Open Access **Baghdad Science Journal** **P-ISSN: 2078-8665**
2021, Vol. 18 No.2 (Suppl. June) **E-ISSN: 2411-7986**

DOI: http://dx.doi.org/10.21123/bsj.2021.18.2(Suppl.).1125

Projective MDS Codes Over $GF(27)$

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Received 4/11/2020, Accepted 13/1/2021, Published 20/6/2021

Abstract:

MDS code is a linear code that achieves equality in the Singleton bound, and projective MDS (*PG*-MDS) is MDS code with independents property of any two columns of its generator matrix. In this paper, elementary methods for modifying a *PG*-MDS code of dimensions 2, 3, as extending and lengthening, in order to find new incomplete *PG*-MDS codes have been used over $GF(27)$. Also, two complete *PG*-MDS codes over $GF(27)$ of length 16 and 28 have been found.

Key words: Conic, Finite field, Finite projective plane, Maximum distance separable codes.

Introduction:

Let $F_q = GF(q)$ denote the Galois field with q elements and let p denote the characteristic of $GF(q)$. An $[n,k]$-code C over F_q is a k-dimensional subspace of F_q^n. The parameter n is called *the length* of C. The weight $wt(x)$ of a vector $v \in F_q^n$ is the number of non-zero coordinates of v. The minimum non-zero weight of all codewords in C is called *the minimum weight* (*Hamming distance*) of C and then an $[n,k]$-code with minimum weight d is called an $[n,k,d]$-code. From Singleton Bound Theorem, the parameter d has maximum value $n-k+1$, when n and k are given (1). The linear code that achieves equality in the Singleton bound is called, or *MDS* codes for short. The code which has minimum weight d that correct e errors that can be accrued is called *e-error correcting* code and $e = \left\lfloor \frac{d-1}{2} \right\rfloor$ ($\lfloor x \rfloor$denotes the smallest integer less than or equal to x, ex. $\left\lfloor \frac{5}{2} \right\rfloor = \left\lfloor \frac{4}{2} \right\rfloor = 2$).

To any an $[n,k]$-code C over F_q, there is also another parameter $r = n-k$ called the *redundancy*, which represent the check digits added to the message to give protection against noise. The orthogonal complement of an $[n,k]$-code C (the set of all vectors which are orthogonal to every vector in C), is called the *dual code* of C with dimension $n-k$, and denoted by $C^\perp$. For more on linear codes, see (1).

Any linear code C over F_q with the three fundamental parameters, its length is n, its dimension is k, and its redundancy r has a natural interpretation of each of these parameters. There are six basic modification techniques to linear codes depending on these three parameters. Each fixes one parameter and increases or decreases the other two parameters accordingly. These techniques are partitioned into three pairs and each member of a pair is the inverse process to the other as summarized below.

(i) Augmenting: Expurgating; Fix n; increase k; decrease r : decrease k; increase r.

(ii) Extending: Puncturing; Fix k; increase n; increase r : decrease n; decrease r.

(iii) Lengthening: Shortening; Fix r; increase n; increase k : decrease n; decrease k.

Since the redundancy of a code is its "dual dimension," each technique also has a natural dual technique.

The motivation of this research is based on two main ideas in the coding theory which can be summarized as follows:

1- It is well known algebraically; that two linear codes with same parameters will have the same efficiency if they are linearly isomorphism (equivalent).

2- The fundamental problem to find codes with two properties:
(i) reasonable information content (length is big enough),
(ii) reasonable error handling ability.

A number of researchers worked on these two ideas recently in the sense of modification, for instance, see (2). Grassl (3) presented a new table with bounds to good codes not MDS code (small length n and large minimum Hamming distance d) for $2 \leq q \leq 9$. Emami and Pedram (4) use punctured and shortened methods to construct codes (optimal Linear codes) with minimum value of n for certain dimension k and minimum Hamming distance d. For further authors whose used shortening or puncturing structure of codes see; (5), (6) and the references therein. Some other researcher concentrate on the generator's entries of a linear code to achieve the singleton bound on the minimum distance; that is, code with maximum ability to correct errors (GM-MDS) (7), (8), (9), (10).

The main aim to this paper is to work with especial type of maximum distance separable (MDS) codes namely, projective MDS code (11) over F_{27}, since they provide the maximum protection against device failure for a given amount of redundancy; that is, the greatest error correcting capability (since error correcting capability is a function of minimum distance).To do that, an extending (dually, Lengthening) technique has been used.

The article is organized as follows. First section provided basic definitions and some properties of MDS and finite projective geometry. In second section, the inequivalent, incomplete projective MDS codes of dimension two have been constructed. Finally, in last section, the inequivalent, incomplete (complete) projective MDS codes of dimension three have been constructed, and three special complete MDS codes of lengths 16 and 28 have been founded using projective conic in the projective plane.

Definitions and Basic Properties

Any linear $[n,k]$-code can be defined by a $(k \times n)$ matrix G or by a $(n-k) \times n$ matrix H whose entries from F_q as defined below.

