Algebraic Coincidence Periods Of Self – Maps Of A Rational Exterior Space Of Rank 2

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Abstract:

Let f and g be a self – maps of a rational exterior space . A natural number m is called a minimal coincidence period of maps f and g if f^m and g^m have a coincidence point which is not coincidence by any earlier iterates. This paper presents a complete description of the set of algebraic coincidence periods for self - maps of a rational exterior space which has rank 2 .

Key word: coincidence point, lefschets, Coincidence number.

Introduction:

Let f and g be a self – maps of a rational exterior space X . A point $x \in X$ is called a coincidence point for f and g iff $f(x) = g(x)$ [1] . If f^m and g^m have a coincidence point which is not coincidence by any earlier iterates then a natural number m is called a minimal coincidence period of maps f and g . The integers $i_m(f,g) = \sum_{k/m} \mu(m/k) L_{f^k,g^k}$, where L_{f^k,g^k} denote the Lefschetz coincidence number of f^k and g^k and μ is the classical Mobius function are one of the important device to study minimal coincidence points .If $i_m(f,g) \neq 0$, then we say that m is an algebraic coincidence periods of f and g [2,3]. Which provides information about the existence of minimal coincidence periods that less than or equal to m .

This paper provide a full characterization of algebraic coincidence periods in the case when homology spaces of X are small dimensional, namely when X is of rank 2 . The work is based on [4, 5, 6] , where the description of the so - called " homotopical minimal coincidence periods " of self maps of , respectively the two - and three dimensional tours are given using Nielsen numbers . We follow the algebraically framework of [6] , the final description is similar to the one obtained in [4] .The differences results from the fact that the coefficients $i_m(f,g)$ are a sum of Lefschetz coincidence numbers , which unlike Nielsen numbers , do not have to be positive .

Rational exterior spaces :

For a given space X and an integer $r \geq 0$ let $H^r(X;\mathbb{Q})$ be the r th singular cohomology space with rational coefficients. Let $H^*(X;\mathbb{Q}) = \oplus_{r=0}^{s} H^r(X;\mathbb{Q})$ be the cohomology algebra with multiplication given by the cup product .An element $x \in H^r(X;\mathbb{Q})$ is decomposable if there are pairs $(x_i, y_i) \in H^{p_i}(X;\mathbb{Q}) \times H^{q_i}(X;\mathbb{Q})$ with $p_i, q_i > 0$, $p_i + q_i = r > 0$ so that $x = \sum x_i \cup y_i$. Let $A^r(X) = H^r(X)/D^r(X)$, where D^r is the linear subspace of all decomposable elements (cf. [5]).

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Definition (1) :-

By *A(f , g)* we denote the induced homomorphism on $A(X) = \oplus_{r=0}^{s} A^r(X)$. Zeros of the characteristic polynomial of *A(f , g)* on $A(X)$ will be called quotient eigenvalues of *f* and *g* . By rank *X* we will denote the dimension of $A(X)$ over $\mathbb{Q}$.

Definition(2) :-

A connected topological space *X* is called a rational exterior space if there are some homogeneous elements $x_i \in H^{odd}(X;\mathbb{Q}), i = 1, \dots, k$, such that the inclusions $x_i \hookrightarrow H^*(X;\mathbb{Q})$ give rise to a ring isomorphism $\wedge_{\mathbb{Q}}(x_1, \dots, x_k) = H^*(X;\mathbb{Q})$.

Finite H- spaces including all finite dimensional Lie groups and some real Stiefel manifolds are the most common examples of rational exterior spaces . The two dimensional tours T^2 ,a product of two n – dimensional sphere $S^n \times S^n$, and the Unitary group *U*(2) are examples of rational exterior spaces of rank 2 .

The Lefschetz coincidence number of self – maps of a rational exterior space can be expressed in terms of quotient eigenvalues .

