Evaluating the Area of a Circle and the Volume of a Sphere by Using Monte Carlo Simulation

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ABSTRACT:

In this reseach Monte Carlo simulation is used to calculate the area of any circle as well as the volume of any sphere numerically. Different functions are considered to test the performance of this method, in addition to some improvement to give more satisfactory results.

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1- Introduction:

The evaluation of plane area, in particular the determination of the area of the circle, is one of the oldest problems in mathematics and seems to have been started in every culture where men have dealt with mathematical problems. The earliest attempts at measuring the area of a circle are thought to have been undertaken by the Babylonian and the Egyptian. The Babylonians discovered the fact that the radius of a circle is the side-length of an inscribed regular hexagon. The Egyptians discovered a good approximation of

the area of a unit circle which is equivalent to 3.1605. The next real progress came during the Greek period, Archimedes proved that the area between a second-order parabola and one of its tangents is a rational number (Engels, 1981).

This paper discusses Monte Carlo simulation, a classical yet unorthodox application of the method which is equivalent to making simulation of a function. The discussion is started briefly with the Monte Carlo simulation basics, then explaining the use of this technique to solve two theoretical problems :

1- The estimation of the area of a circle through random sampling.

2- A generalization of this idea to calculate the estimation of the volume of a sphere.

In order to increase the accuracy of the method used in sampling in this paper, the average of the results of 10 runs for each value of the sample size n are taken.

2- Basic Monte Carlo Simulation:

A Monte Carlo simulation is a generic numerical method using a computer utilizing pseudo-random numbers to simulate stochastic inputs to a process and generate large numbers of possible outcomes. It solves a problem by generating suitable random numbers and observing that fraction of the numbers obeying some property or properties (Sabelfeld, 2004). It is commonly used for simulations in many fields that require solutions for problems that are impractical or impossible to solve by traditional analytical or numerical methods (Yang, 2002). The Monte Carlo simulation is mainly used to predict the final consequence of a series of occurrences, each having its own probability. A major requirement in Monte Carlo simulation is that the mathematical system be described by probability density functions. Once the density functions are known, the Monte Carlo simulation can proceed by random sampling from these density functions for use within the simulation. Given that a particular simulation outcome is based randomness performing the same simulation again would, most

likely result is a different outcome. Many simulations are performed and the desired result is taken as an average over the number of outcomes (Smaldone, 2001).

The main steps of Monte Carlo Simulation are (Stokes, 2004):
1) Model the inputs and process.
2) Draw a vector of random varieties.
3) Evaluate the function of interest.
4) Repeat the last two steps many times, aggregating the results.

3- Simulating Deterministic Behavior:

In this section we illustrate the use of Monte Carlo simulation to model a deterministic behavior, the area of a circle and the volume of a sphere, were Monte Carlo technique can be extended to functions of several variables and become more practical in that situation.

3.1- Simulating the Area of a Circle:

We want to construct a simulation which will estimate the area of a circle. A program is written to play with Monte Carlo methods, it uses the relationship between π and the area of a circle, $A = \pi r^2$. An elegant way to visualite the relationship is to imagine slicing a circle into smaller pieces, then rearranging these pieces into a rectangle. The area of the rectangle is equal to the area of the circle and is equal to its radius **r** multiplied by its length πr, or $A = \pi r^2$.

The trick with Monte Carlo simulation is to find a function that includes your unknown function of interest, then guess values and evaluate if they are included in your function of interest. The fraction that is contained is proportional to the fraction of the area of the unknown function in your known function area.

The discussion of Monte Carlo simulation to model a deterministic behavior (the area of a circle), is as follows:
Consider the following general case equation of the circle

$$(x-a)^2 + (y-b)^2 = r^2 \qquad (1)$$

In order to estimate the area of this circle through random sampling, the following algorithm is suggested.

Monte Carlo area algorithm:

Input: The radius **r**, the center (a,b), and the total number **n** of random points to be generated during the simulation.

Output: Area of the circle.

