{Y, \emptyset, \{1\}, \{3\}\} = S_{\alpha}O(Y)$.

It is clear that (X, τ) is a semi- α -connected space while (Y, τ_Y) is a semi- α -disconnected subspace.

It is known that the union of any family of connected sets whose intersection is nonempty is connected, but this fact is not true in general if we replace connected by semi- α -connected as can be seen from the following example.

4.6 Example

Let $X = \{1, 2, 3\}$, $\tau = \{X, \emptyset, \{1\}, \{2\}, \{1, 2\}\}$, $S_{\alpha}O(X) = \tau \cup \{\{1, 3\}, \{2, 3\}\}$. Let $A = \{1, 3\}$, $\tau_A = \{A, \emptyset, \{1\}\} = S_{\alpha}O(A)$. Let $B = \{2, 3\}$, $\tau_B = \{B, \emptyset, \{2\}\} = S_{\alpha}O(B)$.

It is clear that A and B are semi- α -connected space and $A \cap B \neq \emptyset$, but $A \cup B = X$ which is semi- α -disconnected space.

It is known A is τ -disconnected if and only if A is τ_Y -disconnected, where (Y, τ_Y) is a subspace of a topological space (X, τ) , but this fact is not true in general if we replace disconnectedness by semi- α -disconnectedness.

4.7 Remark

Let Y be a subspace of a topological space (X, τ) and $A \subseteq Y$. If A is a semi- α -disconnected space in Y , then A need not to be semi- α -disconnected space in X as we seen in the following example.

4.8 Example

Let $X = \{1, 2, 3, 4\}$, $\tau = \{X, \emptyset, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$, $S_{\alpha}O(X) = \tau \cup \{\{1, 4\}, \{1, 2, 4\}, \{1, 3, 4\}\}$.

Let $Y = \{2, 3, 4\}$, $\tau_Y = \{Y, \emptyset, \{2\}, \{3\}, \{2, 3\}\}$

$S_{\alpha}O(Y) = \tau_Y \cup \{\{2, 4\}, \{3, 4\}\}$.

Let $A = \{2, 3\} \subseteq Y$.

It is clear that A is semi- α -disconnected space in Y , but it is semi- α -connected in X .

4.9 Definition

Let (X, τ) be a topological space. Two nonempty subsets A and B of X are said to be semi- α - τ -separated set if and only if $A \cap \text{semi-}\alpha\text{-cl}(B) = \emptyset$ and $\text{semi-}\alpha\text{-cl}(A) \cap B = \emptyset$.

4.10 Remark

Every τ -separated sets are semi- α - τ -separated, but the converse is not true in general.

4.11 Example

Let $X = \{1, 2, 3\}$, $\tau = \{X, \emptyset, \{1\}\}$, let $A = \{2\}$ and $B = \{3\}$.

It is clear that A and B are semi- α - τ -separated sets, since $\text{semi-}\alpha\text{-cl}(A) = \{2\}$ and $\text{semi-}\alpha\text{-cl}(B) = \{3\}$, but they are not τ -separated sets since $\text{cl}(A) = \{2, 3\} = \text{cl}(B)$.

4.12 Remark

Every semi- α - τ -separated sets are disjoint, but the converse is not true in general.

4.13 Example

Let $X = \{1, 2, 3\}$, $\tau = \{X, \emptyset, \{1\}, \{2\}, \{1, 2\}\}$, let $A = \{1, 2\}$ and $B = \{3\}$.

It is clear that A and B are disjoint, but they are not semi- α -separated sets, since $\text{semi-}\alpha\text{-cl}(A) = X$.

It is known that if Y is the union of two nonempty τ -separated sets, then Y is disconnected space but this fact may be false in semi- α -connected as we see in the following remark.

4.14 Remark

Let (X, τ) be a topological space and Y is a subspace of X . If Y is the union of two nonempty semi- α - τ -separated sets, then Y need not to be semi- α -disconnected subspace.

4.15 Example

Let $X = \{1, 2, 3, 4\}$, $\tau = \{X, \emptyset, \{1\}\}$, let $Y = \{2, 3, 4\}$, $\tau_Y = I$, let $A = \{2\}$ and $B = \{3, 4\}$.

It is clear that Y is the union of two nonempty semi- α - τ -separated sets, but Y is a semi- α -connected subspace.

It is known that $\text{cl}(A)$ in $Y = \text{cl}(A)$ in $X \cap Y$, where $A \subseteq Y$ and Y is a subspace of (X, τ) , but this fact is no more true if we replace cl by $\text{semi-}\alpha\text{-cl}$ as it can be seen by the following example.

4.16 Example

Let $X = \{1, 2, 3, 4\}$, $\tau = \{X, \emptyset, \{1\}\}$, let $Y = \{2, 3, 4\}$, $\tau_Y = I$, let $A = \{2\} \subseteq Y$.

$\text{Semi-}\alpha\text{-cl}(A)$ in $X = \{2\}$ and $\text{semi-}\alpha\text{-cl}(A)$ in $Y = Y$. Then $\text{semi-}\alpha\text{-cl}(A)$ in $Y \neq \text{semi-}\alpha\text{-cl}(A)$ in $X \cap Y$.

4.17 Definition

Let (X, τ) be a topological space and (Y, τ_Y) be a subspace of X . Two nonempty subsets A and B of Y are said to be semi- α - τ_Y -separated sets if

and only if $A \cap \text{semi-}\alpha\text{-cl}(B)$ in $Y = \phi$ and $\text{semi-}\alpha\text{-cl}(A)$ in $Y \cap B = \phi$.

4.18 Remark

It is known that A and B are τ -separated sets, where A and B are subsets of Y and (Y, τ_Y) is a subspace of (X, τ) if and only if A and B are τ_Y -separated sets, but this fact is no more true if we replace τ -separated (resp. τ_Y -separated) sets by $\text{semi-}\alpha\text{-}\tau$ -separated (resp. $\text{semi-}\alpha\text{-}\tau_Y$ -separated) sets as we see in the following example.

4.19 Example

See example (4.13).

It is clear that A and B are $\text{semi-}\alpha\text{-}\tau$ -separated sets, but they are not τ_Y - $\text{semi-}\alpha$ -separated sets since $\text{semi-}\alpha\text{-cl}(A)$ in $Y = \text{semi-}\alpha\text{-cl}(B)$ in $Y = Y$.

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حول الفضاء الجزئي شبه المتصل- α

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الخلاصة:

لقد قمنا في هذا البحث باستخدام مفهوم المجموعة شبه المفتوحة- α في تعريف مفهوم جديد من الاتصال القوي على الفضاءات الجزئية يدعى شبه الاتصال- α . كما برهننا ان هذا النوع من الاتصال هو صفة تبولوجية وقمنا باعطاء مثال كون هذه الخاصية ليست وراثية. كما برهننا ان صورة الفضاء شبه المتصل- α بفعل الدالة من النمط شبه المتردد- α هو فضاء شبه متصل- α .