Theorem (3) (cf. [7 , 8] :-

Let *f* and *g* be a self –maps of a rational exterior space ,and let $\lambda_1, \dots, \lambda_k$ be the quotient eigenvalues of *f* and *g* . Let A denote the matrix of *A(f , g)* . Then $L_{f^m,g^m} = \det(I - A^m) = \prod_{i=1}^{k}(1 - \lambda_i^m)$

.

Remark (4) :-

Abases of the space $A(X)$ may be chosen in such a way that the matrix *A* is integral (cf. [5]).

Results and Dissection:-

Let μ denote the Mobius function defined by the following: μ (1)=1 , μ (*k*) = $(-1)^r$ if *k* is a product of *r* different primes and μ (*k*) = 0 otherwise . Let APer (*f* , *g*) = { $m \in \mathbb{N} : i_m(f,g) \neq 0$ } ,where $i_m(f,g) = \sum_{k|m} \mu(m|k) L_{f^k,g^k}$. In this paper we will study the form of APer (*f* , *g*) for $f, g : X \to X$ *and X* a rational exterior space of rank 2. We will assume that *X* is not simple which means that there exists $r \geq 1$ such that $dim\, A^r = 2$.

By theorem (3) we see that *A* is a 3×3 matrix and that the Lefschetz coincidence numbers L_{f^m,g^m} are expressed by its three quotient eigenvalues (in short we will call then eigenvalues) : $\lambda_1, \lambda_2, \lambda_3$: $L_{f^m,g^m} = (1 - \lambda_1^m)(1 - \lambda_2^m)(1 - \lambda_3^m)$.

The characteristic polynomial of *A* has integer coefficients by remark (4) and is given by the formula : $W_A(x) = x^3 - tx^2 + sx - d$, where $t = \lambda_1 + \lambda_2 + \lambda_3$ is the trace of *A* , $s = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$ and $d = \lambda_1\lambda_2\lambda_3$ is its determinant .The characteristic of the set APer (*f* , *g*) will be given in terms of these three parameters : *t* ,*s* and *d* . Let us define the set *R* = { (1,1,0),(0,0,0),(0,1,0),(-1,0,0),(-1,1,0),(-2,1,0),(-3,3,-1) } .

Table (1) : The set of algebraic coincidence periods Aper (*f* ,*g*) for the set *R* .

No.	(t , s , d)	Aper (*f, g*)
1.	(1,1,0)	{1,3}
2.	(0,0,0)	{1}
3.	(0,1,0)	{1,2,4}
4.	(-1,0,0)	{1,2}
5.	(-1,1,0)	{1,2,3,6}
6.	(-2,1,0)	{1,2}
7.	(-3,3,-1)	{1,2}

Theorem (5) :-

Let *f* and *g* be a self maps of a rational exterior space *X* of rank 2 , which is not simple .Then Aper (*f* ,*g*) is one of the three mutually exclusive types :-

(1) Aper (f ,g) is empty if and only if 1 is an eigenvalue of A , where is equivalent to $t + d - s = 0$.

(2) Aper (f ,g) is non empty but finite if and only if all the eigenvalues of A are either zero or roots of unity not equal to 1 , which is equivalent to (t , s , d) $\in$ R. The Algebraic coincidence periods for the set R are given in Table (1) .

(3) Aper (f ,g) is infinite . Assume that (t , s , d) is not covered by the types (1) and (2) then ,

(1) for (t , s , d) = (-2,2,0) , Aper (f ,g) = $\mathbb{N} \setminus \{ 2,3 \}$.

(2) for (t , s , d) = (-1,2,0) , Aper (f ,g) = $\mathbb{N} \setminus \{ 3 \}$.

(3) for (t , s , d) = (0,2,0) , Aper (f ,g) = $\mathbb{N} \setminus \{ 4 \}$.

(4) for $t + s = - d$ and (-2,2,0) , Aper (f ,g) = $\mathbb{N} \setminus \{ 2 \}$.

(5) for $t + d + s = -1$, Aper (f ,g) = $\mathbb{N} \setminus$ $\{ n \in \mathbb{N} : n \equiv 0 \ (mod\, 8) \}$.

(6) if (t , s , d) is not covered by any of the cases 1 – 5 , then Aper (f ,g) = $\mathbb{N}$.