Definition 1: (1) A generator matrix G of an $[n,k]$-code C is a $k \times n$ matrix whose rows form a basis for C. The standard form of a generator matrix G is $[I_k A]$. A linear code for which any two columns of a generator matrix are linearly independent is called a *projective code* (*PG-code*). A linear code which cannot extend by adding columns to its generator matrix is called a *complete code*, otherwise it is called *incomplete code*.

Definition 2: (1) A parity check matrix H of an $[n,k]$-code C is a $(n-k) \times n$ matrix whose rows forma basis for $C^{\perp}$. The standard form of a parity check matrix H is $\left[-A^T I_{(n-k)}\right]$.

Theorem 1: (1) An $[n,k]_q$-code C is MDS if and only if the dual code $C^{\perp}$, $[n,n-k]$ is MDS.

Let $PG(2,q)$ denote the 2-dimensional projective space over F_q (finite projective plane).

Definition 3: (12) A non-singular plane quadric (form of degree two) in $PG(2,q)$ is called a conic. A conic consists of $q+1$ points no three of which are collinear.

During the paper, the notation *PG*-MDS will briefly refer to a projective MDS code.

PG-MDS Code of Dimension 2 over F_{27}

The technique used in this paper to check whether that two codes are projectively equivalent or not is as follows:
The $s \times r$ matrices A is called *projectively equivalent* to $s \times r$ matrices B, and denoted by $A \cong_P B$ if there exist a non-singular $s \times s$ matrix T such that matrix TA transformed to B by performing the following operations:
(i) make the last position of each row of TA, 0 or 1;
(ii) a permutation of the columns on TA.

The matrix $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ is only standard matrix generate the *PG*-MDS code [2,2,1]. This matrix can be extended to create a *PG*-MDS code [3,2,2] by adding the column $\begin{bmatrix}1\\1\end{bmatrix}$.

The *PG*-MDS code [3,2,2] generated by the matrix $G = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\end{bmatrix}$ in standard form is unique up to equivalence since, if $\hat{G} = \begin{bmatrix}x & y & y\\0 & 1 & 1\end{bmatrix}$ is a generator matrix of another *PG*-MDS code [3,2,2], $x,y,z \in F_q$, then there is a 2×2 non-singular matrix T transform the matrix G to $\hat{G}$ as follows.

Let $\mathcal{T} = \begin{bmatrix}a & b\\c & d\end{bmatrix} \in GL(2,q)$ such that $\mathcal{T}G = \hat{G}$. So, the following equations are deduced:
$a = \lambda x, \quad b = \beta y, \quad c = \lambda, \quad d = \beta, \quad \lambda x + b\beta = \gamma z,$
$\lambda + \beta = \gamma,$

Baghdad Science Journal
P-ISSN: 2078-8665
E-ISSN: 2411-7986

where $\lambda, \beta, z \in GF(q)\backslash\{0\}$. So, $\mathcal{T} = \begin{bmatrix} \mathcal{S}x & \mathcal{J}y \\ \mathcal{S} & \mathcal{J} \end{bmatrix}$, where $\mathcal{S} = \frac{\begin{vmatrix} z & y \\ 1 & 1 \end{vmatrix}}{(x-y)}\gamma$ and $\mathcal{J} = \frac{\begin{vmatrix} x & z \\ 1 & 1 \end{vmatrix}}{(x-y)}\gamma$.

Let $G = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = [\, I_2 A_{2\times 1}]$. The matrix G is of rank 2; that is, any two columns are linearly independents, so this gives the incomplete *PG*- MDS code with the parameters [3,2,2]. The 2×3 matrix G can be extended in to 2×4 matrices, ${G^*}_i$ by adding appropriate 25 columns $\begin{bmatrix} \alpha^i \\ 1 \end{bmatrix}$ to G from right side of G, where $\alpha^i \in F_{27}$, $i = 1,2, \dots, 25$, such that the rows of the new matrix ${G^*}_i$ still linearly independent and any two columns is linearly independent. So, these 25 matrices, gives raise to 25 generated matrices ${G^*}_i$ of *PG*-MDS codes. Among these 25 matrices, only 5 of them are non inequivalent as given the next theorem.

Theorem 2: Over F_{27}, there are only five inequivalent, incomplete *PG*-MDS codes with parameters [4,2,3] and error correcting $e = 1$.

Proof: By searching for a non-singular 2×2 matrix, the equivalents generator matrices have been identified as given below.