Step 1: Generate **n** uniform random sequences

$$x_i \in [a-r, a+r] \qquad\qquad (2)$$

Step 2: Calculate the simulation function $\quad f(x_i) = 2\sqrt{r^2 - (x_i - a)^2}$ which correspons to the equation (1) for the random x_i coordinates.

Step 3: Calculate $\quad Area\ of\ the\ circle = \frac{2r}{n}\sum_{i=1}^{n} f(x_i) \qquad (3)$

 Step 4: Stop.

Table (1) summarizes the results of 10 different simulations to obtain the estimation of the area of a circle defined by the equation:

$$(x-3)^2 + (y-2)^2 = 9 \qquad\qquad (4)$$

using drand48 (seed = 1) as a random number generators. Approximations were made using n = 100, 500, 1000, 5000, and 10000 terms in the summation, each approximation was compared to the exact value of the area (= 28.274 cm^2).

Table.1: Monte Carlo approximation to the area of a circle, drand48.

Run Number	Length of simulation run	Estimated Area	Absolute Error
1	100	28.132	0.142
	500	28.110	0.164
	1000	28.347	0.073
	5000	28.156	0.118
	10000	28.256	0.018
2	100	28.629	0.354
	500	28.312	0.038
	1000	28.159	0.116
	5000	28.356	0.081
	10000	28.251	0.023
3	100	27.823	0.451
	500	28.237	0.037
	1000	28.239	0.035
	5000	28.313	0.039
	10000	28.331	0.056
4	100	28.397	0.122
	500	28.125	0.149

	1000	28.228	0.047
	5000	28.394	0.119
	10000	28.286	0.012
	100	28.537	0.262
	500	28.431	0.157
5	1000	28.193	0.081
	5000	28.281	0.006
	10000	28.273	0.001
	100	27.941	0.333
	500	28.105	0.169
6	1000	28.260	0.015
	5000	28.252	0.022
	10000	28.264	0.010
	100	28.429	0.154
	500	28.383	0.109
7	1000	28.189	0.085
	5000	28.123	0.151
	10000	28.382	0.108
	100	28.529	0.255
	500	28.165	0.110
8	1000	28.349	0.075
	5000	28.319	0.044
	10000	28.294	0.020
	100	27.724	0.551
	500	28.490	0.216
9	1000	28.177	0.097
	5000	28.172	0.102
	10000	28.259	0.016
	100	27.985	0.289
	500	28.569	0.295
10	1000	28.305	0.030
	5000	28.183	0.092
	10000	28.229	0.045

From table (1) we observe the following:

1) The estimates of the area approach the exact value (= 28.274 cm^2) as the number of points generated (or length of the simulation run) increases.

2) The 10 simulation runs do yield different estimates given the same *n*. Each run actually may be regarded as an observation in the simulation experiment.

Table 2 illustrates our results for various choices of centers (a, b), radiuses *r*. Approximations were made using *n* = 100, 1000, and 10000 terms in the summation, each approximation was compared to the exact value of the area.

Table 2: Monte Carlo approximation to the area of 5 circles; rand.

Center (a,b)	Radius r	Simulation function	Exact area	Sample size	Approximated area	Absolute error
(0,0)	1	$2\sqrt{1-x^2}$	3.142	100	3.214	0.072
				1000	3.148	0.006
				10000	3.140	0.002
(-1,-2)	2	$2\sqrt{3-2x-x^2}$	12.566	100	12.581	0.015
				1000	12.578	0.012
				10000	12.565	0.001
(1,-2)	4	$2\sqrt{15+2x-x^2}$	50.265	100	50.424	0.159
				1000	50.259	0.006
				10000	50.263	0.002
(2,3)	4.5	$2\sqrt{20.25-(x-2)^2}$	63.617	100	62.528	1.089
				1000	63.442	0.175
				10000	63.610	0.007
(-2,3)	5.25	$2\sqrt{27.563-(x-2)^2}$	86.590	100	86.096	0.494
				1000	86.401	0.189
				10000	86.586	0.004

From table 2 we observe that the estimates of the area of each circle approach to the exact value of the area as the number of points generated increases.