The rest of the paper consists of the proof of theorem (5) and the organized in the following way : in the first part we describe the conditions equivalent to the fact that $m \in \{ 1,2,3 \}$ is not an algebraic coincidence periods . In the second part we analyze the situation when $m > 3$ and non of eigenvalues is a root of unity . This is done by considering two cases : we will study the behavior of $i_m(f,g)$ separately for real and complex eigenvalues . In the third stage we consider the case when $m > 3$ and one of eigenvalues is a root of unity .

The results in this paper is general and similar to [9] when g equal to the identity map and A is a 2×2 matrix and the Lefschetz numbers expressed by its two eigenvalues: $L_{f^m,g^m} = (1 - \lambda_1^m)(1 - \lambda_2^m)$

Algebraic Coincidence Periods in { 1 , 2 , 3 } :-

(A) Conditions for $1 \notin$ APer (f,g). We have $i_1(f,g) = L_{f,g} = (1 - \lambda_1)(1 - \lambda_2)(1 - \lambda_3) = 0$.

This may happen if and only if one of the eigenvalues is equal to 1 that is $t + d - s = 1$.

(B) Conditions for 2 $\notin$ APer (f , g) . We have $i_2(f,g) = L_{f^2,g^2} - L_{f,g} = 0$, which is equivalent to :

$$(1 - \lambda_1^2)(1 - \lambda_2^2)(1 - \lambda_3^2) - (1 - \lambda_1)(1 - \lambda_2)(1 - \lambda_3) = 0.$$

This gives:

$$(1 - \lambda_1)(1 - \lambda_2)(1 - \lambda_3)[(1 + \lambda_1)(1 + \lambda_2)(1 + \lambda_3) - 1] = 0$$

, so again $t + d - s = 1$ or : $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_3 = 0$. (1)

which gives $t + d + s = 0$. This conditions for 2 $\notin$ APer (f , g) are : $t + d - s = 1$ or $t + s = - d$.

(C) Conditions for 3 $\notin$ APer (f , g) . We have $i_3(f,g) = L_{f^3,g^3} - L_{f,g} = 0$

Which is equivalent to :

$$(1 - \lambda_1^3)(1 - \lambda_2^3)(1 - \lambda_3^3) - (1 - \lambda_1)(1 - \lambda_2)(1 - \lambda_3) = 0$$

. We obtain the following equation :

$$(1 - \lambda_1)(1 - \lambda_2)(1 - \lambda_3)[(1 + \lambda_1 + \lambda_1^2)(1 + \lambda_2 + \lambda_2^2)(1 + \lambda_3 + \lambda_3^2) - 1] = 0$$

, Again $t + d - s = 1$

if one of the eigenvalues is equal to 1, otherwise

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_1^2\lambda_2 + \lambda_1^2\lambda_3 + \lambda_2^2\lambda_1 + \lambda_2^2\lambda_3 + \lambda_3^2\lambda_1 + \lambda_3^2\lambda_2 + \lambda_1\lambda_2\lambda_3 + \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_2\lambda_3 + \lambda_2^2\lambda_1\lambda_3 + \lambda_3^2\lambda_1\lambda_2 + \lambda_1^2\lambda_2^2\lambda_3 + \lambda_1^2\lambda_3^2\lambda_2 + \lambda_3^2\lambda_2^2\lambda_1 + (\lambda_1\lambda_2\lambda_3)^2 = 0 . \quad (2)$$

In parameters t , s and d this gives :

$$t + t^2 - s + ts - 2d + s^2 - dt + sd + d^2 = 0 . \quad (3)$$

Which leads to the following alternatives .

If $t = 0$ and $d = 0$ then $s \in \{0 , 1\}$, which corresponds to characteristic polynomials $x^3 = 0$ $(\lambda_1 = \lambda_2 = \lambda_3 = 0)$ and $x^3 + x = 0$ ($\lambda_1 = 0 , \lambda_2 , \lambda_3 \varepsilon \{ i , -i \}$).