${G^*}_i \cong_P {G^*}_j$	row 1, row 2 of T	
${G^*}_1 \cong_P {G^*}_2$	α^{12} 1,	0 1
${G^*}_1 \cong_P {G^*}_{10}$	α^{23} α^{10},	0 1
${G^*}_1 \cong_P {G^*}_{16}$	α^{23} α^{10},	0 1
${G^*}_1 \cong_P {G^*}_{24}$	α^{10} α^{24},	0 1
${G^*}_1 \cong_P {G^*}_{25}$	α^{25} 0,	0 1
${G^*}_3 \cong_P {G^*}_4$	α^{17} α^{4},	0 1
${G^*}_3 \cong_P {G^*}_6$	α^{13} 1,	0 1
${G^*}_3 \cong_P {G^*}_{20}$	α^{4} α^{20},	0 1
${G^*}_3 \cong_P {G^*}_{22}$	α^{13} 1,	0 1
${G^*}_3 \cong_P {G^*}_{23}$	α^{23} 0,	0 1
${G^*}_5 \cong_P {G^*}_7$	α^{13} 1,	0 1
${G^*}_5 \cong_P {G^*}_{11}$	α^{19} α^{11},	0 1
${G^*}_5 \cong_P {G^*}_{15}$	α^{8} 1,	0 1
${G^*}_5 \cong_P {G^*}_{19}$	α^{6} α^{19},	0 1
${G^*}_5 \cong_P {G^*}_{21}$	α^{21} 1,	0 1
${G^*}_8 \cong_P {G^*}_9$	α^{14} α^{9},	0 1
${G^*}_8 \cong_P {G^*}_{12}$	α^{13} 1,	0 1
${G^*}_8 \cong_P {G^*}_{14}$	α α^{14},	0 1
${G^*}_8 \cong_P {G^*}_{17}$	α^{5} 1,	0 1
${G^*}_8 \cong_P {G^*}_{18}$	α^{18} 1,	0 1

Therefore, the only inequivalent ones are summarized below.

G_i	Row 1				Row 2			
G_1	1	0	1	α^{13}	0	1	1	1
G_2	1	0	1	α	0	1	1	1
G_3	1	0	1	α^{3}	0	1	1	1
G_4	1	0	1	α^{5}	0	1	1	1
G_5	1	0	1	α^{8}	0	1	1	1

■

To extend each matrix G_i, an appropriate column $\begin{bmatrix} \alpha^j \\ 1 \end{bmatrix}$ is added to G_i for which α^j does not belong to the first row of G_i. So, there are $(q-3)$ possibilities for $\begin{bmatrix} \alpha^j \\ 1 \end{bmatrix}$; that is, 24 possibility. Therefore, by this way, $5(q-3) = 120$ cods can be constructed. This procedure will be used to extend the codes in this paper.

In the next theorems, only the inequivalent codes are presented.

Theorem 3: Over F_{27}, there are eight inequivalent *PG*-MDS codes with parameters [5,2,4] and error correcting $e = 2$. These codes are given below.

M_i	Row 1					Row 2				
M_1	1	0	1	α^{13}	α	0	1	1	1	1
M_2	1	0	1	α	α^{2}	0	1	1	1	1
M_3	1	0	1	α	α^{3}	0	1	1	1	1
M_4	1	0	1	α	α^{6}	0	1	1	1	1
M_5	1	0	1	α	α^{7}	0	1	1	1	1
M_6	1	0	1	α	α^{12}	0	1	1	1	1
M_7	1	0	1	α^{3}	α^{7}	0	1	1	1	1
M_8	1	0	1	α^{3}	α^{9}	0	1	1	1	1

■

From the eight M_i *PG*-MDS codes, $8(q-4) = 184$ projective cods can be constructed. These codes are given in the next theorem.

Theorem 4: Over F_{27}, there are 34 inequivalent, incomplete *PG*-MDS codes with parameters [6,2,5] and error correcting $e = 2$ as given in Table 1.

Table 1. Inequivalent, incomplete *PG*-MDS codes N_i of length 6

N_i	Row 1						Row 2					
N_1	1	0	1	α^{13}	α	α^{2}	0	1	1	1	1	1
N_2	1	0	1	α^{13}	α	α^{3}	0	1	1	1	1	1
N_3	1	0	1	α^{13}	α	α^{4}	0	1	1	1	1	1
N_4	1	0	1	α^{13}	α	α^{5}	0	1	1	1	1	1
N_5	1	0	1	α^{13}	α	α^{6}	0	1	1	1	1	1
N_6	1	0	1	α^{13}	α	α^{7}	0	1	1	1	1	1
N_7	1	0	1	α^{13}	α	α^{8}	0	1	1	1	1	1
N_8	1	0	1	α^{13}	α	α^{11}	0	1	1	1	1	1
N_9	1	0	1	α^{13}	α	α^{12}	0	1	1	1	1	1
N_{10}	1	0	1	α^{13}	α	α^{14}	0	1	1	1	1	1
N_{11}	1	0	1	α^{13}	α	α^{15}	0	1	1	1	1	1
N_{12}	1	0	1	α^{13}	α	α^{16}	0	1	1	1	1	1
N_{13}	1	0	1	α^{13}	α	α^{19}	0	1	1	1	1	1