Figure 1-a plots the estimated area of the circle defined by equation (4) for run 1 (blue curve) and run 2 (red curve) given in table 1 and the exact value (= 28.274 cm^2 green   curve) against the length of the simulation run $\boldsymbol{n}$.
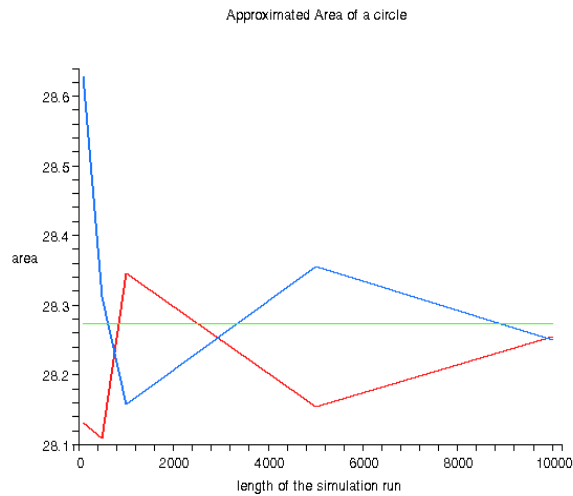
Approximated Area of a circle

Figure 1-a: Approximated area for run 1 and 2.

From figure 1-a we can see that initially, the estimates oscillate around the exact value before eventually stabilizing a condition normally reached after the simulation experiment is repeated a sufficiently large number of times.

Figure 1-b plots the absolute error given in table 1 for runs 1 (blue curve) and 2 (red curve) against the length of the simulation run n.
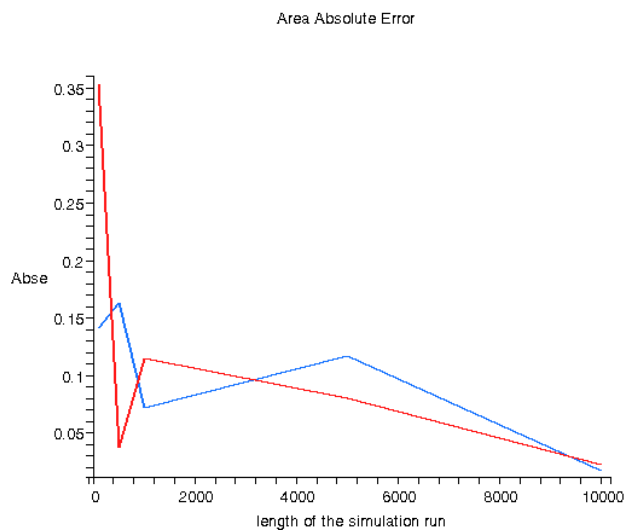
Area Absolute Error

Figure 1-b: Area absolute error for run 1 and

From figure 1-b we observe that initially, the absolute error decreases as the number of points generated increase.

3.2- Estimation of the Volume of a Sphere by Simulation:

Let's consider finding part of the volume of the sphere $x^2 + y^2 + z^2 \leq 1$ that lies in the first octant ; $x > 0$, $y > 0$, $z > 0$. The methodology to approximate the volume is very similar to that of finding the area under a closed curve, however, now we will use an approximation for the volume under the surface by the following rule:

$$\frac{volume\ under\ surface}{volume\ of\ box} \cong \frac{number\ of\ points\ counted\ below\ surface\ in\ 1st\ octant}{total\ number\ of\ points}$$