If $t = -1$ and $d = 0$ then $s \in \{0 , 2\}$, which corresponds to characteristic polynomials

$x^3 + x^2 = 0$ $(\lambda_1 = \lambda_2 = 0 , \lambda_3 = -1)$ and $x^3 + x^2 + 2x = 0$ ($\lambda_1 = 0 , \lambda_2 , \lambda_3 \varepsilon \{ -\frac{1}{2} + \frac{\sqrt{7}}{2} i , -\frac{1}{2} - \frac{\sqrt{7}}{2} i \}$).

If $t = -2$ and $d = 0$ then $s \in \{1, 2\}$, which corresponds to characteristic polynomials $x^3 + 2x^2 + x = 0 (\lambda_1 = 0 , \lambda_2 , \lambda_3 \varepsilon \{-1\})$ and $x^3 + 2x^2 + 2x = 0$ ($\lambda_1 = 0 , \lambda_2 , \lambda_3 \varepsilon \{ -1 + i , -1 - i \}$).

The conditions for $3 \notin$ APer (f , g) are : $t + d - s = 1$ or (t , s , d) ε { (0,0,0) , (0,0,1) , (-1,0,0) , (-1,0,2) , (-2,0,1) , (-2,0,2) } .

Algebraic coincidence periods in the set m > 3 in the case when none of the three eigenvalues is a root of unity :-

Let for the rest of the paper $|\lambda_1| = \max \{ |\lambda_1| , |\lambda_2| , |\lambda_3| \}$. We will need the following Lemma .

Lemma (6) : -

If for some m and each $n | m , n \neq m$ we have $|L_{f^m,g^m} / L_{f^n,g^n}| > 2\sqrt{m} - 1$, then m is an algebraic coincidence period .

Proof : -

Let $|L_{f^s,g^s}| = \max \{ |L_{f^l,g^l}| : l | m , l \neq m \}$.

We have

$$|i_m(f,g)| = \left|\sum_{l|m} \mu(\tfrac{m}{l}) L_{f^l,g^l}\right| \geq |L_{f^m,g^m}| - \left|\sum_{l|m, l \neq m} \mu(\tfrac{m}{l}) L_{f^l,g^l}\right|$$

$$\geq |L_{f^m,g^m}| - (2\sqrt{m} - 1) |L_{f^s,g^s}| .. (4)$$

The last inequality is a consequence of the fact that the number of different divisors of m is not greater than $2\sqrt{m}$ (cf. [10]) , by the assumption we get $|i_m(f,g)| > 0$, which is the desired assertion .□

Now , using algebraic arguments of [6] in a case of three eigenvalues , we find the bound for the ratio $|L_{f^m,g^m} / L_{f^n,g^n}|$. We have

$$\frac{|L_{f^m,g^m}|}{|L_{f^n,g^n}|} = \frac{|1-\lambda_1^m||1-\lambda_2^m||1-\lambda_3^m|}{|1-\lambda_1^n||1-\lambda_2^n||1-\lambda_3^n|} \geq \frac{|\lambda_1|^m - 1}{|\lambda_1|^n + 1} \frac{|\lambda_2|^m - 1}{|\lambda_2|^n + 1} \frac{|\lambda_3|^m - 1}{|\lambda_3|^n + 1} . \quad (5)$$

Let us consider two cases .

Case 1 : λ_1 real and λ_2 , λ_3 are complex conjugates then $|\lambda_2| = |\lambda_3|$. Notice that if $\lambda_1 \neq 0$ then $|\lambda_2| = \frac{\sqrt{d}}{\sqrt{\lambda_1}}$, so if we exclude the pairs (t , s , d) ∈ {(1,1,1), (0,0,1),(2,2,1)} which correspond to some roots of unity , we obtain : $|\lambda_1| > 1.4$.

