

# Local and Global Uniqueness Theorems of the N-th Order Partial Differential Equations

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## Abstract:

In this paper, we consider inequalities in which the function is an element of n-th partially order space. Local and Global uniqueness theorem of solutions of the n-th order Partial differential equation Obtained which are applications of Gronwall's inequalities.

**Key words:** Local and global Uniqueness theorem, N-th order partial differential equations.

## Introduction:

Consider the differential equation of the type

$$U(x_1, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_n, U)$$

$$U(x_0, x_i) = g(x)$$

$$\text{With } U(x_i, x_0) = h(x)$$

$$g(x_i) = h(x_0)$$

$$i=0, 1, 2, \dots, n \dots (1)$$

$$\text{where } J=[0, a], a>0,$$

$f \in [J \times R^n, R]$  and  $R^n$  denotes the real n-dimensional Euclidean space,  $x_i, i=0, 1, \dots, n-1$  is a real positive constants and both  $g(x)$  and  $h(x)$  are continuous functions, Liu and Ge[2] based on the coincidence degree method of Gaines and Mawhin [3].

Proved that (1) has at least one solution  $U$ . Elias [4] proved the existence of global at least one solution to (1).

In this paper, Bihari's inequality is applied to obtain local uniqueness and Gronwall's inequality to obtain global uniqueness of solution to (1).

It is important mentioning that it was shown by Baihov D. and Simeonov [1] that the solution of (1) is of the form:

$$U(x) = \sum_{i=0}^{n-1} \frac{U_{i-1}}{i!} + \int_{x_0}^{x_1} \int_{x_2}^{x_2} \dots \int_{x_{n-1}}^{x_n} f(v, s, \dots, t, U) dv ds \dots dt$$

$$x \in J^{0,1,2,\dots,n} \dots (2)$$

with  $0 < x_i < a, a > 0$  and

$$U(x_0, x_i) = g(x)$$

$$U(x_i, x_0) = h(x)$$

$$g(x_i) = h(x_0)$$

are initial constant.

## - Local Uniqueness:

In this section, a local uniqueness result is proved by applying Bihari's inequality theorem.

## Bihari's Inequality Theorem [1]

Suppose the following conditions holds:

1.  $a(t)$  is positive continuous function in  $J = [\alpha, \beta)$

2.  $K_j(t, s), j = 1, 2, 3, \dots, n$ , are non negative continuous functions for  $\alpha \leq s \leq t \leq \beta$  which are no decreasing in  $t$  for any fixed  $s$ .

3.  $g_j(u), j = 1, 2, \dots, n$  are non decreasing continuous functions in  $R_+$ , with  $g_j(u) > 0$  for  $u > 0$  and

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$g(au) \leq r(a)w(u)$ , for  $a > 0, u \geq 0$   
 where  $r(a)$  is non negative continuous  
 function in  $R_+$ , which is positive for  
 $u > 0$ .

$4. u(t)$  is non negative continuous  
 function in  $J$  and

$$u(t) \leq a(t) + \sum_{j=1}^n \int_{\alpha}^t k_j(t,s) g_j(u(s)) ds, t \in J$$

then

$$r(t) \leq a(t) \psi_{n-1}(t) G_n^{-1} \left[ G_n(1) + \frac{r_n(a(t)) \psi_{n-1}(t)}{a(t)} \right] \int_0^t k_n(t,s) ds$$

, where

$$G_n(u) = \int_{u_n}^u \frac{dx}{g_n(x)}, u > 0, (u_n > 0)$$

### **Theorem 1: (Local Uniqueness)**

The initial value problem (1)  
 has a unique solution on the interval  
 $0 < u < a$ , if the function  $f$  is continuous  
 in the region

$$0 < x < a, |(x_1, x_2, \dots, x_n, u) - (x_1, x_2, \dots, x_n, v)| \leq b$$

and such that

$$|f(x_1, x_2, \dots, x_n, u) - F(x_1, x_2, \dots, x_n, v)| \leq \sum_{i=0}^{n-1} \phi_i(|U^i - V^i|)$$

where  $\phi(z)$  is a continuous non  
 decreasing function on  $0 < z < A$ , with  
 $\phi(0) = 0$ ,  $b > 0$  and  $A$  is a positive  
 constant.

Proof:

Let  $U(x)$  and  $V(x)$  be two  
 solutions to (1) which are defined in  
 neighborhood at the right of  $x_0$ . That is

$$\begin{aligned} U(x) &= \sum_{i=0}^{n-1} (g(x_i) + h(x_i) - g(x_0)) \\ &+ \int_{x_0}^{x_1} \int_{x_1}^{x_2} \dots \int_{x_{n-1}}^{x_n} f(v, s, \dots, t, U) dv ds \dots dt \\ V(x) &= \sum_{i=0}^{n-1} (G(x_i) + H(x_i) - G(x_0)) \\ &+ \int_{x_0}^{x_1} \int_{x_1}^{x_2} \dots \int_{x_{n-1}}^{x_n} F(v, s, \dots, t, V) dv ds \dots dt \end{aligned}$$