Figure (2) shows the sequence of calculations required to employ Monte Carlo techniques to find approximate volume under a surface.
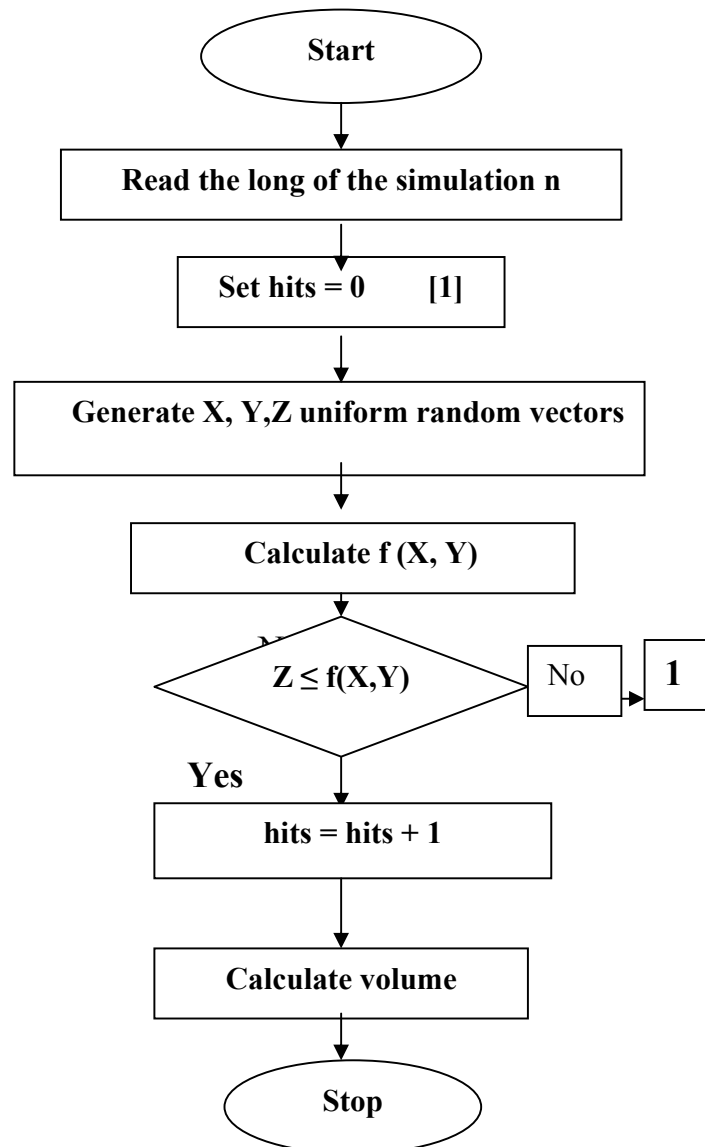
Figure 2: Flow chart for Monte Carlo simulation volume algorithm.

      Table 3 summarizes the results of several different simulations to obtain the estimation of the volume of the sphere $x^2 + y^2 + z^2 \leq 1$ using rand as a random number generator. Approximations were made using ***n*** =100, 500, 1000, 5000 and 10000 terms in the summation, each approximation was compared to the exact value of the volume (=4.189 cm^3).

Table 3: Monte Carlo approximation to the volume of a sphere, rand.

Run Number	Length of Simulation run	Estimated volume	Absolute Error
1	100	3.377	0.812
	500	3.590	0.599
	1000	3.908	0.281
	5000	4.185	0.004
	10000	4.187	0.002
2	100	3.227	0.962
	500	3.404	0.785
	1000	3.932	0.257
	5000	4.186	0.003
	10000	4.190	0.001
3	100	3.369	0.820
	500	3.770	0.449
	1000	3.905	0.284
	5000	4.185	0.004
	10000	4.188	0.001
4	100	3.367	0.822
	500	3.525	0.664
	1000	3.979	0.210
	5000	4.183	0.006
	10000	4.186	0.003
5	100	3.343	0.846
	500	3.523	0.666
	1000	3.918	0.271
	5000	4.184	0.005
	10000	4.191	0.002
6	100	3.355	0.834
	500	3.520	0.669
	1000	3.958	0.231
	5000	4.182	0.007
	10000	4.190	0.002
7	100	3.397	0.792
	500	3.624	0.565
	1000	3.899	0.290
	5000	4.183	0.006
	10000	4.188	0.001
8	100	3.455	0.734
	500	3.591	0.598
	1000	3.884	0.295
	5000	4.184	0.005
	10000	4.188	0.001
9	100	3.343	0.846
	500	3.414	0.775
	1000	3.994	0.195
	5000	4.183	0.006
	10000	4.190	0.001
10	100	3.481	0.708
	500	3.702	0.487
	1000	3.919	0.270
	5000	4.183	0.006
	10000	4.188	0.001

From table 3 we observe the following:

1. The estimates of the volume approach the exact value (= 4.189 cm^3) as the number of points generated increases.
2. The 10 simulation runs do yield different estimates given the same n. Each run actually may be regarded as an observation in the simulation experiment.