N_{14}	1	0	1	α^{13}	α	α^{22}	0	1	1	1	1	1
N_{15}	1	0	1	α^{13}	α	α^{24}	0	1	1	1	1	1
N_{16}	1	0	1	α^{13}	α	α^{25}	0	1	1	1	1	1
N_{17}	1	0	1	α	α^{2}	α^{3}	0	1	1	1	1	1
N_{18}	1	0	1	α	α^{2}	α^{4}	0	1	1	1	1	1
N_{19}	1	0	1	α	α^{2}	α^{7}	0	1	1	1	1	1
N_{20}	1	0	1	α	α^{2}	α^{9}	0	1	1	1	1	1
N_{21}	1	0	1	α	α^{2}	α^{10}	0	1	1	1	1	1
N_{22}	1	0	1	α	α^{2}	α^{11}	0	1	1	1	1	1
N_{23}	1	0	1	α	α^{3}	α^{4}	0	1	1	1	1	1
N_{24}	1	0	1	α	α^{3}	α^{6}	0	1	1	1	1	1
N_{25}	1	0	1	α	α^{3}	α^{7}	0	1	1	1	1	1
N_{26}	1	0	1	α	α^{3}	α^{12}	0	1	1	1	1	1
N_{27}	1	0	1	α	α^{6}	α^{7}	0	1	1	1	1	1
N_{28}	1	0	1	α	α^{6}	α^{11}	0	1	1	1	1	1
N_{29}	1	0	1	α	α^{6}	α^{20}	0	1	1	1	1	1
N_{30}	1	0	1	α	α^{7}	α^{8}	0	1	1	1	1	1
N_{31}	1	0	1	α	α^{7}	α^{12}	0	1	1	1	1	1
N_{32}	1	0	1	α	α^{7}	α^{15}	0	1	1	1	1	1
N_{33}	1	0	1	α	α^{7}	α^{19}	0	1	1	1	1	1
N_{34}	1	0	1	α^{3}	α^{9}	α^{12}	0	1	1	1	1	1

New inequivalent, incomplete *PG*-MDS codes for fixed dimension $k = 2$ and length $7 \leq n \leq 14$ over F_{27} can be constructed by means of a combinatorial computer search, and using the same technique in Theorem 2, 3 and 4. In the next theorem, the full details about these codes are given.

Let $*_n$ denote the number of all codes with length n and $\#_n$ denote the number of inequivalent ones.

Theorem 5: Over F_{27}, for fixed $k = 2$ the inequivalent, incomplete *PG*-MDS codes with parameters given below exist.

$*_n = \#_{n-1}(q-(n-2))$	$\#_n$	n	k	d	e
748	73	7	2	6	2
1533	196	8	2	7	3
3920	382	9	2	8	3
7258	745	10	2	9	4
13410	1142	11	2	10	4
19414	1665	12	2	11	5
26640	1976	13	2	12	5
29640	2170	14	2	13	6

■

Example 1: In this example, *PG*-MDS codes are given for each length n, $8 \leq n \leq 14$.

(i) *PG*-MDS code with parameters [7,2,6] and $e = 2$.

$$\mathcal{K}_7 = \begin{bmatrix} 1 & 0 & 1 & \alpha^{13} & \alpha & \alpha^2 & \alpha^3 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} N_1 & \begin{vmatrix} \alpha^3 \\ 1 \end{vmatrix} \end{bmatrix}.$$

(ii) *PG*-MDS code with parameters [8,2,7] and $e = 3$.

$$\mathcal{K}_8 = \begin{bmatrix} 1 & 0 & 1 & \alpha^{13} & \alpha & \alpha^2 & \alpha^3 & \alpha^6 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \mathcal{K}_7 & \begin{vmatrix} \alpha^6 \\ 1 \end{vmatrix} \end{bmatrix}.$$

(iii) *PG*-MDS code with parameters [9,2,8] and $e = 4$.

$$\mathcal{K}_9 = \begin{bmatrix} 1 & 0 & 1 & \alpha^{13} & \alpha & \alpha^2 & \alpha^3 & \alpha^6 & \alpha^8 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \mathcal{K}_8 & \begin{vmatrix} \alpha^8 \\ 1 \end{vmatrix} \end{bmatrix}.$$

(iv) *PG*-MDS code with parameters [10,2,9] and $e = 4$.

$$\mathcal{K}_{10} = \begin{bmatrix} 1 & 0 & 1 & \alpha^{13} & \alpha & \alpha^2 & \alpha^3 & \alpha^6 & \alpha^8 & \alpha^9 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \mathcal{K}_9 & \begin{vmatrix} \alpha^9 \\ 1 \end{vmatrix} \end{bmatrix}.$$

(v) *PG*-MDS code with parameters [11,2,10] and $e = 5$.

$$\mathcal{K}_{11} = \begin{bmatrix} 1 & 0 & 1 & \alpha^{13} & \alpha & \alpha^2 & \alpha^3 & \alpha^6 & \alpha^8 & \alpha^9 & \alpha^{14} \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \mathcal{K}_{10} & \begin{vmatrix} \alpha^{14} \\ 1 \end{vmatrix} \end{bmatrix}.$$

(vi) *PG*-MDS code with parameters [12,2,11] and $e = 5$.

$$\mathcal{K}_{12} = \begin{bmatrix} 1 & 0 & 1 & \alpha^{13} & \alpha & \alpha^2 & \alpha^3 & \alpha^6 & \alpha^8 & \alpha^9 & \alpha^{14} & \alpha^5 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \mathcal{K}_{11} & \begin{vmatrix} \alpha^5 \\ 1 \end{vmatrix} \end{bmatrix}.$$

(vii) *PG*-MDS code with parameters [13,2,12] and $e = 6$.