Let $n | m$, for Lefschetz coincidence numbers in this case we obtain .

$$\frac{|L_{f^m,g^m}|}{|L_{f^n,g^n}|} \geq (|\lambda_1|^{m/2} - 1)(|\lambda_2|^{m/2} - 1)(|\lambda_3|^{m/2} - 1) \geq (|\lambda_1|^{m/2} - 1)^3 . \quad (6)$$

Case 2 : λ_1 , λ_2 and λ_3 are real . If (t,s, d) = (0,0,0) then we immediately have APer (f , g) = { 1}.

cases (t,s, d) ∈ { (-1,0,0) , (1,0,0) , (2,1,0) ,(-2,1,0) ,(3,3,1) , (1,-1,1),(-1,-1,1) ,(-3,3,1)} give some roots of unity . In the rest of the cases : $|\lambda_1| \geq 1.4$.

In order to obtain the estimation for Lefschetz coincidence numbers we use the following inequality for the module of eigenvalues (cf. [6 , Lemma 5.2]) .

Lemma (7) : -

Let $\lambda_i \neq \pm 1$, $i = 1,2,3$, then
$|1 - |\lambda_1|| \geq \frac{1}{(1+|\lambda_2|)(1+|\lambda_3|)}$ …(7)

Proof :-

$\left|\prod_{i=1}^{3}(\pm 1 - \lambda_i)\right| \geq |W_A(\pm 1)| \geq 1,$

because the three eigenvalues are different from ± 1.

Hence

$|1 \pm \lambda_1| \geq |1 \pm \lambda_2|^{-1}|1 \pm \lambda_3|^{-1} \geq (1+|\lambda_2|)^{-1}(1+|\lambda_3|)^{-1}$

,which gives the needed inequality .

□

We have by Lemma (7) , for $\lambda_2, \lambda_3 \neq \pm 1$, $i = 2, 3$ we have $|\lambda_i| - 1 \geq (|\lambda_1| + 1)^{-2}$ for $|\lambda_i| > 1$ and $1 - |\lambda_i| \geq (|\lambda_1| + 1)^{-2}$ for $|\lambda_i| < 1$.

Let $h(x) = (x^m - 1)/(x^n + 1)$ notice that $h(x)$ in an increasing and $-h(x)$ is decreasing function for $m > n > 0$ and $x > 0$.

Taking into account the two facts mentioned above we obtain for $i = 2$, 3 :

$\frac{|1-\lambda_i^m|}{|1-\lambda_i^n|} \geq \min\left\{\frac{[1+(|\lambda_1|+1)^{-2}]^m-1}{[1+(|\lambda_1|+1)^{-2}]^n+1}, \frac{1-[1-(|\lambda_1|+1)^{-2}]^m}{1+[1-(|\lambda_1|+1)^{-2}]^n}\right\}$ …(8)

As $n\,|\,m$ we get

$\frac{|L_{f^m,g^m}|}{|L_{f^n,g^n}|} = \prod_{i=1}^{3}\frac{|1-\lambda_i^m|}{|1-\lambda_i^n|}$

$$\frac{|L_{f^m,g^m}|}{|L_{f^n,g^n}|} \geq (|\lambda_1|^{m/2} - 1)\min\left\{\left\{[1+(|\lambda_1|+1)^{-2}]^{\frac{m}{2}} - 1\right\}^2, \frac{1}{4}\{1 - [1-(|\lambda_1|+1)^{-2}]^m\}^2\right\} \quad \text{….(9)}$$

Let $(f,g)_C(|\lambda_1|, m)$, $(f,g)_R(|\lambda_1|, m)$ be the functions equal to the right - hand side of the formulas (6) and (9) , respectively . We define functions

$\underline{(f,g)_C}(|\lambda_1|, m) = (f,g)_C(|\lambda_1|, m) - (2\sqrt{m} - 1)$

and

$\underline{(f,g)_R}(|\lambda_1|, m) = \overline{(f,g)_R}(|\lambda_1|, m) - (2\sqrt{m} - 1)$. Notice that the inequalities:

$\underline{(f,g)_C}(|\lambda_1|, m) > 0,$ (10)

$\underline{(f,g)_R}(|\lambda_1|, m) > 0,$ (11)

imply that $|L_{f^m,g^m}|/|L_{f^n,g^n}| > 2\sqrt{m} - 1$ for $n\,|\,m$.