Figure 3-a plots the volume estimates for runs 1 (blue curve) and 2 (red curve) and the exact value (= 4.189 cm^3 green curve) against the length of the simulation run n.
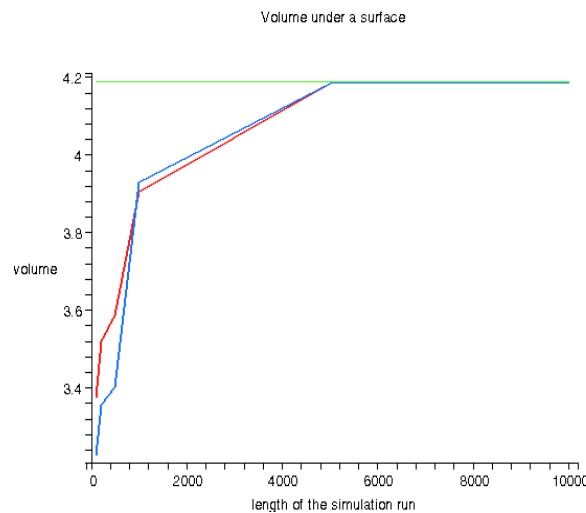

Figure 3-a: Approximated volume for run 1 and2.

From figure 3-a we see that initially, the estimates increase reaching the exact value as the number of points generated increases.

Figure 3-b plots the absolute error given in table 3 for run 1 (blue curve) and 2 (red curve) against the length of the simulation run n.
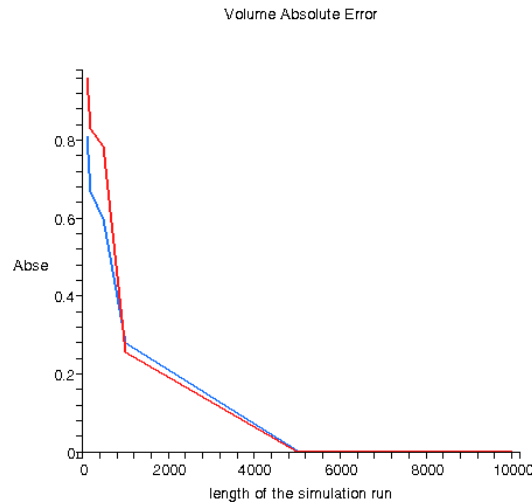
Figure 3-b: Volume absolute error for run 1 and 2.

From figure 3-b we observe that initially, the absolute error decreases as the number of points generated increase.

4- Improved Monte Carlo Simulation:

A straightforward method for speeding up the Monte Carlo simulation algorithm is to use a large sample size and a long simulation run. One of the most important methods to speed the convergence of the Monte Carlo area and volume algorithm discussed in section 3.1 and 3.2 is to average the results of the simulation runs. Table 4 summarizes the mean and the absolute error of the results given in table (1) of the 10 runs for each n = 100, 500, 1000, 5000, and 10000.