$$\mathcal{K}_{13} = \begin{bmatrix} 1 & 0 & 1 & \alpha^{13} & \alpha & \alpha^2 & \alpha^3 & \alpha^6 & \alpha^8 & \alpha^9 & \alpha^{14} & \alpha^5 & \alpha^7 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \mathcal{K}_{12} & \begin{vmatrix} \alpha^7 \\ 1 \end{vmatrix} \end{bmatrix}.$$

(viii) *PG*-MDS code with parameters [14,2,13] and $e = 6$.

$$\mathcal{K}_{14} = \begin{bmatrix} 1 & 0 & 1 & \alpha^{13} & \alpha & \alpha^2 & \alpha^3 & \alpha^6 & \alpha^8 & \alpha^9 & \alpha^{14} & \alpha^5 & \alpha^7 & \alpha^{10} \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \mathcal{K}_{13} & \begin{vmatrix} \alpha^{10} \\ 1 \end{vmatrix} \end{bmatrix}.$$

Corollary 1: Over F_{27}, for fixed $k = 2$ the inequivalent, incomplete *PG*-MDS codes with length $15 \leq n \leq 26$ exist.

Proof:

The first row of each generator matrix Q of a *PG*-MDS code C, $[n', k]$ except the first element which is 1, in Theorems 2,3,4, and 5 has distinct elements from the base field F_{27}. So, the matrix

$\mathcal{H} = \begin{bmatrix} \alpha^{i_1} & \alpha^{i_2} & \dots & \alpha^{i_{(28-n')}} \\ 1 & 1 & \dots & 1 \end{bmatrix}$,
where α^{i_j} belong to the complement of the first row of the generator matrix Q, forming a generator matrix of a *PG*-MDS codes, $[28-n', 2]$, $2 \le n' \le 14$. Therefore, the inequivalent, incomplete *PG*-MDS codes of the following parameters exist.

$\#_n$	n	k	d	e
1976	15	2	14	6
1665	16	2	15	7
1142	17	2	16	7
745	18	2	17	8
382	19	2	18	8
196	20	2	19	9
73	21	2	20	9
34	22	2	21	10
8	23	2	22	10
5	24	2	23	11
1	25	2	24	11
1	26	2	25	12

Corollary 2: Over F_{27} for fixed distance $d = 3$ and $e = 1$, the inequivalent *PG*-MDS codes with parameters given below exist.

$\#_n$	n	k	d	e
5	4	2	3	1
8	5	3	3	1
34	6	4	3	1
73	7	5	3	1
196	8	6	3	1
382	9	7	3	1
745	10	8	3	1
1142	11	9	3	1
1665	12	10	3	1
1976	13	11	3	1
2170	14	12	3	1
1976	15	13	3	1
1665	16	14	3	1
1142	17	15	3	1
745	18	16	3	1
382	19	17	3	1
196	20	18	3	1
73	21	19	3	1
34	22	20	3	1
8	23	21	3	1
5	24	22	3	1
1	25	23	3	1
1	26	24	3	1

Proof:
From Theorem 1, the dual code $C^{\perp}$ of each code C in Theorem 5 is also inequivalent *PG*-MDS. Since each generator matrix Q of a *PG*-MDS code C, $[n,k]$ in Theorems 2,3,4,5 is in standard form; that is, $Q = [I_2\ A_{2\times(n-2)}]$, $4 \le n \le 26$, where $A = \begin{bmatrix} 1 & \alpha^{i_1} & \dots & \alpha^{i_{(n-3)}} \\ 1 & 1 & \dots & 1 \end{bmatrix}$, so the parity-check matrix is $H = [-A^T{}_{(n-2)\times 2} I_{(n-2)}]$. So, for any $a, b \in GF(27)\backslash\{0\}$, if $-a(1, \alpha^{i_1}, \dots, \alpha^{i_{(n-3)}}) \pm b(1,1,\dots,1) = 0$, then $a + b = 0$ and $-a\alpha^{i_j} - b = 0$, $1 \le j \le (n-3)$. Thus, $b\alpha^{i_j} - b = b(\alpha^{i_j} - 1) = 0$. But $b \ne 0$; that is, $\alpha^{i_j} = 1$ for all j which is contradicted with that each α^{i_j} are distinct. Therefore, any two columns are linearly independent; that is, $C^{\perp}$ is *PG*-code. ■

It clear that, each code, $[n, n-k, 3]$ in Corollary 1 is a sub code of $[n+1, (n+1)-k, 3]$, $n > k$.
All the *PG*-codes in Theorems 2,3,4,5 are embedded in the complete *PG*-MDS code [28,2,27] which is generated by the matrix
$\mathfrak{D} = \begin{bmatrix} 1 & \beta_{i_1} & \cdots & \beta_{i_{27}} \\ 0 & 1 & \cdots & 1 \end{bmatrix}$,
where all $\beta_{i_j} \in F_{27}$ are distinct. Therefore, all *PG*-codes in Theorems 2,3,4,5 are incomplete codes.

***PG*- MDS Code of Dimension 3 over F_{27}**

In this section, *PG*-MDS codes are constructed from these ones in section two by transferring the generator matrices into other matrices which each one formed a generator matrix to a *PG*-MDS code $[n, 3, n-2]$, $3 \le n \le 28$.