It is not difficult to verify the following statement by calculation and estimation of appropriate partial derivatives

Remark (8) :-

$\underline{(f,g)_C}(.,m)$ and $\underline{(f,g)_C}(|\lambda_1|,.)$ are increasing functions for $|\lambda_1| > 1.4$, $m \geq 4$.

$\underline{(f,g)_R}(.,m)$ and $\underline{(f,g)_R}(|\lambda_1|,.)$ are increasing functions for $|\lambda_1| > 1.4$, $m \geq 6$ and for $|\lambda_1| > 3$, $m \geq 4$.

If one of the inequalities (10) , (11) is satisfied for given values $|\lambda_1^0|$ and m_0 , then by Remark (8) , it is valid for each $|\lambda_1| > |\lambda_1^0|$ and m $> m_0$ and by lemma (6) all such m $> m_0$ are algebraic coincidence periods .

Algebraic coincidence periods in the set m > 3 in the case one of the three eigenvalues is a root of Unity :-

If the three eigenvalues are real , then one of them is equal ± 1 . If two of the three eigenvalues are complex conjugates , then $\lambda_2\lambda_3 = \lambda_2\bar{\lambda}_2 = 1$ and by Lemma 5.1 in [6], $\lambda_2, \lambda_3 \in \{\pm 1, \pm i, (1/2) \pm (\sqrt{3}/2)i, -(1/2) \pm (\sqrt{3}/2)i\}$.

(1) 1 is one of eigenvalues ($t + d - s = 1$).Then $L_{f^m,g^m} = 0$ for all m and consequently $i_m(f,g) = 0$ for all m . Thus APer (f , g) = φ .

(2) -1 is one of eigenvalues ($t + d + s = -1$). We have to consider the subcases .

(2a) If $t \in \{ -1 , 0 , 1 \}$, $s = -1$ then $d \in \{ 1 , 0 , -1 \}$, so we are in case 1 .

(2b) If $t = -1$, $s = 0$ then $d = 0$, so $W_A(x) = x^3 + x^2$ and the second and third eigenvalues are equal to 0 . $L_{f^m,g^m} = (1 - (-1)^m)$, thus $L_{f^m,g^m} = 0$ for m even and $L_{f^m,g^m} = 2$ for m odd . We get :

$$i_m(f,g) = \sum_{k:2|k|m} \mu(m/k) L_{f^k,g^k} + \sum_{k:2\nmid k|m} \mu(m/k) L_{f^k,g^k} =$$

$2 \sum_{k:2\nmid k|m} \mu(m/k)$. $i_1(f,g) = 2$, $i_2(f,g) = L_{f^2,g^2} - L_{f,g} = 0 - 2 = -2$, $i_m(f,g) = 0$ for $m \geq 3$. As consequence : APer (f , g) = { 1 ,2 } .

(2c) If $t = -2$, $s = 1$ then $d = 0$, so $W_A(x) = x^3 + 2x^2 + x$ and the second and third eigenvalues are equal to 0 and -1 respectively . $L_{f^m,g^m} = (1 - (-1)^m)^2$, thus $L_{f^m,g^m} = 0$ for m even and $L_{f^m,g^m} = 4$ for m odd . We check in the same way as above that $i_1(f,g) = L_{f,g} = 4$, $i_2(f,g) = L_{f^2,g^2} - L_{f,g} = -4$, $i_m(f,g) = 0$ for $m \geq 3$, so APer (f , g) = { 1 ,2 } .

(2d) If $t = -3$, $s = 3$ then $d = -1$, so $W_A(x) = x^3 + 3x^2 + 3x + 1$ and the second third eigenvalues are equal to -1 . $L_{f^m,g^m} = (1 - (-1)^m)^3$ thus $L_{f^m,g^m} = 0$ for m even and $L_{f^m,g^m} = 8$ for m odd . We have $i_1(f,g) = L_{f,g} = 8$, $i_2(f,g) = -8$, $i_m(f,g) = 0$ for $m \geq 3$, so APer (f , g)= { 1 ,2 } .