This leads easily to

$$|U(x) - V(x)| \leq \sum_{i=0}^{n-1} \left[ |g(x_i) - G(x_i)| + |h(x_0) - H(x_0)| + |g(x_0) - G(x_0)| \right] + \int_{x_0}^{x_1} \int_{x_1}^{x_2} \dots \int_{x_{n-1}}^{x_n} \left| \frac{f(v, s, \dots, t, U) - F(v, s, \dots, t, V)}{F(v, s, \dots, t, V)} \right| dv ds \dots dt$$

$$\text{If } \prod_{i=0}^{n-1} (x_{i-1} - x_i) \geq 0 \quad \text{and}$$

$$|f(v, s, \dots, t, U) - F(v, s, \dots, t, V)| < \epsilon$$

$$\text{and } \sum_{i=0}^{n-1} \left[ |g(x_i) - G(x_i)| + |h(x_0) - H(x_0)| + |g(x_0) - G(x_0)| \right] < \epsilon$$

$$|U(x) - V(x)| \leq \sum_{i=0}^{n-1} \epsilon_i + \epsilon \prod_{i=0}^{n-1} (x_{i-1} - x_i)$$

$$+ \int_{x_0}^{x_1} \int_{x_1}^{x_2} \dots \int_{x_{n-1}}^{x_n} K |U - V| dv ds \dots dt$$

Let  $r(x)$  be the right hand side of the  
 above inequality,

$$\begin{aligned} r(x) &\leq \sum_{i=0}^{n-1} \epsilon_i + \epsilon \prod_{i=0}^{n-1} (x_{i-1} - x_i) \\ &+ \int_{x_0}^{x_1} \int_{x_1}^{x_2} \dots \int_{x_{n-1}}^{x_n} \epsilon_i + (v - x_0)(s - x_1) \dots (t - x_n) K z(v, s, \dots, t) dv ds \dots dt \end{aligned}$$

If only  $\prod_{i=0}^{n-1} (x_{i-1} - x_i) \geq 0$  on a compact  
 space, the last equation is bounded by a  
 constant  $M$ .  $|U - V| \leq M$  on this set  
 .consequently, the solution of such  
 boundary value problem equation (1)  
 depends continuously on  $f$  and the  
 boundary data .if  $\epsilon \rightarrow 0, |U - V| \rightarrow 0$ ,  
 on the compact set.

By using Bihari's inequality we have

$$r(x) \leq \psi_{n-1}(x) G_n^{-1} \left[ G_n(1) + \frac{r_n(\epsilon) \psi_{n-1}(x)}{\epsilon} \right]$$

### **Global Uniqueness**

The global uniqueness for the  
 initial value problem (1) will be  
 discussed with the aid of Gronwall's  
 inequality, which seems by the  
 following theorem.

### Gronwall's Inequality Theorem [6]

Let  $a(t)$ ,  $b(t)$  and  $u(t)$  be continuous functions in  $J=[\alpha, \beta]$  and let  $b(t)$  be a nonnegative in  $J=[\alpha, \beta]$  and  $a(t)$  is nondecreasing in  $J=[\alpha, \beta]$  suppose

$$u(t) \leq a(t) + \int_{\alpha}^t b(s)u(s) ds, \quad t \in J$$

Then

$$u(t) \leq a(t)e^{\int_{\alpha}^t b(s)ds}, \quad t \in J$$

### Theorem (2) (Global uniqueness theorem)

Assume that:

1.  $f$  is a continuous function in the region  $R = \{(s(x_1, x_2, \dots, x_n, U)) : 0 < x < a, |(x_1, x_2, \dots, x_n, u) - (x_1, x_2, \dots, x_n, v)| \leq b\} \subset \Omega$  where  $\Omega$  is an open  $(x_0, x_1, x_2, \dots, x_n, u)$  in  $R^{n+1}$  with  $a, b > 0$ .
2.  $f$  satisfy Lipschitz condition with respect to  $(x_0, x_1, x_2, \dots, x_n, u)$ ,

$$|f(x_1, x_2, \dots, x_n, u) - F(x_1, x_2, \dots, x_n, v)| \leq$$

$$L \sum_{i=0}^{n-1} |U^i - V^i|$$

For some positive constant  $L$ , then the solution of (1) is unique.