If $\mathcal{W} = [c_1\ \dots\ c_n] = \begin{bmatrix} 1 & 0 & \cdots & \theta \\ 0 & 1 & \cdots & 1 \end{bmatrix}$ is *PG*-MDS code $[n, 2, n-1]$, where c_i are the columns of $\mathcal{W}$, then it could transfer to a *PG*-code $[n, 3, n-2]$ with generator matrix $\mathcal{W}^+ = [c_1{}^+\ \dots\ c_n{}^+]$ using the one to one map, T^* as follows:

$$c_i{}^T = [x, y] \xrightarrow{T^*} \begin{cases} c_i{}^+ = [(x/y)^2 - \alpha^{14}(x/y), \alpha^{10}(1 - \alpha^{12}x/y), \alpha^{16}x/y] & if\ y \ne 0 \\ c_i{}^+ = [1,0,0] & if\ y = 0 \end{cases}$$

Therefore, the generator matrix $G = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ of the unique *PG*-MDS code [3,2,2] transfer to the matrix $G^+ = \begin{bmatrix} r_1 \\ r_3 \\ r_3 \end{bmatrix} = [c_1 c_2 c_3] = \begin{bmatrix} 1 & 0 & \alpha^{19} \\ 0 & 1 & \alpha^2 \\ 0 & 0 & 1 \end{bmatrix}$. It is not difficult to prove that the rows r_1, r_2 and r_3 are linearly independents and the columns $c_{i,} c_j$ are also linearly independents. Here, d=1, Since, $w(r_1) = w(r_2) = 1$. So, G^+ formed generator matrix of the unique *PG*-MDS code [3,3,1] .

Theorem 6: Over F_{27}, the inequivalent *PG*-MDS codes with parameters in Table 2 are exists.

Table 2. *PG*- MDS Code of parameter $[n, 3, n-2]$

$\#_n$	n	k	d	e
1	4	3	2	0
8	5	3	3	1
34	6	3	4	1
73	7	3	5	2
196	8	3	6	2
382	9	3	7	3
745	10	3	8	3
1142	11	3	9	4
1665	12	3	10	4
1976	13	3	11	5
2170	14	3	12	5
1976	15	3	13	6
1665	16	3	14	6
1142	17	3	15	7
745	18	3	16	7
382	19	3	17	8
196	20	3	18	8
73	21	3	19	9
34	22	3	20	9
8	23	3	21	10
1	24	3	22	10
1	25	3	23	11
1	26	3	24	11

Proof:

- $n = 4$**:** From Theorem 2, five 3×4 matrices, ${G_i}^+$ are constructed by transferring the generator matrix $G_i, i = 1,2, \dots,5$ using the map T^* as given below.

	T^*		
${G_i}^+$	Row1	Row2	Row3
${G_1}^+$	$10\alpha^{19}\alpha^{13}$	$01\alpha^2\alpha^9$	0011
${G_2}^+$	$10\alpha^{19}\alpha^{24}$	$01\alpha^2\alpha^6$	0011
${G_3}^+$	$10\alpha^{19}\alpha^6$	$01^2\alpha^{12}$	0011
${G_4}^+$	$10\alpha^{19}\alpha^3$	$01\alpha^2\alpha^7$	0011
${G_5}^+$	$10\alpha^{19}\alpha^{15}$	$01\alpha^2\,\alpha^{16}$	0011

But ${G_i}^+ \cong_P {G_j}^+$, for each $1 \le i \ne j \le 5$ as shown below.

${G^+}_i \cong_P {G^+}_j$	$row\ 1, row\ 2, row3$ of T
${G^+}_1 \cong_P {G^+}_2$	$\alpha^3 0\alpha^{15},\quad 0\alpha^{15}\alpha^{23},\quad 001$
${G^+}_1 \cong_P {G^+}_3$	$\alpha^9 0\alpha^7,\quad 0\alpha^{19}\alpha^{13},\quad 001$
${G^+}_1 \cong_P {G^+}_4$	$\alpha^{20} 0\alpha^{23},\quad 0\alpha^{2\alpha^{13}},\quad 001$
${G^+}_1 \cong_P {G^+}_5$	$\alpha^{25} 0\alpha^{21},\quad 0\alpha^4\alpha^{22},\quad 001,$

Therefore, there is unique PG- MDS code parameters $[4,3,2]$ with generator matrix G^+:

$$G^+ = \begin{bmatrix} 1 & 0 & \alpha^{19} & \alpha^{13} \\ 0 & 1 & \alpha^2 & \alpha^9 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Also, this means that there is a unique PG-MDS code parameters with parameters $[24,3,22]$

- $n = 5$: From Theorem 3, the eight generators M_i are transformed by T^* into the following matrices ${\mathrm{M}_i}^+$ of rank 3 as shown below:

${\mathrm{M}_i}^+$	Row1	Row2	Row3
${\mathrm{M}_1}^+$	$10\alpha^{19}\alpha^{13}\alpha^{24}$	$01\alpha^2\alpha^9\alpha^6$	00111
${\mathrm{M}_2}^+$	$10\alpha^{19}\alpha^{24}\alpha^{20}$	$01\alpha^2\alpha^6\alpha$	00111
${\mathrm{M}_3}^+$	$10\alpha^{19}\alpha^{24}\alpha^6$	$01\alpha^2\alpha^6\alpha^{12}$	00111
${\mathrm{M}_4}^+$	$10\alpha^{19}\alpha^{24}\alpha^2$	$01\alpha^2\alpha^6\alpha^5$	00111
${\mathrm{M}_5}^+$	$10\alpha^{19}\alpha^{24}\alpha^{22}$	$01\alpha^2\alpha^6\alpha^{24}$	00111
${\mathrm{M}_6}^+$	$10\alpha^{19}\alpha^{24}\alpha^{21}$	$01\alpha^2\alpha^6\alpha^{18}$	00111
${\mathrm{M}_7}^+$	$10\alpha^{19}\alpha^6\alpha^{22}$	$01\alpha^2\alpha^{12}\alpha^{24}$	00111
${\mathrm{M}_8}^+$	$10\alpha^{19}\alpha^6 1$	$01\alpha^2\alpha^{12}1$	00111

These matrices, ${\mathrm{M}_i}^+$ can be transformed into the matrices ${\mathcal{L}_i}^+$of rank 3 as in Table 3.

Table 3. Equivalent matrices

	${\mathcal{L}_i}^+$
${\mathrm{M}_i}^+ \cong_P {\mathcal{L}_i}^+$	$\left[\begin{array}{c\|c} G^+ & \begin{matrix} x \\ y \\ z \end{matrix} \end{array}\right]$
${\mathrm{M}_1}^+ \cong_P {\mathcal{L}_1}^+$	$\left[\begin{array}{c\|c} G^+ & \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \end{array}\right]$
${\mathrm{M}_2}^+ \cong_P {\mathcal{L}_2}^+$	$\left[\begin{array}{c\|c} G^+ & \begin{matrix} 0 \\ \alpha \\ 1 \end{matrix} \end{array}\right]$
${\mathrm{M}_3}^+ \cong_P {\mathcal{L}_3}^+$	$\left[\begin{array}{c\|c} G^+ & \begin{matrix} \alpha^2 \\ \alpha^{14} \\ 1 \end{matrix} \end{array}\right]$
${\mathrm{M}_4}^+ \cong_P {\mathcal{L}_4}^+$	$\left[\begin{array}{c\|c} G^+ & \begin{matrix} \alpha \\ \alpha^{16} \\ 1 \end{matrix} \end{array}\right]$
${\mathrm{M}_5}^+ \cong_P {\mathcal{L}_5}^+$	$\left[\begin{array}{c\|c} G^+ & \begin{matrix} \alpha^{18} \\ \alpha^{23} \\ 1 \end{matrix} \end{array}\right]$
${\mathrm{M}_6}^+ \cong_P {\mathcal{L}_6}^+$	$\left[\begin{array}{c\|c} G^+ & \begin{matrix} \alpha^7 \\ \alpha^7 \\ 1 \end{matrix} \end{array}\right]$
${\mathrm{M}_7}^+ \cong_P {\mathcal{L}_7}^+$	$\left[\begin{array}{c\|c} G^+ & \begin{matrix} \alpha^5 \\ 0 \\ 1 \end{matrix} \end{array}\right]$
${\mathrm{M}_8}^+ \cong_P {\mathcal{L}_8}^+$	$\left[\begin{array}{c\|c} G^+ & \begin{matrix} \alpha^8 \\ \alpha^{12} \\ 1 \end{matrix} \end{array}\right]$

Therefore, there are eight unique PG-MDS code with parameters $[5,3,3]$ and generator matrix ${\mathcal{L}_i}^+$.

By using the same technique the other results will deduce. ■

Theorem 7: There is a unique, complete PG-MDS code $\mathcal{C}^*$ with parameters $[28,3,26]$ over F_{27}.

Proof:

The 28×3 matrix $M_{\mathcal{C}^*}$ with following rows has rank three and satisfies the projective conic $XY + \alpha^6 XZ +$

$\alpha^{24}YZ$; that is, the rows of $M_{\mathcal{C}^*}$ formed complete arc of degree 2 with 28 points which is Segre bound. Therefore, $\mathcal{C}^*$ has $d = n - k + 1 = 26$.

$M_{\mathcal{C}^*}$ Matrix

$C(1)$	$C(2)$	$C(3)$	$C(4)$	$C(5)$	$C(6)$	$C(7)$
1	0	0	α^{14}	α^{3}	α^{8}	α^{13}
0	1	0	α^{13}	α^{7}	α^{23}	α^{9}
0	0	1	1	1	1	1

$C(8)$	$C(9)$	$C(10)$	$C(11)$	$C(12)$	$C(13)$	$C(14)$
α^{10}	α^{12}	α^{21}	α^{24}	α^{18}	α^{20}	α
α^{3}	α^{17}	α^{18}	α^{6}	α^{8}	α	α^{21}
1	1	1	1	1	1	1

$C(15)$	$C(16)$	$C(17)$	$C(18)$	$C(19)$	$C(20)$	$C(21)$
α^{7}	α^{23}	α^{19}	α^{5}	α^{6}	α^{2}	α^{25}
α^{25}	α^{10}	α^{2}	α^{22}	α^{12}	α^{5}	α^{11}
1	1	1	1	1	1	1

$C(22)$	$C(23)$	$C(24)$	$C(25)$	$C(26)$	$C(27)$	$C(28)$
α^{17}	1	α^{15}	α^{9}	α^{22}	α^{16}	α^{4}
α^{15}	1	α^{16}	α^{20}	α^{24}	α^{4}	α^{14}
1	1	1	1	1	1	1

Corollary 3: All the codes in Theorem 6 are incomplete.