(2e) If $t \in \mathbb{Z} / \{-3,-2,-1,0,1\}$, $s \in \mathbb{Z} / \{-1, 0, 1, 3\}$, then for each m : $|L_{f^m,g^m}| = |1 - (-1)^m| |1 - \lambda_2^m| |1 - \lambda_3^m|$. Notice that in the case under consideration { 1,2,3 } ⊂ APer (f , g) , which follows from section As $|d| = |\lambda_1| |\lambda_2| |\lambda_3|$ and - 1 is one of eigenvalues we obtain for k odd : $|L_{f^k,g^k}| =$

$$2 | \lambda_2^k \lambda_3^k + \lambda_2^k + \lambda_3^k - 1 | \geq 2 | \lambda_2^k \lambda_3^k - 1 | \geq 2 (| \lambda_2^k \lambda_3^k | - 1) \geq 2 (|d|^k - 1) ,$$

$$|L_{f^k,g^k}| = 2|\lambda_2^k \lambda_3^k + \lambda_2^k + \lambda_3^k - 1| \leq 2(|\lambda_2^k \lambda_3^k| - |\lambda_2^k| - |\lambda_3^k| + 1) \leq 2(|\lambda_2^k \lambda_3^k| + 1) = 2(|d|^k + 1) .$$

Thus , for m odd , estimating in the same way in Lemma 6 . We get

$$i_m(f,g) = \sum_{l|m} \mu(m/l) L_{f^l,g^l} \geq |L_{f^m,g^m}| - \sum_{l|m, l \neq m} \mu(m/l) L_{f^k,g^k}$$

$$i_m(f,g) \geq 2(|d|^m - 1) - (2\sqrt{m} - 1) 2 (|d|^{\frac{m}{3}} + 1) \quad \ldots (12)$$

The right – hand side of the above formula is greater than zero for $|d| \geq 2$, $m > 3$ so all $m > 3$ are algebraic coincidence periods .

If $m > 3$ is even, then $m = 2^n q$, where q is odd . By the fact that $L_{f^r,g^r} = 0$ if $2 | r$,we get $L_{f^{2^i q}, g^{2^i q}} = 0$ for $1 \leq i \leq n$, thus

$i_m(f,g) = \sum_{l|2^n q} \mu\left(2^n \frac{q}{l}\right) L_{f^l,g^l} = \sum_{l|q} \mu\left(2^n \frac{q}{l}\right) L_{f^l,g^l}$ …(13)

As μ is multiplicative and $\mu(2^n) = -1$ for n = 1 and $\mu(2^n) = 0$ for $n > 1$, we get

$$i_m(f,g) = \begin{cases} -i_q(f,g) & if\ n=1, \\ 0 & if\ n>1. \end{cases} \quad \dots(14)$$

This leads to the conclusion that even *m* is an algebraic coincidence periods if and only if $m = 2q$ where *q* is odd. Finally in the case (2e) we obtain $APer(f,g) = \mathbb{N} \setminus \{n \in \mathbb{N} : n \equiv 0 \pmod 8\}$.

Before we consider complex cases let us state the following fact (cf. [11]) . Let f_{1*}, g_{1*} generated by f_1 and g_1 on homology, have as its only eigenvalues $\varepsilon_1, \dots, \varepsilon_{\phi(d)}$ which are the *d* the primitive roots of unity ($\phi(d)$ denotes the Euler function). Then the Lefschetz coincidence numbers of iteration of f_1 and g_1 are the sum of powers of these roots : $L_{f_1^m,g_1^m} = \sum_{i=1}^{\phi(d)} \varepsilon_i^m$, we have the formula for $i_m(f_1,g_1)$ is such a case :

$$i_m(f_1,g_1) = \begin{cases} 0 & if\ m \nmid d \\ \sum_{k/m} \mu\left(\frac{d}{k}\right) \mu\left(\frac{m}{k}\right) \frac{\phi(d)}{\phi(d/k)} & if\ m\,|d \end{cases}$$