#### Proof

Let  $U(x)$  and  $V(x)$  be two solutions to (1) then

$$U(x) = \sum_{i=0}^{n-1} (g(x_i) + h(x_i) - g(x_0)) +$$

$$\int_{x_0}^{x_1} \int_{x_2}^{x_2} \dots \int_{x_{n-1}}^{x_n} f(v, s, \dots, t, U) dv ds \dots dt$$

$$V(x) = \sum_{i=0}^{n-1} (G(x_i) + H(x_i) - G(x_0)) +$$

$$\int_{x_0}^{x_1} \int_{x_2}^{x_2} \dots \int_{x_{n-1}}^{x_n} F(v, s, \dots, t, V) dv ds \dots dt$$

$$x \in J$$

From which we get

$$\begin{aligned} |U(x) - V(x)| &\leq \sum_{i=0}^{n-1} \left[ |g(x_i) - G(x_i)| + |h(x_0) - H(x_0)| + |g(x_0) - G(x_0)| \right] + \\ &\quad \int_{x_0}^{x_1} \int_{x_2}^{x_2} \dots \int_{x_{n-1}}^{x_n} \left| f(v, s, \dots, t, U) - F(v, s, \dots, t, V) \right| dv ds \dots dt \\ &\leq \sum_{i=0}^{n-1} \left[ |g(x_i) - G(x_i)| + |h(x_0) - H(x_0)| + |g(x_0) - G(x_0)| \right] + \\ &\quad \int_{x_0}^{x_1} \int_{x_2}^{x_2} \dots \int_{x_{n-1}}^{x_n} L \left| U(v, s, \dots, t) - V(v, s, \dots, t) \right| dv ds \dots dt \end{aligned}$$

Let

$$z(v, s, \dots, t) = |U(v, s, \dots, t) - V(v, s, \dots, t)| L$$

Then

$$\begin{aligned} |U(x) - V(x)| &\leq \sum_{i=0}^{n-1} \left[ |g(x_i) - G(x_i)| + |h(x_0) - H(x_0)| + |g(x_0) - G(x_0)| \right] + \\ &\quad + \int_{x_0}^{x_1} \int_{x_2}^{x_2} \dots \int_{x_{n-1}}^{x_n} z(v, s, \dots, t) dv ds \dots dt \end{aligned}$$

Let  $r(x)$  equal to the right hand side of the above inequality, then

$$z(v, s, \dots, t) \leq r(v, s, \dots, t)$$

$$\begin{aligned} r(x) &\leq \sum_{i=0}^{n-1} \left[ |g(x_i) - G(x_i)| + |h(x_0) - H(x_0)| + |g(x_0) - G(x_0)| \right] + \\ &\quad + \int_{x_0}^{x_1} \int_{x_2}^{x_2} \dots \int_{x_{n-1}}^{x_n} z(v, s, \dots, t) dv ds \dots dt \end{aligned}$$

By the above inequality (Gronwall's inequality)

$$r(x) \leq \sum_{i=0}^{n-1} \left[ |g(x_i) - G(x_i)| + |h(x_0) - H(x_0)| + |g(x_0) - G(x_0)| \right] e^{\int_{x_0}^{x_1} \int_{x_2}^{x_2} \dots \int_{x_{n-1}}^{x_n} z(v, s, \dots, t) dv ds \dots dt}$$

Since

$$g(x_i) = G(x_i), h(x_0) = H(x_0) \text{ and}$$

$$g(x_0) = G(x_0) \text{ then } r(x) \leq 0 \text{ since}$$

$$|u(x_0, x_1, \dots, x_n) - v(x_0, x_1, \dots, x_n)| \leq r(x) \leq 0$$

Then

$$|u(x_0, x_1, \dots, x_n) - v(x_0, x_1, \dots, x_n)| \leq 0$$

and since the absolute value larger than or equal to zero then

$$U(x) = v(x) \quad x \in J$$

**Example:** consider the two boundary value problems

$$u_{xy} = f(x, y, u), \quad 0 < x, y < a, a > 0 \dots (3)$$

$$u_{(x_0, y)} = g(y) = e^y,$$

$$u_{(x, y_0)} = h(x) = e^x$$

$$g(y_0) = e^{y_0} = h(x_0) = e^{x_0}$$

and

$$U_{xy} = F(x, y, U),$$

$$U(x_0, y) = G(y) = \cos y,$$

$$U(x, y_0) = H(x) = \cos x$$

$$G(y_0) = \cos y_0 = H(x_0) = \cos x_0$$

Where all the functions are continuous and  $f$  satisfies a Lipchitz condition with constant  $K$ .  $K > 0$

If and only if  $(x - x_0)(y - y_0) \geq 0$  on a compact set

### Conclusion:

1- It is easy to note that the uniqueness of a special cases solution ( $n=1$  or  $n=2$ ) can be obtain by using Bihari's and Gronwall's inequality which is give the work more accuracy and easier.

2-The quantity between the brackets in the above example is bounded by the constant  $M$ , hence  $|u - U| \leq M \in$  on this set.

3-the solution of such boundary value problem eq(3) depends consequently on  $f$  and the boundary data .

If  $\epsilon \rightarrow 0$  then  $|u - U| \rightarrow 0$  on the

compact set.

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## نظريتي الوحدانية العامة والمحلية للمعادلات التفاضلية الجزئية من الرتبة النونية

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### الخلاصة:

لقد تم الاعتماد على المترجمات التي تساعد مشتقات الدوال الجزئية ذات الرتبة النونية للحصول على نظرية الوحدانية العامة والمحلية لحل المعادلة التفاضلية الجزئية وهي تطبيق لمترجمات كرانول.