Table 4: Average results of the 10 runs.

n	100	500	1000	5000	10000
Mean	28.123	28.293	28.245	28.255	28.286
Abs error	0.151	0.019	0.029	0.019	0.012

From table 4 we observe that it is interesting that the effect of transient conditions is dampened when we average the results of the 10 runs. This result can be seen clearly in figure 4, where we plot

the mean (red curve), and the exact value (=28.274 cm^2 blue curve) against the length of the simulation run **n**.
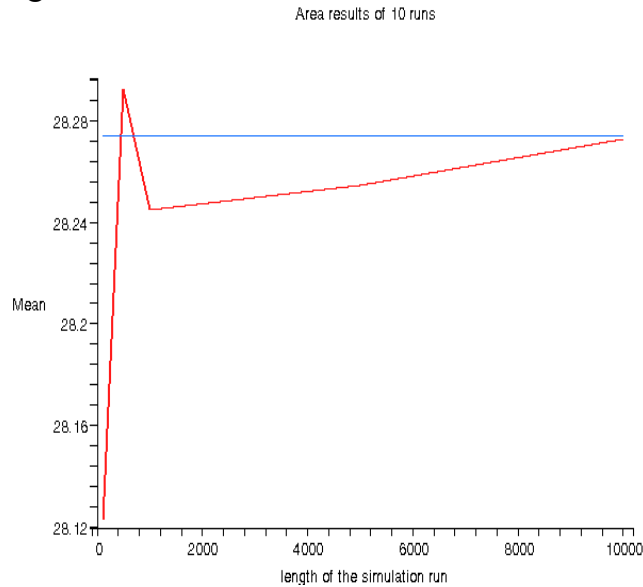

Figure 4: Mean of 10 runs.

Figure 4 shows that the mean of the results of the 10 runs increases rapidly as **n** increases. Table 5 summarizes the mean and the absolute error of the results given in table (3) of the 10 runs for each **n** = 100, 200, 500, 1000, 5000, and 10000.

Table 5: Average results of the 10 runs.

n	100	200	500	1000	5000	10000
Mean	3.371	3.434	3.566	3.930	4.184	4.189
Abs error	0.818	0.755	0.623	0.256	0.005	0.000

From table 5 we observe that it is interesting that the effect of transient conditions is dampened when we average the results of the 10 runs. This result can be seen clearly in figure 5, where we plot the mean (red curve), the absolute error (blue curve), and the exact value (=4.189 cm^3 green curve) against the length of the simulation run **n**.
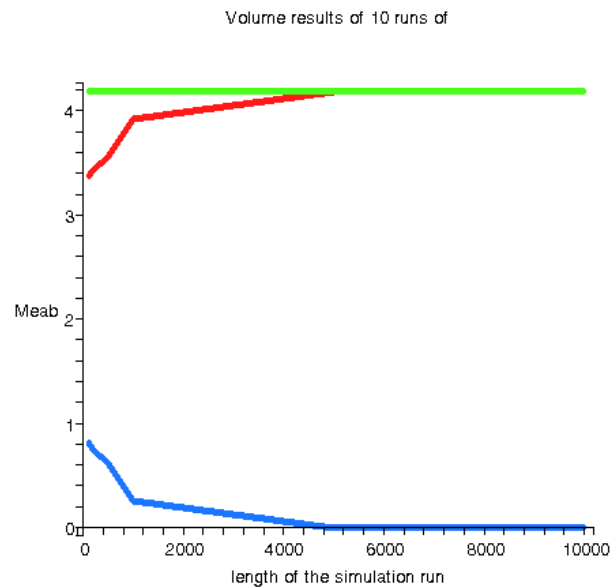
Figure 5: Mean and absolute error of 10 runs.

Figure 5 shows that the mean of the results of the 10 runs increases rapidly as the length of the simulation run n increases, while the absolute error decreases rapidly as n increases.

5- Discussion:

The present work deals with Monte Carlo simulation problem. Two methods are presented to calculate the area of a circle and the volume of a sphere. Many programs are prepared according to the above mentioned methods and their application on a set of corresponding functions for the general formula of circle equation and the equation of a sphere centered at the origin with unit radius. The suggested Monte Carlo simulation algorithms was accelerated by using two techniques, the first is the increase of sample size used, the second which is more important is to find the average of simulation results. The work concluded that these two techniques are necessary for speeding up the rate of convergence of Monte Carlo simulation algorithms used to calculate the area of a circle and the volume of a sphere, and finally to get more accurate estimators with less possible absolute error.

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