Proof:

Each generator matrix of the code in Theorem 6 is projectively equivalent to a sub matrix of $M_{\mathcal{C}^*}$, and this done by computer computation. For example, $G^+ = [C(1) \quad C(2) \quad C(17) \quad C(7)]$, $\mathcal{L}_1{}^+ = [C(1) \quad C(2) \quad C(17) \quad C(7) \quad C(3)]$. ■

Theorem 8: There are two inequivalent, complete *PG*-MDS codes over F_{27} with parameters [16,3,14].

Proof: By choosing appropriated 15 columns of the matrix $M_{\mathcal{C}^*}$ in Theorem 7, the following two 3×16 matrices $\mathfrak{U}_1$ and $\mathfrak{U}_2$ have been got:

Columns of $\mathfrak{U}_1$:

$C(1)$	$C(2)$	$C(3)$	$C(4)$	$C(5)$	$C(6)$	$C(7)$	$C(8)$
1	0	α^{3}	α^{13}	α^{24}	α^{18}	α^{11}	α
0	1	α^{7}	α^{9}	α^{6}	α^{8}	α^{19}	α^{21}
0	0	1	1	1	1	1	1

$C(9)$	$C(10)$	$C(11)$	$C(12)$	$C(13)$	$C(14)$	$C(15)$	$C(16)$
α^{7}	α^{6}	α^{25}	α^{17}	1	α^{22}	α^{16}	α^{4}
α^{25}	α^{12}	α^{11}	α^{15}	1	α^{24}	α^{4}	α^{14}
1	1	1	1	1	1	1	1

Columns of $\mathfrak{U}_2$:

$C(1)$	$C(2)$	$C(3)$	$C(4)$	$C(5)$	$C(6)$	$C(7)$	$C(8)$
1	0	0	α^{14}	α^{8}	α^{10}	α^{12}	α^{21}
0	1	0	α^{13}	α^{23}	α^{3}	α^{17}	α^{18}
0	0	1	1	1	1	1	1

$C(9)$	$C(10)$	$C(11)$	$C(12)$	$C(13)$	$C(14)$	$C(15)$	$C(16)$
α^{11}	α^{20}	α^{23}	α^{19}	α^{5}	α^{2}	α^{15}	α^{9}
α^{19}	α	α^{10}	α^{2}	α^{22}	α^{5}	α^{16}	α^{20}
1	1	1	1	1	1	1	1

From Theorem 7, any two columns are linearly independents. So, it is enough to prove the completeness for $\mathfrak{U}_1(\mathfrak{U}_2)$. To do this a computer program has been used to prove that each extra column, $c = [a, b, 1]$ to $\mathfrak{U}_1(\mathfrak{U}_2)$ will be linearly dependent with other two columns of $\mathfrak{U}_1(\mathfrak{U}_2)$. So, the rank of $\mathfrak{U}_1(\mathfrak{U}_2)$ will be less than 3. ■

Conclusion:

In this paper the extending and lengthening are used to conclude the existence of incomplete, projective MDS codes of dimension two and three over the finite field of order twenty-seven. Where if $k = 2$, codes of length n, $4 \leq n \leq 26$ and distance d, $3 \leq d \leq 25$ are founded. Also, if k =3, codes of length n, $4 \leq n \leq 26$ and distance d, $2 \leq d \leq 24$ are founded. Two complete, projective MDS have been computed of dimension three and length sixteen.

Acknowledgements

The author thanks the University of Mustansiriyah for their support; also, thanks the Department of Mathematics at College of Science for providing a good environment to do this research.

Author's declaration:

- Conflicts of Interest: None.
- I hereby confirm that all the Figures and Tables in the manuscript are mine. Besides, the Figures and images, which are not mine, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Mustansiriyah.

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ترميزات MDS الأسقاطية على $GF(27)$

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الخلاصة:
الترميز MDS هو الترمز الخطي الذي يحقق المساواة في القيد المفرد، و MDS الاسقاطي (PG-MDS) هو ترميز MDS مع خاصية الاستقلالية لأي عمودين من المصفوفة المولدة الخاصة به .في هذه البحث، تم استخدام الطرق الأولية لتعديل الترمز PG-MDS للأبعاد 2 ، 3 ، مثل الامتداد والاطالة ، من أجل ايجاد ترميزات PG-MDS جديدة غير مكتملة معرفة على الحقل $GF(27)$. أيضا ، تم ايجاد اثنين من الترميزات PG-MDS كاملة على الحقل $GF(27)$ بطول 16 و 28.

الكلمات المفتاحية: المخروط، الحقل المنتهي، المستوي الاسقاطية المنتهية، ترميزات المسافة القصوى القابلة للفصل.