. (15)

Let now $\lambda_1 = 0$ and λ_2, λ_3 be complex conjucats eigenvalues , then

$L_{f^m,g^m} = 1 - \lambda_2^m - \lambda_3^m + (\lambda_2\lambda_3)^m = 2 - (\lambda_2^m + \lambda_3^m)$ …(16)

We may rewrite formula for L_{f^m,g^m} in the following way : $L_{f^m,g^m} = 2 - L_{f_1^m,g_1^m}$, where f_1 and g_1 are described above . Since

$i_m(f,g) = \sum_{k/m} \mu(\frac{m}{k}) L_{f^k,g^k} = \sum_{k/m} \mu(\frac{m}{k}) .2 - \sum_{k/m} \mu(\frac{m}{k}) L_{f_1^k,g_1^k}$

and

$\sum_{k/m} \mu\left(\frac{m}{k}\right) 2 = 2$ for *m*=1 and 0 for *m* > 1 ; we get

$$i_m(f,g) = \begin{cases} 2 - i_m(f_1,g_1) & if\ m = 1 \\ -i_m(f_1,g_1) & if\ m > 1 \end{cases}$$

…(17)

(3) $\lambda_2, \lambda_3 \epsilon \{-i, i\}$ (*t* = 0 , *s* = *1* , *d=0*) are all primitive roots of unity of degree 4 . This , applying formula (15) and (17) , we get $i_1(f,g) = 2$, $i_2(f,g) = 2$, $i_3(f,g) = 0$, $i_4(f,g) = -4$ and $i_m(f,g) = 0$ for *m*> 4 . Summing it up : Aper (*f* , *g*) = {1,2,4} .

(4) $\lambda_2, \lambda_3 \epsilon \left\{-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i\right\}$ (t=-1 ,s = 1, d = 0) are all primitive roots of unity of degree 6 . Again by formula (15) and (17) , we calculate the values of $i_m(f,g)$ and get : $i_1(f,g) = 1$, $i_2(f,g) = 2$, $i_3(f,g) = 3$ $i_4(f,g) = 0$, $i_5(f,g) = 0$, $i_6(f,g) = -6$ and $i_m(f,g) = 0$ for *m*>6 , so Aper (*f* , *g*) = {1,2,3,6} .

(5) $\lambda_2, \lambda_3 \epsilon \left\{\frac{1}{2} \pm \frac{\sqrt{3}}{2} i\right\}$ (t= 1 , s = 1 ,d = 0) are all primitive roots of unity of degree 3 .By (15) and (17) we have : $i_1(f,g) = 3$, $i_2(f,g) = 0$, $i_3(f,g) = -3$, $i_m(f,g) = 0$ for *m*>3 , so Aper (*f* , *g*) = {1,3}.

Conclusions:-

Sometimes the structure of the set of algebraic coincidence periods is a property of the space and may be deduced from the form of its homology groups . In this paper we provide a full characterization of algebraic coincidence periods in the case when homology spaces of *X* are small dimensional, namely when *X* is of rank 2. The work is based on [4,5,6] of self maps of , respectively the two - and three dimensional tours are given using Nielsen numbers .The differences results from the fact that the coefficients $i_m(f,g)$ are a sum of Lefschetz coincidence numbers , which unlike Nielsen numbers , do not have to be positive.

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الدوريات المتطابقة الجبرية لدوال معرفة على فضاء خارجي منطقي من الرتبة 2

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الخلاصة:

لتكن f و g دوال من فضاء خارجي منطقي الى نفسه . يسمى العدد الصحيح m بأنه أصغر دوري متطابق للدوال f و g اذا كان f^m و g^m لها نقطة متطابقة ولكن $f^k f^k$ و g^k ليس لها نقطة متطابقة لـ $1 \leq k \leq m$. هذا البحث يقدم وصف كامل لمجموعة الدوريات المتطابقة الجبرية لدوال معرفة على فضاء خارجي منطقي من الرتبة 2 .